Closed Weak G-Supplemented Modules

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Abstract

A module M is called closed weak g-supplemented if for any closed submodule N of M, there is a submodule K of M such that $M = K + N$ and $K \cap N \ll_g M$ (*i.e.* K is a weak g-supplement of N in M). In this work many various properties of closed weak g-supplemented modules are investigated. We will prove a module M is closed weak g-supplemented if and only if M/X is closed weak g-supplemented for any closed submodule X of M. So, any direct summand of closed weak g-supplemented module is also closed weak g-supplemented. Every nonsingular homomorphic image of a closed weak g-supplemented module is closed weak g-supplemented. We define and study also modules, in which every cofinite closed submodule of it have weak g-supplements, namely, cofinitely closed weak g-supplemented.

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1. Introduction

Throughout this article, all rings are associative with identity and all modules are unitary left R-modules, unless otherwise stated. A submodule N of a module M is called essential if $N \cap K \neq 0$ for any nonzero submodule K of M. If $N + K \neq M$ for any proper submodule K of M, then N is called a small submodule. A closed submodule N of M, is a submodule which has no proper essential extensions inside M [1]. A module M is called supplemented (weak suppl-emented) if for any submodule N of M, there is a submodule K of M such that $M = K + N$ and $K \cap N$ is small in N (resp. in M), see [2], [3]. Recall from [4] that a module M is called closed weak supplemented if for any closed submodule N of M, there is a submodule K of M such that $M = K + N$ and $K \cap N$ is small in M. A submodule N of M is said to be g-small if for every essential submodule K of M with $M = N + K$ implies $K = M$ (in [5], it is called an e-small submodule of M and denoted by $N \ll_e M$). Let U and V be submodules of a module M. If $M = U + V$ and $M = U + X$ with X is essential in V, implies $X = V$, or equivalently, $M = U + V$ and $U \cap V$ is g-small in V, then V is called a g-supplement of U in M. If every submodule of M has a g-supplement in M, then M is called a g-supplemented module [6]. If $M = U + V$ and $U \cap V$ is g-small in M, then V is called a weak g-supplement of U in M.

If every submodule of M has a weak g-supplement in M, then M is called a weakly g-supplemented module [7]. Recall that a module M is said to be extending (or CS-module) if for any submodule N of M, there exists a decomposition $M = M_1 \oplus M_2$ such that N is essential in M_1, or equivalently, a module M is extending if and only if every closed submodule of M is a direct summand [8]. In this article, we replace the condition of extending modules by the condition that the closed submodule has a weak g-supplemented, such as modules, called closed weak g-supplemented modules, which is a proper generalization for both extending and weakly g-supplemented modules. In Section 2, we define and investigate the class of closed weak g-supplemented modules, and give various properties of them. In Section 3, we stated some conditions that make homomrphic image of a closed weak g-supplemented module is also closed weak g-supplemented. For an integral domain R, we prove that a torsion free homomorphic image R-module of a closed weak g-supplemented module is closed weak g-supplemented. A submodule N of a module M is called cofinite if M/N is finitely generated. [7], have defined cofinitely weak g-supplemented modules as a proper generalization of weakly g-supplemented. A module M is said to be cofinitely weak g-supplemented if every cofinite submodule of M has a weak g-supplement in M. In Section 4, we introduce the notion of cofinitely closed weak g-supplemented modules as a proper generalization of closed weak g-supplemented and cofinitely weak g-supplemented modules. If every cofinite and closed submodule of M has (is) a weak g-supplement in M, then M is called a cofinitely closed weak g-supplemented module. We provide some properties of these modules. For a left R-module M, the notations $N \subseteq M$, $N \leq M$, $N \leq_e M$, $N \ll_g M$, $N \leq^c M$ or $N \leq^{cc} M$ mean that N is a subset, a submodule, an essential submodule, a g-small submodule, a closed submodule, or cofinite and closed submodule of M, respectively . We will denotes $l_R(M) = \{r \in R \mid rm = 0 \text{ for all } m(\neq 0) \in M\}$.

First, we will state some well-known properties of g-small submodules in [5] which needed in this work.

Lemma 1.1. Let $K \leq N$ and L_i (for $1 \leq i \leq n$) be submodules of a module M. Then the following conditions are hold.

(i) If $N \ll_g M$, then $K \ll_g M$ and $N/K \ll_g M/K$.

(ii) $\sum_{i=1}^{n} L_i \ll_g M$ if and only if $L_i \ll_g M$ for $1 \leq i \leq n$.

(iii) If $\grave{M}$ is another module and let $\varphi: M \to \grave{M}$ be a homomorphism, then $\varphi(K) \ll_g \grave{M}$ where $K \ll_g M$. In particular, if $K \ll_g N \leq M$ then $K \ll_g M$.

(iv) Assume that $K_1 \leq M_1$ and $K_2 \leq M_2$, where $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2 \ll_g M_1 \oplus M_2$ if and only if $K_i \ll_g M_i$ for $i = 1,2$.

2. Closed Weak G-supplemented Modules

Recall [7] that a module M is called weakly g-supplemented if for every submodule N of M, there is a submodule K of M such that $M = K + N$ and $K \cap N \ll_g M$. In this section, we generalized the concept of weak g-supplemented modules to the notion of closed weak g-supplemented modules. Several properties of this class has been discussed.

Definition 2.1. A module M is said to be closed weak g-supplemented if for all closed submodule N of M, there is a submodule K of M such that $M = K + N$ and $K \cap N \ll_g M$ (*i.e.* K is a weak g-supplement of N in M).

Clearly, all of simple, extending, uniform and semisimple are closed weak g-supplemented modules. Also, we see that closed weak supplemented modules are closed weak g-supplemented, and weakly g-supplemented modules are closed weak g-supplemented, but the converse need not be true, in general. Before, we need the following Proposition which appeared in [9].

Proposition 2.2. Let M be an indecomposable R-module. Then a proper submodule N of M is small if and only if it is g-small.

Now we have the following example.

Example 2.3. Assume $R = M = Z$, and $N = 2Z \leq M$. Since M is a uniform R-module, so it is closed weak g-supplemented. On the other hand, $K = 3Z$ is the only submodule of M such that $K + N = Z$, but $K \cap N = 6Z$ is not g-small in Z as Z-module, by apply Proposition 2.2 (in fact, $6Z$ is not small in Z, and Z is an indecomposable as Z-module), this mean $N = 2Z$ does not have a weak g-supplement in Z, hence Z as Z-module is not weakly g-supplemented.

Recall that a module M is called Hollow (e-Hollow) if every proper submodule of M is small (resp. g-small) [Hadi & Aidi, 2015]. It is clear that Hollow modules are e-Hollow, and hence it is weakly g-supplemented. So we have the following implications :

↱ e-Hollow → Weakly G-supplemented → Closed Weak G-supplemented

Hollow

↳ Weak supplemented → Closed Weak supplemented → Closed Weak G-supplemented

Proposition 2.4. Let M be an indecomposable R-module. Then M is closed weak supplemented if and only if M is closed weak g-supplemented.

Proof. It follows directly from Proposition 2.2.

For an R-module M, the set $Z(M) = \{m \in M \mid l_R(m) \leq_e R\}$ is called a singular submodule. A module M is called singular if $Z(M) = M$, and it is called nonsingular if

$Z(M) = 0$ [1]. A submodule N of M is said to be δ-small (briefly $N \ll_\delta M$), if $M = K + N$ with M/K singular, implies that $K = M$, where $K \leq M$ [10]. Notice, every δ-small submodule is g-small (in fact, M/K is singular, whenever $K \leq_e M$), but not conversely. Recall that a module M is called closed weak δ-supplemented if for each $N \leq^c M$, there is a submodule K of M such that $M = K + N$ and $K \cap N \ll_\delta M$ [11].

Proposition 2.5. Let M be a nonsingular R-module. Then a proper submodule N of M is δ-small if and only if it is g-small.

Proof. See [12].

[5], proved that if a module is projective, then δ-small and g-small submodules are equivalent. So, we have the following result.

Proposition 2.6. If M is a nonsingular (or projective) R-module. Then M is closed weak δ-supplemented if and only if M is closed weak g-supplemented.

The following Lemma is appeared in [7].

Lemma 2.7. Let M be an R-module, $X \leq U \leq M$ and V be a weak g-supplement of U in M. Then $(V + X)/X$ is a weak g-supplement of U/X in M/X.

Proposition 2.8. Let M be a module and $X \leq^c M$. If M is a closed weak g-supplemented module, then the factor module M/X is closed weak g-supplemented.

Proof. Assume that U/X is a closed submodule of M/X, so U is closed in M (because $X \leq^c M$). Since M is a closed weak g-supplemented module, then there is a weak g-supplement V of U in M, hence U/X has a weak g-supplement $(V + X)/X$ in M/X by previous Lemma. Therefore M/X is a closed weak g-supplemented module.

Corollary 2.9. Let M be a module. Then M is closed weak g-supplemented if and only if M/X is closed weak g-supplemented, for any closed submodule X of M.

Corollary 2.10. Let M be a closed weak g-supplemented module. Then any direct summand of M is closed weak g-supplemented.

Proof. Let N be a direct summand of M (*i.e.* $M = N \oplus K$ for some $K \leq M$), then $K \leq^c M$. Since M is a closed weak g-supplemented module, then by Proposition 2.8, M/K is also closed weak g-supplemented. But $M/K = (N \oplus K)/K \cong N$. Hence N is closed weak g-supplemented.

Lemma 2.11. Let N and L be submodules of a module M with H is a weak g-supplement of $N+L$ in M, and G is a weak g-supplement of $(H+L)\cap N$ in N. Then L has a weak g-supplement $H+G$ in M.

Proof. Since H is a weak g-supplement of $N+L$ in M, and G is a weak g-supplement of $(H+L)\cap N$ in N, then we get $M=(N+L)+H$, $(N+L)\cap H \ll_g M$, $N=\big((H+L)\cap N\big)+G$ and $\big((H+L)\cap N\big)\cap G=(H+L)\cap G \ll_g N$ (also in M). Thus, $(H+G)\cap L \leq \big((H+L)\cap G\big)+\big((G+L)\cap H\big) \leq \big((H+L)\cap G\big)+\big((N+L)\cap H\big) \ll_g M$ by Lemma 1.1(ii), and also $M=N+(L+H)=\big((H+L)\cap N\big)+G+(L+H)=(H+G)+L$. Hence $H+G$ is a weak g-supplement

of L in M.

Proposition 2.12. Let $M=M_1 \oplus M_2$ with M_i is a closed weak g-supplemented module for $i=1,2$. Suppose that $M_i\cap(M_j+L)\leq^c M_i$ and $M_j\cap(L+K)\leq^c M_j$, where K is a weak g-supplement of $M_i\cap(M_j+L)$ in M_i, and L is any closed submodule of M with $(i\neq j)$. Then M is a closed weak g-supplemented module.

Proof. Assume that L is any closed submodule of $M=M_1\oplus M_2$. Trivially $M=M_1+(M_2+L)$ has a weak g-supplement 0 in M. By hypothesis, $M_1\cap(M_2+L)\leq^c M_1$ and M_1 is closed weak g-supplemented, then there is a submodule K of M_1 such that $M_1=K+(M_1\cap(M_2+L))$ and $K\cap\big(M_1\cap(M_2+L)\big)=K\cap(M_2+L)\ll_g M_1$. By Lemma 2.11, we get $0+K=K$ is a weak g-supplement of M_2+L in M. Since $M_2\cap(L+K)\leq^c M_2$ and M_2 is a closed weak g-supplemented module, hence $M_2\cap(L+K)$ has a weak g-supplement P of M_2. Again by Lemma 2.11, $K+P$ is a weak g-supplement of L in M. Therefore M is a closed weak g-supplemented module.

Proposition 2.13. Let $M=M_1+M_2$ be a module such that M_1 is a closed weak g-supplemented module and M_2 is any module. Suppose that $N\cap M_1\leq^c M_1$ for any closed submodule N of M. Then M is a closed weak g-supplemented module if and only if for any closed submodule N of M with $M_2\nsubseteq N$ has a weak g-supplement.

Proof. ⇒) Clear.

⇐) Let N be a closed submodule of M. If $M_2\nsubseteq N$, so by a condition, N has a weak g-supplement. Now, if $M_2\subseteq N$ then $M=M_1+N$ and it has a weak g-supplement $H=0$ in M. Since M_1 is closed weak g-supplemented and $N\cap M_1\leq^c M_1$, then $N\cap M_1$ has a weak g-supplement G in M_1. By applying Lemma 2.11, N has a weak g-supplement $H+G=G$ in M. From two cases, M is a closed weak g-supplemented module.

Following [1], a submodule N of a module M is called $\mathfrak{J}$-closed if M/N is non-singular. Goodearl K.R. in Proposition 2.2.4, proved that every $\mathfrak{J}$-closed submodule of a module is closed. The converse holds, whenever a module is nonsingular. However we have the following Corollary.

Corollary 2.14. Let $M = M_1 + M_2$ be a nonsingular module such that M_1 is a closed weak g-supplemented module and M_2 is any module. Then M is a closed weak g-supplemented module if and only if for any closed submodule N of M with $M_2 \nsubseteq N$ has a weak g-supplement.

Proof. Let M be a nonsingular module. If $N \leq^c M$, then N is a $\mathfrak{J}$-closed submodule of M (since M is nonsingular); that is $\frac{M}{N}$ is nonsingular. Since $\frac{M_1+N}{N} \leq \frac{M}{N}$ then $\frac{M_1+N}{N}$ is so a nonsingular module. But we have $\frac{M_1+N}{N} \cong \frac{M_1}{N \cap M_1}$, so $\frac{M_1}{N \cap M_1}$ is nonsingular, this mean $N \cap M_1$ is a $\mathfrak{J}$-closed submodule of M_1, hence $N \cap M_1 \leq^c M_1$ for any closed submodule N of M. Thus the result is obtained by Proposition 2.13.

Proposition 2.15. Let M be a module has the property $N \cap L \leq^c M$ for all submodules N, L of M. If L is a closed weak g-supplemented module, then N has a weak g-supplement inside L where $M = N + L$.

Proof. Assume that $M = N + L$. By assumption, $N \cap L \leq^c M$ but $N \cap L \leq L$ in M, so we have $N \cap L \leq^c L$. Since L is a closed weak g-supplemented module, then there is a weak g-supplement K of $N \cap L$ in L, *i.e.* $K + (N \cap L) = L$ and $K \cap (N \cap L) = K \cap N \ll_g L$. Thus $L \leq K + N$, and hence $M = N + L \leq K + N$. Thus $M = K + N$ and $K \cap N \ll_g M$ where K is a submodule of L. Therefore N has a weak g-supplement K in L.

Corollary 2.16. If M is a semisimple module and let N, L be submodules of M, then there is a weak g-supplement of N inside L where $M = N + L$.

Theorem 2.17. Let $M = M_1 \oplus M_2$ be an R-module such that $l_R(M_1) + l_R(M_2) = R$. Then M is closed weak g-supplemented if and only if each M_i, $i \in \{1,2\}$, is closed weak g-supplemented.

Proof. The necessity follows directly from Corollary 2.10. Conversely, let M_1 and M_2 are both closed weak g-supplemented R-modules. If $N \leq^c M$, and since $l_R(M_1) + l_R(M_2) = R$, then by [12] $N = A \oplus B$ where $A \leq M_1$ and $B \leq M_2$. By transitivity for closed submodules, we get $A \leq^c M_1$ and $B \leq^c M_2$. Since M_i is a closed weak g-supplemented module, for $i \in \{1,2\}$, we have $A_1 + A = M_1$, $A_1 \cap A \ll_g M_1$, $B_1 + B = M_2$ and $B_1 \cap B \ll_g M_2$ for some $A_1 \leq M_1$ and $B_1 \leq M_2$. Put $K = A_1 \oplus B_1$, so $K \leq M$. Thus $K + N = (A_1 \oplus B_1) + (A \oplus B) = (A_1 + A) \oplus (B_1 + B) = M_1 \oplus M_2 = M$, and $K \cap N = (A_1 \oplus B_1) \cap (A \oplus B) = (A_1 \cap A) \oplus (B_1 \cap B) \ll_g M_1 \oplus M_2 = M$ by Lemma 1.1 (iv), this mean K is a weak g-supplement of N in M. Hence $M = M_1 \oplus M_2$ is a closed weak g-supplemented R-module.

A submodule N of a module M is called distributive if $N \cap (X + Y) = (N \cap X) + (N \cap Y)$ for all $X, Y \leq M$. A module M is called distributive if for all its submodules are distributive [13]. Recall that an R-module M is called duo if for any submodule N of M, and for all $\varphi \in End(M)$, $\varphi(N) \subseteq N$ (*i.e.* N is fully invariant) [18]. We know that every duo module is distributive. However, we have the following Theorem.

Theorem 2.18. Let $M = M_1 \oplus M_2$ be a distributive R-module. Then, each M_i, $i \in \{1,2\}$, is closed weak g-supplemented if and only if M is closed weak g-supplemented.

Proof. Assume that M_1 and M_2 are closed weak g-supplemented R-modules, and let $A \leq^c M$. We claim that $A \cap M_i \leq^c M_i$ for each $i \in \{1,2\}$ as follows: let $A \cap M_i \leq_e B$ in M_i, and since M is a distributive R-module, then we have $A = A \cap M = A \cap (M_1 \oplus M_2) = (A \cap M_1) \oplus (A \cap M_2) \leq_e B \oplus (A \cap M_2)$ in M, but $A \leq^c M$ this implies $A = (A \cap M_1) \oplus (A \cap M_2) = B \oplus (A \cap M_2)$, and so $B = A \cap M_1$, thus $A \cap M_1 \leq^c M_1$. Similarly, $A \cap M_2 \leq^c M_2$. Since M_1 and M_2 are closed weak g-supplemented, then there is a submodule N_i of M_i such that $M_i = N_i + (A \cap M_i)$ and $N_i \cap (A \cap M_i) = N_i \cap A \ll_g M_i$ for each $i \in \{1,2\}$. Put $K = N_1 \oplus N_2$. So, we get $M = M_1 \oplus M_2 = (N_1 + (A \cap M_1)) \oplus (N_2 + (A \cap M_2)) = (N_1 \oplus N_2) + \big((A \cap M_1) \oplus (A \cap M_2)\big) = (N_1 \oplus N_2) + (A \cap (M_1 \oplus$

$M_2) = (N_1 \oplus N_2) + (A \cap M) = (N_1 \oplus N_2) + A = K + A$, since M is a distributive module. Also, $K \cap A = (N_1 \oplus N_2) \cap A = (N_1 \cap A) \oplus ((N_2 \cap A) \ll_g M_1 \oplus M_2 = M$ (because M is distributive, and by Lemma 1.1 (iv)) this mean K is a weak g-supplement of A in M, and hence M is closed weak g-supplemented. The converse, follows directly by Corollary 2.10.

Corollary 2.19. Let $M = \oplus_{i=1}^{n} M_i$ be a distributive (or duo) R-module. Then M is closed weak g-supplemented if and only if for all M_i, $i \in \{1,2,\dots,n\}$, is closed weak g-supplemented

A ring R is said to be closed weak g-supplemented if, R is a left closed weak g-supplemented as R-module.

Next, we will discuss the relation between closed weak g-supplemented rings and modules. In the following two results, R is a commutative ring.

Lemma 2.20. Let M be a finitely generated faithful multiplication R-module. Then the following assertions are hold.

(i) $L_1 \leq_e L_2$ in M if and only if $I_1 \leq_e I_2$ in R, where $L_i = I_i M$ for $i \in \{1,2\}$.

(ii) $L \ll_g M$ if and only if $I \ll_g R$, where $L = IM$.

Proof. (i) Assume $L_1 \leq_e L_2$ in M , to prove $I_1 \leq_e I_2$ in R, where $L_i = I_i M$ for $i \in \{1,2\}$. Let $J \leq I_2$ such that $I_1 \cap J = 0$, then $L_1 \cap JM = I_1 M \cap JM = (I_1 \cap J)M = 0.M = 0$, but $L_1 \leq_e L_2$ and $JM \leq L_2$ implies that $JM = 0$, so $J \subseteq l_R(M) = 0$, and hence $J = 0$. So $I_1 \leq_e I_2$ in R. Conversely, let $K \leq L_2$ such that $L_1 \cap K = 0$, so there is $J \leq I_2$ in R with $K = JM$. Thus, we have $(I_1 \cap J)M = I_1 M \cap JM = L_1 \cap K = 0$ implies $I_1 \cap J \subseteq l_R(M) = 0$, but $I_1 \leq_e I_2$ and $J \leq I_2$, hence $J = 0$, $K = JM = 0$. Therefore $L_1 \leq_e L_2$ in M.

(ii) Suppose that $L \ll_g M$. If $I + J = R$ with $J \leq_e R$, then $M = RM = (I + J)M = IM + JM = L + JM$, where $JM \leq_e M$ by (i). As $L \ll_g M$, we get $M = RM = JM$ implies that

$J = R$, and so $I \ll_g R$. Conversely, assume $L + K = M$, where $K \leq_e M$. So by (i), there is $J \leq_e R$ such that $K = JM$. Thus, we have $(I + J)M = IM + JM = L + K = M = RM$, then $I + J = R$, but $J \leq_e R$ and $I \ll_g R$, so $J = R$ implies $K = JM = RM = M$. Therefore $L \ll_g M$.

Theorem 2.21. Let M be a finitely generated faithful multiplication R-module. Then M is closed weak g-supplemented if and only if R is closed weak g-supplemented.

Proof. Assume that M is a closed weak g-supplemented R-module, and let $I \leq^c R$. It is easy to see that $L = IM \leq^c RM = M$, so there is $N = JM \leq M$ where $J \leq R$, such that $N + L = M$ and $N \cap L \ll_g M$, as M is closed weak g-supplemented. Hence $(I + J)M = IM + JM = L + N = M = RM$, so $I + J = R$, also $(I \cap J)M = L \cap N \ll_g M = RM$, so by Lemma 2.20 (ii), $I \cap J \ll_g R$, hence I has a weak g-supplement J of R, and so R is closed weak g-supplemented. Conversely, let $N = JM \leq^c M$ where $J \leq R$, so it is easy to see that $J \leq^c R$. There is $I \leq R$ such that $I + J = R$ and $I \cap J \ll_g R$, as R is closed weak g-supplemented. It follows, $M = RM = (I + J)M = IM + JM = IM + N$, and by Lemma 2.20 (ii), $IM \cap N = (I \cap J)M \ll_g RM = M$, that is IM is a weak g-supplement of N in M. Hence M is a closed weak g-supplemented R-module.

Next, we shall discuss the behavior of closed weak g-supplemented modules under localization. Firstly, we prove the following Lemma.

Lemma 2.22. Let M be an R-module and S be a multiplicative closed subset of R, provided $S^{-1}A = S^{-1}B$ *iff* $A = B$ for each $A, B \leq M$. Then the following assertions hold.

(i) $N \leq_e K$ in R-module M if and only if $S^{-1}N \leq_e S^{-1}K$ in $S^{-1}R$-module $S^{-1}M$.

(ii) $N \ll_g K$ in R-module M if and only if $S^{-1}N \ll_g S^{-1}K$ in $S^{-1}R$-module $S^{-1}M$.

(iii) K is a weak g-supplement of N in R-module M if and only if $S^{-1}K$ is a weak g-supplement of $S^{-1}N$ in $S^{-1}R$-module $S^{-1}M$.

Proof. (i) Assume that $N \leq_e K \leq M$ as R-module. If $S^{-1}L \leq S^{-1}K$ such that $S^{-1}N \cap S^{-1}L = S^{-1}(0)$, where $L \leq K$, then $S^{-1}(N \cap L) = S^{-1}(0)$, so by assumption we have $N \cap L = 0$ implies $L = 0$, as $N \leq_e K$ and $L \leq K$. Thus $S^{-1}L = S^{-1}(0)$, and hence $S^{-1}N \leq_e S^{-1}K \leq S^{-1}M$ as $S^{-1}R$-module. Conversely, let $N \cap L = 0$ where $L \leq K$, then $S^{-1}N \cap S^{-1}L = S^{-1}(N \cap L) = S^{-1}(0)$, where $S^{-1}L \leq S^{-1}K$, this implies $S^{-1}L = S^{-1}(0)$, as $S^{-1}N \leq_e S^{-1}K$, so by assumption $L = 0$. Therefore $N \leq_e K \leq M$ as R-module.

(ii) Let $N \ll_g K \leq M$. Suppose that $S^{-1}N + S^{-1}L = S^{-1}K$ where $S^{-1}L \leq_e S^{-1}K$. So, $S^{-1}K = S^{-1}(N + L)$ implies that $N + L = K$ by assumption. Since $S^{-1}L \leq_e S^{-1}K$, then $L \leq_e K$, by (i), that is, we have $N + L = K$ and $L \leq_e K$, so $L = K$ (since $N \ll_g K$), and hence $S^{-1}L = S^{-1}K$. Conversely, assume that $N + L = K$ where $L \leq_e K$. Thus $S^{-1}N +$

$S^{-1}L = S^{-1}(N + L) = S^{-1}K$ and $S^{-1}L \leq_e S^{-1}K$ by (i), so $S^{-1}L = S^{-1}K$, since $S^{-1}N \ll_g S^{-1}K$. By assumption, $L = K$ and hence $N \ll_g K$.

(iii) Assume K is a weak g-supplement of N in R-module M, then $K + N = M$ and $K \cap N \ll_g M$, so $S^{-1}K + S^{-1}N = S^{-1}(K + N) = S^{-1}M$ and $S^{-1}K \cap S^{-1}N = S^{-1}(K \cap N) \ll_g S^{-1}M$ by (ii), this means $S^{-1}K$ is a weak g-supplement of $S^{-1}N$ in $S^{-1}R$-module $S^{-1}M$. The converse, by a similar way.

Lemma 2.23. Let M be an R-module, $N \leq M$ and let S be a multiplicative closed subset of R. Then N is closed in M as R-module if and only if $S^{-1}N$ is closed in $S^{-1}M$ as $S^{-1}R$-module, provided $S^{-1}A = S^{-1}B$ *iff* $A = B$ for each $A, B \leq M$.

Proof. See [14]

The next Theorem is a consequence of the previous two Lemma's.

Theorem 2.24. Let M be an R-module and let S be a multiplicative closed subset of R. Then M is a closed weak g-supplemented as R-module if and only if $S^{-1}M$ is a closed weak g-supplemented as $S^{-1}R$-module, provided $S^{-1}A = S^{-1}B$ *iff* $A = B$ for each $A, B \leq M$.

Corollary 2.25. Let M be an R-module. For each maximal ideal P of R, M is a closed weak g-supplemented R-module if and only if M_P is a closed weak g-supplemented R_P-module.

3. The Homomorphic Images

In this section, we will consider the conditions for which the homomorphic images of closed weak g-supplemented modules are also closed weak g-supplemented. We know that any image of a weakly g-supplemented module is weakly g-supplemented, see [Nebiyev & Okten, 2017, Cor.5]. However, we start with the following definition.

Definition 3.1. [14] Let M and N be R-modules. M is called relatively c-Rickart to N if for any $\varphi \in Hom_R(M, N)$, $Ker\varphi$ is closed in M.

Corollary 3.2. Let $\varphi: M \to N$ be an R-homomorphism. If M is closed weak g-supplemented and relatively c-Rickart to N, then $Im\varphi$ is closed weak g-supplemented.

Proof. Assume that $\varphi \in Hom_R(M, N)$. Since M is relatively c-Rickart to N, then $Ker\varphi \leq^c M$. So by Proposition 2.8, $M/Ker\varphi$ is closed weak g-supplemented. But we have $M/Ker\varphi \cong Im\varphi$, therefore $Im\varphi$ is a closed weak g-supplemented module.

Let M be a module over an integral domain R. If the set $T(M) = \{m \in M | rm = 0$ for some $r(\neq 0) \in R\}$ is equal zero, then M is called torsion free [15].

Proposition 3.3. Let $\varphi: M \to N$ be an R-epimorphism with N is a torsion free module over an integral domain R. If M is a closed weak g-supplemented R-module, so is N.

Proof. Assume M is a closed weak g-supplemented R-module, and $K \leq^c N$. Since N is torsion free, then by [19] Lemma 3.1, there is a submodule H of M such that $Ker\varphi \leq H \leq^c M$ and $K \cong H/Ker\varphi$. Since M is closed weak g-supplemented and $H \leq^c M$, then H has a weak g-supplement L of M, and so $(L + Ker\varphi)/Ker\varphi$ is a weak g-supplement of $H/Ker\varphi \cong K$ in $M/Ker\varphi \cong N$, by Lemma 2.7. Therefore N is closed weak g-supplemented.

Lemma 3.4. Let $\varphi: M \to N$ be an R-homomorphism, and $L \leq^c N$. If N is a nonsingular module, then $H = \varphi^{-1}(L)$ is a closed submodule of M.

Proof. See [4].

Theorem 3.5. Any nonsingular homomorphic image of a closed weak g-supplemented module is also closed weak g-supplemented.

Proof. Let M be a closed weak g-supplemented module, $\varphi: M \to N$ be an R-epimorphism with $Im\varphi = N$ is a nonsingular module. Let $L \leq^c N$, then by previous Lemma, $H = \varphi^{-1}(L) \leq^c M$, so there is a submodule K of M such that $M = K + H$ and $K \cap H \ll_g M$, as M is a closed weak g-supplemented module. Thus, we have $N = \varphi(M) = \varphi(K) + \varphi(H) = \varphi(K) + L$. Since $Ker\varphi = \varphi^{-1}(0) \subseteq H$, then $\varphi(K \cap H) = \varphi(K) \cap \varphi(H) = \varphi(K) \cap L \ll_g N$, by Lemma 1.1($iii$). So L has a weak g-supplement $\varphi(K)$ in N, and hence N is a closed weak g-supplemented module.

In above theorem, the condition of nonsingularity of N is not necessary, as example: we know that Z is a closed weak g-supplemented Z-module, and let $\pi: Z \to Z_p$ be a natural map, for any prime p. Since Z_p is a simple Z-module, so it is closed weak g-supplemented. Note that Z_p is a singular Z-module, for any prime p.

Corollary 3.6. A nonsingular factor module of a closed weak g-supplemented module is also closed weak g-supplemented.

Let R be a ring, it is well known that R is left nonsingular if and only if all left projective R-modules are nonsingular. However, we have the following Corollary.

Corollary 3.7. Let R be a left nonsingular ring. Then the following statements are equivalent.

(i) Every projective left R-module is closed weak g-supplemented.

(ii) Every nonsingular left R-module is closed weak g-supplemented.

Proof. Assume (i), let M be a left nonsingular R-module, so M is a homomorphic image of a free R-module F, then F is projective, and hence F is a left closed weak g-supplemented R-module, by (i). This mean M is a nonsingular homomorphic image of a closed weak g-supplemented module F, so M is also closed weak g-supplemented by Theorem 3.5. Conversely, is clear.

Corollary 3.8. Let R be a left nonsingular ring. Then the following statements are equivalent.

(i) R is a left closed weak g-supplemented ring.

(ii) Every left nonsingular cyclic R-module is closed weak g-supplemented.

(iii) Every principal left ideal of R is closed weak g-supplemented.

Proof. $(i) \Rightarrow (ii)$ Let $M = Rm$ be a left nonsingular R-module, where $m(\neq 0) \in M$. Consider an epimorphism $\varphi: R \to Rm$ which defined by $\varphi(r) = rm$ for all $r \in R$. So, we have $M = Rm$ is a nonsingular homomorphic image of R which is a left closed weak g-supplemented R-module, hence M is closed weak g-supplemented, by Theorem 3.5.

$(ii) \Rightarrow (iii)$ Let $I = Ra$ be a principal left ideal of R, where $a(\neq 0) \in R$. Since R is a left non-singular ring (by assumption), then I is so nonsingular; this means I is a nonsingular cyclic R-module, so by (ii), I is closed weak g-supplemented.

$(iii) \Rightarrow (i)$ Clearly, R is generated by identity 1 (*i.e.* R is a principal ideal of itself), so by (iii), R is a left closed weak g-supplemented ring.

Let M be a module. In [5], define the set $Rad_g(M) = \bigcap\{N \leq_e M \mid N \text{ is maximal in } M\} = \sum\{N \subseteq M \mid N \ll_g M\}$. However, we give a condition under which the concepts extending and closed weak g-supplemented modules are coincide.

Proposition 3.9. Let M be an R-module such that $Rad_g(M) = 0$. Then M is extending if and only if M is closed weak g-supplemented.

Proof. Assume that M is a closed weak g-supplemented module. Let L be a closed submodule of M, so L has a weak g-supplement K of M (*i.e.* $M = K + L$ and $K \cap L \ll_g M$), thus $K \cap L \subseteq Rad_g(M) = 0$, this mean $M = K \oplus L$, so L is a direct summand of M, and hence M is extending. The converse, is clear.

Corollary 3.10. Let M be an R-module such that $\frac{M}{Rad_g(M)}$ is nonsingular. If M is a closed weak g-supplemented module, then $\frac{M}{Rad_g(M)}$ is extending.

Proof. Suppose M is a closed weak g-supplemented module. By a natural mapping $M \to \frac{M}{Rad_g(M)}$, we have $\frac{M}{Rad_g(M)}$ is a nonsingular homomorphic image of M, so by Theorem 3.5, $\frac{M}{Rad_g(M)}$ is so closed weak g-supplemented. But $Rad_g\left(\frac{M}{Rad_g(M)}\right) = Rad_g(M) = 0_{\frac{M}{Rad_g(M)}}$, hence $\frac{M}{Rad_g(M)}$ is extending, by Proposition 3.9. ❑

Theorem 3.11. Let R be a left nonsingular ring with $Rad_g(M) = 0$ for all left R-modules M. Then the following statements are equivalent.

(i) Every projective left R-module M is closed weak g-supplemented.

(ii) Every nonsingular left R-module M is closed weak g-supplemented.

(iii) Every nonsingular left R-module M is extending.

(iv) Every nonsingular left R-module M is projective.

Proof. $(i) \Leftrightarrow (ii)$ It follows by Corollary 3.7, $(ii) \Leftrightarrow (iii)$ it follows by Proposition 3.9.

$(i) \Rightarrow (iv)$ Let M be a nonsingular R-module, so there is a free (it is projective) R-module F such that $M \cong F/L$ for some submodule L of F. Since M is nonsingular, then F/L is so nonsingular (*i.e.* L is a $\mathfrak{Z}$-closed submodule of F), hence L is closed in F. On the other hand, F is a closed weak g-supplemented R-module, by (i). By assumption $Rad_g(F) = 0$ implies F is extending, by Proposition 3.9. Hence L is a direct summand of F (*i.e.* $F = L \oplus K$ for some $K \le M$). Thus $M \cong F/L \cong K$, this mean M isomorphic to a direct summand of a free R-module F, therefore M is projective, by [16].

$(iv) \Rightarrow (ii)$ Let M be a nonsingular R-module, and $N \le^c M$. By [1], N is $\mathfrak{Z}$-closed in M; that is M/N is nonsingular, so M/N is projective by (iv). Consider the natural epimorphism $\pi: M \to M/N$. Since M/N is projective, then π splits (*i.e.* $Ker\pi = N$ is a direct summand of M), hence M is extending. Therefore M is a closed weak g-supplemented R-module.

4. Cofinitely Closed Weak G-supplemented Modules

A module M is called a cofinitely weak supplemented (g-supplemented) module if for every cofinite submodule of M has (is) a weak supplement (resp. weak g-supplement), see [17] , [7]. A submodule N of a module M is said to be cofinite if the factor module M/N is finitely generated. In this section we define and study a special type of cofinitely weak g-supplemented and closed weak g-supplemented modules, namely, cofinitely closed weak g-supplemented modules as follows :

Definition 4.1. Let M be an R-module. Then M is called cofinitely closed weak g-supplemented if every cofinite closed submodule N of M has (is) a weak g-supplement (*i.e.* for each $N \le^{cc} M$, $M = K + N$ and $K \cap N \ll_g M$ for some $K \le M$).

So, we clearly have the following implications for modules :

Cofinitely Weak supplemented

↓

Cofinitely Weak G-supplemented

↓

Cofinitely Closed Weak G-supplemented

↑

Closed Weak G-supplemented

Proposition 4.2. Let M be a finitely generated module. If M a cofinitely closed weak g-supple mented module, then M is closed weak g-supplemented.

Proof. Let $N \leq^c M$. Since M is a finitely generated module, then M/N is so finitely generated; that is, N is a cofinite submodule of M, thus N has a weak g-supplement in M, as M is cofinitely closed weak g-supplemented. Therefore M is closed weak g-supplemented.

Example 4.3. Suppose $M = Z$ as Z-module. By Example 2.3, Z is a closed weak g-supplemented Z-module, so it is cofinitely closed weak g-supplemented. But, a cofinite submodule $N = 2Z$ does not have a weak g-supplement in Z, this mean that Z is not a cofinitely weak g-supplemented as Z-module.

By using Lemma 2.11, the following two results are to prove immediately.

Proposition 4.4. For cofinitely closed weak g-supplemented modules M_1, M_2 with $M = M_1 \oplus M_2$. Suppose $M_i \cap (M_j + L) \leq^{cc} M_i$ and $M_j \cap (L + K) \leq^{cc} M_j$, where K is a weak g-supplement of $M_i \cap (M_j + L)$ in M_i, $(i \neq j)$, and for any $L \leq^{cc} M$. Hence M is a cofinitely closed weak g-supple mented module.

Proof. Analogous of proof Proposition 2.12.

Proposition 4.5. For any R-module M_2, let $M = M_1 + M_2$ be a module, where M_1 is a cofinitely closed weak g-supplemented R-module. Suppose that $N \cap M_1 \leq^{cc} M_1$ for any $N \leq^{cc} M$. Then M is cofinitely closed weak g-supplemented if and only if for every $N \leq^{cc} M$ with $M_2 \nsubseteq N$, N has a weak g-supplement.

Proof. Analogous of proof Proposition 2.13.

Corollary 4.6. For any R-module M_2, let $M = M_1 + M_2$ be a nonsingular R-module, where M_1 is finitely generated and cofinitely closed weak g-supplemented R-module. Then M is a cofinitely closed weak g-supplemented module if and only if for all $N \leq^{cc} M$ with $M_2 \nsubseteq N$, N has a weak g-supplement.

Proof. Let $N \leq^{cc} M$, where M be a nonsingular R-module. Then, we have $N \cap M_1 \leq^{c} M_1$ (see Corollary 2.14). Since M_1 is finitely generated, then $M_1/N \cap M_1$ is so finitely generated; that is, $N \cap M_1$ is cofinite in M_1, hence $N \cap M_1 \leq^{cc} M_1$ for each $N \leq^{cc} M$. Hence, the result is follow, by Proposition 4.5.

Proposition 4.7. If M is a cofinitely closed weak g-supplemented module and let $X \leq^{c} M$, then the factor module M/X is cofinitely closed weak g-supplemented.

Proof. Let $U/X \leq^{cc} M/X$, so U is closed in M (because $X \leq^{c} M$). Also, U/X is cofinite in M/X, implies $\frac{M/X}{U/X} \cong \frac{M}{U}$ is finitely generated, thus $U \leq^{cc} M$. By hypothesis, U has a weak g-supplement V in M. Thus, the result is follow, by Lemma 2.7.

Corollary 4.8. Any direct summand of a cofinitely closed weak g-supplemented module is also cofinitely closed weak g-supplemented.

Proposition 4.9. Let M be an R-module with $Rad_g(M) = 0$. Then, every cofinite closed submodule is a direct summand of M if and only if M is cofinitely closed weak g-supplemented.

Proof. ⇒) Clear.

⇐) Let N be any cofinite closed submodule of M. Since M is a cofinitely closed weak g-supple- mented module, then $M = L + N$ and $L \cap N \ll_g M$ for some $L \leq M$, so $L \cap N \subseteq Rad_g(M) = 0$, hence $M = L \oplus N$. Thus N is a direct summand of M.

Corollary 4.10. Let M be a finitely generated R-module with $Rad_g(M) = 0$. Then M is extending if and only if M is cofinitely closed weak g-supplemented.

Proof. Clear.

Theorem 4.11. Any nonsingular homomorphic image of a cofinitely closed weak g-supplemented module is cofinitely closed weak g-supplemented.

Proof. Let $\varphi: M \to N$ be an epimorphism such that M is cofinitely closed weak g-supplemented, and $Im\varphi = N$ is a nonsingular module. Let $L \leq^{cc} N$, so $\varphi^{-1}(L) = H \leq^{c} M$, by Lemma 3.4. On the other hand, we have $\frac{M}{Ker\varphi} \cong N$ and $\frac{H}{Ker\varphi} \cong \varphi(H) = L$, thus $\frac{M}{H} \cong \frac{M/Ker\varphi}{H/Ker\varphi} \cong \frac{N}{L}$ is finitely generated, hence $\varphi^{-1}(L) = H$ is a cofinite submodule of M, thus $\varphi^{-1}(L) \leq^{cc} M$. Since M is cofinitely closed weak g-supplemented, then $\varphi^{-1}(L)$ has a weak g-supplement K in M. By some steps of proof Theorem 3.5, we get $\varphi(K)$ is a weak g-supplement of L in N. Therefore N is a cofinitely closed weak g-supplemented module.

Corollary 4.12. Let M be an R-module such that $\frac{M}{Rad_g(M)}$ is finitely generated and nonsingular. If M is a cofinitely closed weak g-supplemented module, then $\frac{M}{Rad_g(M)}$ is extending.

Proof. It follows by Theorem 4.11 and Corollary 4.10.

Corollary 4.13. Let R be a left nonsingular ring. Then the following statements are equivalent.

(i) Every projective left R-module is cofinitely closed weak g-supplemented.

(ii) Every nonsingular left R-module is cofinitely closed weak g-supplemented.

Let R be a ring, if R as R-module is cofinitely closed weak g-supplemented, then R is called a cofinitely closed weak g-supplemented ring.

Corollary 4.14. Let R be a left nonsingular ring. Then the following statements are equivalent.

(i) R is a left cofinitely closed weak g-supplemented ring.

(ii) Every left nonsingular cyclic R-module is cofinitely closed weak g-supplemented.

(iii) Every principal left ideal of R is cofinitely closed weak g-supplemented.

We end this work with the following Proposition.

Proposition 4.15. Let M be a cofinitely closed weak g-supplemented R-module. If for every $U \leq^{cc} M$, and V is a weak g-supplement of U in M, $V \cap U$ has a g-supplement in V. Then U has a g-supplement in M. [20]

Proof. Let $U \leq^{cc} M$, so there is a submodule V in M such that $M = V + U$ and $V \cap U \ll_g M$ (*i.e.* V is a weak g-supplement of U in M), as M is a cofinitely closed weak g-supplemented R-module. By hypothesis, $V \cap U$ has a g-supplement submodule L in V (*i.e.* $V = L + (V \cap U)$ and $L \cap (V \cap U) = L \cap U \ll_g L$. Now, $M = V + U = L + (V \cap U) + U = L + U$. Hence U has a g-suppl ement L in M.

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الخلاصة

يُدعى المقاس M بأنه مغلق ضعيف داعم من النمط g أذا كان لكل مقاس جزئي مغلق N من M, يوجد مقاس جزئي K من M بحيث ان $M = K + N$ و $K \cap N \ll_g M$. في هذا العمل, العديد من الخواص المختلفة للمقاسات المغلقة الضعيفة الداعمة من النمط g قد تحققت. نحن سوف نبرهن أن المقاس M هو مغلق ضعيف داعم من النمط g أذا وفقط أذا كان M/X هو مقاس مغلق ضعيف داعم من النمط g لكل مقاس جزئي مغلق X من M. كل حد مباشر من مقاس مغلق ضعيف داعم من النمط g يكون كذلك. كل صورة غير منفردة لمقاس مغلق ضعيف داعم من النمط g تكون مغلق ضعيف داعم من النمط g. نحن عرَفنا و درسنا أيضا المقاسات التي يكون فيها كل مقاس جزئي مضاد منتهي و مغلق يمتلك مقاس جزئي ضعيف داعم من النمط g, سُميت, بالمقاسات المضادة المنتهية المغلقة الضعيفة الداعمة من النمط g.

الكلمات المفتاحية: المقاسات الجزئية المغلقة, المقاسات الجزئية الصغيرة من النمط g, المقاسات المغلقة الضعيفة الداعمة من النمط g, المقاسات غير المنفردة, المقاسات الجزئية المضادة المنتهية, المقاسات المضادة المنتهية المغلقة الضعيفة الداعمة من النمط g.