A New Modulus of Smoothness for Uniform Approximation

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Abstract

The estimates of best approximation using classical modulus of smoothness is not uniform. Also we sometimes need to improve the degree of best approximation near the end points. Thus we need to improve this classical modulus of smoothness. Here we define a new modulus of smoothness to achieve uniform estimates of best approximation and an improvement of a degree of such version of best approximation. Our modulus of smoothness is for k-monotone functions. Estimates for using our modulus of smoothness are introduced. Applications for these estimates are also introduced.

Keywords: modulus of smoothness, k-monotone function, best approximation.

1. Introduction

Different moduli of smoothness were introduced. But the moduli of smoothness that have the most attention in the recent years are the Ditizian–Totik modulus of smoothness [3], and Ivanov τ modulus of smoothness [4] For more see [2]. The estimates of best approximation using classical modulus of smoothness is not uniform. Sometimes we need to improve the degree of best approximation near the end points, Hear come the needness of uniform approximation if we want to get uniform estimate then we choose modulus of smoothness other than the ordinary modulus of smoothness. We define several types of moduli of smoothness of k-monotone function in L_p, spaces for $p < 1$. We introduce estimates for these moduli of smoothness and then applications for these estimates. Let $L_p\ [a,b],\ p < 0$ denote the space of $f : [a,b] \to \mathcal{R}$

$$\| f \|_p = \| f \|_{l_{p[a,b]}} = \left[\int_a^b |f(x)|^p dx\right]^{1/p}, \quad 0 < p < \infty$$

The k^{th} divided difference [1] of functions defined on the interval [a, b] is

$$f[x_0, x_1, \ldots, x_k] = \sum_{j=0}^{k} f(x_j)/\omega'(x_j),$$

where

$$\omega(x) = \prod_{j=0}^{k}(x - x_j) \qquad \text{for } k \in \{0\} \cup \mathbb{N}.$$

The k-monotone function [1] defined on the interval $[a, b]$ are those function which the kth divided difference of them are non-negative for any $k+1$ points $x_0, x_1, \ldots\ldots x_k$ in the interval $[a, b]$. If $f \in \mathrm{C}^k[a, b]$, f is k-monotone on $[a, b]$ $iff\ f^{(k)}(x) \geq 0\ for\ x \in [a, b]$. Where $\mathrm{C}^k[a, b]$ is the space of all continuous function which k-continuous derivatives.

Let us now define the k[th] order modulus of smoothness for functions f in $l_{p(j)}$, as

$$\omega_k(f, \delta, j)_p = sup\ _{0<h\leq\delta} \parallel \Delta_h^k(f, ., j) \parallel_{l_{p(j)}}$$

where

$$\Delta_h^k(f, x, j) = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f\left(x - \frac{kh}{2}\right), \quad if\ x \pm \frac{kh}{2} \epsilon j$$

Is the k[th] symmetric difference .

2. The Auxiliary Results

Here we introduce the results that we need to prove our main theorems.

If f is piecewise k-monotone function , then the following result is true.

Lemma 2.1

For f is piecewise k-monotone function, $f \epsilon L_{p[a,b]}$ we have

$$\omega_1(f, \delta, [a, b])_q \leq c(q, p, s)\ \delta^{\frac{1}{q}-\frac{1}{p}} \|f\|_{l_{p[a,b]}},\ \ 0 \leq q < p \leq 1$$

Proof

$$\begin{aligned}\omega_1(f, \delta, [a, b])_q &= sup\ _{0<h\leq\delta} \left\|\Delta_h^k(f, ., j)\right\|_q \\ &= \sup\ _{|h|\leq\delta} \left\| f\left(x + \frac{h}{2}\right) - f(x - \frac{h}{2})\right\| \\ &\leq \delta^{\frac{1}{q}-1} \left(\parallel f\left(x + \frac{h}{2}\right) \parallel_q - \parallel f\left(x - \frac{h}{2}\right) \parallel_q \right) \\ &\leq \delta^{\frac{1}{q}-1} \left[\delta^{1/p} \|f\|_{\ p}\right]\end{aligned}$$

$$\omega_1(f, \delta, [a, b])_q \ \leq\ c(p, q, s) \delta^{\frac{1}{q}-\frac{1}{p}} \|f\|_{l_{p[a,b]}} \qquad (2.1)$$

where $c(p, q, s)$ is appositive constant depends on $p, q\ and\ s$ ■

Remark 2.2

We cannot make the constant $c(p, q, s)$ in (2.1) depending on the set S since this make (2.1) not valid for any function f in $L_p[a, b]$, where S is the number of changes of k-monotonicity off in [a, b].

Lemma 2.3

$$f_\beta(x) = sin\pi\beta x, \qquad 0 \le x \le 1$$

where $\beta \in \mathbb{N}$ we have $\omega_1(f,\delta) \le c(p)\beta\delta$

Proof

$$\|\Delta_h^1(f,x)\|_p = \left\|f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)\right\|_p$$

$$= \int_{\frac{h}{2}}^{1-\frac{h}{2}} \left|sin\pi\beta\left(x+\frac{h}{2}\right) - sin\pi\beta(x-\frac{h}{2})\right|^p dx$$

$$=2^p|sin\pi\beta h/2|^p \int_{\frac{h}{2}}^{1-\frac{h}{2}}|cos\pi\beta x|^p dx$$

$$\le c(p)|sin\pi\beta h/2|^p$$

$$\omega_1(f,\delta,[0,1]) \le c(p)sup_{0<h\le\delta}|sin\pi\beta h/2|$$

$$\le c(p)|sin\pi\beta\delta/2|$$

$$\le c(p)\beta\delta$$

especially when $\delta = 1/\beta$, $\beta \to \infty$ so $\delta^{-\frac{1}{q}+\frac{1}{p}}\omega_1(f_\beta,\delta,[0,1])_q \to \infty$ there for this has no upper bounded with bounded by $\|f_\beta\|_{l_{p[0,1]}}$ ■

Remark 2.4

$$\omega_1(f,\delta,[a,b])_q \le c(p,q,s)\delta^{\frac{1}{q}-\frac{1}{p}}\|f\|_{l_{p[a,b]}} \qquad (2.1)$$

We cannot increased the power in (2.1)

Proof

For $p < \infty$, $f_\alpha = x^{-\alpha}$, $0 < \alpha < \frac{1}{p}$

f_α is monotone

$$\|f\|_{l_{p[0,1]}} = \left[\int_0^1 |f(x)|^p dx\right]^{\frac{1}{p}} = \left[\int_0^1 |x^{-\alpha}|^p dx\right]^{\frac{1}{p}}$$

$$= \left[\frac{1}{-\alpha p+1}\right]^{\frac{1}{p}}$$

$$\omega_1(f_\alpha,\delta,[0,1])_q = sup\left\|f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)\right\|$$

$$= \sup\left\|(x+\frac{h}{2})^{-\alpha} - (x-\frac{h}{2})^{-\alpha}\right\|$$

$$\ge sup_{0<h\le\delta}(\|x+h/2)^{-\alpha}\| - \|(x-h/2)^{-\alpha}\|)$$

$$\ge sup_{0<h\le\delta}\left[\int_0^1 \left|\left(x+\frac{h}{2}\right)^{-\alpha}\right|^q dx - \int_0^1 \left|(x-\frac{h}{2})^{-\alpha}\right|^q dx\right]^{\frac{1}{q}}$$

$$\ge \left[\left(\frac{1+\delta/2)^{-\alpha q+1}}{-\alpha q+1} - \frac{\left(\frac{\delta}{2}\right)^{-\alpha q+1}}{-\alpha q+1}\right]^{\frac{1}{q}} - \left[\frac{\left(1-\frac{\delta}{2}\right)^{-\alpha q+1}}{-\alpha q+1} - \frac{(\frac{\delta}{2}^{-\alpha q+1}}{-\alpha q+1}\right]^{\frac{1}{q}}$$ ■

Lemma 2.5 [1]

For f is k-monotone function $L_p[-1,1]$ space. so we can find a polynomial q_n with degree not exceeding n, satisfies

$$\|f-\mathrm{q}_n\|_p \le c(p,k)\omega_k^{\varphi}(f,n^{-1})_p \qquad (2.2)$$

Lemma 2.6 **[1]**

If $f \in L_p(I),\ 0<p<1$

$$E(\Delta^k L_p, \Pi_n \cap \Delta^k)_p \le c(p)n^{-(k+1)}E_{n-k}(f^k)\ .$$

Lemma 2.7 **[1]**

For $\mathrm{P}_n(x) \in \Pi_n\ and\ 0<p<\infty$

$$E_n(f)_p \le c(p)\omega_r^{\varphi}(f,n^{-1})_p$$

3. The Main Results

Here let us introduce our modulus of smoothness and its related results.

Definition 3.1

for is f k-monotone function and $f \in L_p[\mathrm{a},\mathrm{b}]$ the new measure of smoothness we have

$$\gamma_\delta(f,q,p) = \delta^{q-p}$$

Remark 3.2

$$\gamma_\delta(f,q,p) = \delta^{q-p}$$

We cannot increased the power of δ.

Proof

$$f_\epsilon(x) = (1+x)^{\epsilon-\frac{1}{p}} \qquad \epsilon>0\ ,\epsilon-\frac{1}{p}\notin \mathbb{N}_0$$

There $f$and $-f$ are k-monotone

$$\|f_\epsilon\|_{l_{p[0,1]}} = \left[\int_0^1 |f(x)|^p dx\right]^{\frac{1}{p}}$$

$$= \left[\int_0^1 \left|(1+x)^{\epsilon-\frac{1}{p}}\right|^p dx\right]^{\frac{1}{p}}$$

$$\ge \left[\frac{1^{\epsilon p}}{\epsilon p}\right]^{\frac{1}{p}} \ge c(\in,p)$$

$$\omega_1(f_\epsilon,\delta,[0,1])_q = sup\left\|f\left(x+\frac{h}{2}\right)-f\left(x-\frac{h}{2}\right)\right\|$$

$$= sup\left\|(1+(x+\tfrac{h}{2})^{\epsilon-1/p}-(1+(x-\tfrac{h}{2})^{\epsilon-1/p}\right\|$$

$$\ge sup_{0<h\le\delta}\left\|(1+(x+\frac{h}{2})^{\epsilon-1/p}\right\| - \left\|(1+(x-\frac{h}{2})^{\epsilon-1/p}\right\|$$

$$= sup_{0<h\le\delta}\left[\int_0^1\left|(1+(x+\tfrac{h}{2})^{\epsilon-\frac{1}{p}}\right|^q\right]^{\frac{1}{q}} - \left[\int_0^1\left|(1+(x-\tfrac{h}{2})^{\epsilon-\frac{1}{p}}\right|^q\right]^{\frac{1}{q}}$$

$$\geq sup_{0<h\leq\delta}\left[\int_0^1\left|(1+(x+\frac{h}{2})^{\epsilon-1/p}\right|^q\right]^{\frac{1}{q}}dx-\left[\int_0^1\left|(1+x)^{\epsilon-1/p}\right|^q\right]^{\frac{1}{q}}dx$$

$$=sup_{0<h\leq\delta}\left[\left[\frac{(1+x+\frac{h}{2})^{\epsilon q-\frac{q}{p}+1}}{\epsilon q-\frac{q}{p}+1}\right]_0^1\right]^{\frac{1}{q}}-\left[\left[\frac{(1+x)^{\epsilon q-\frac{q}{p}+1}}{\epsilon q-\frac{q}{p}+1}\right]_0^1\right]^{1/q}$$

$$=sup_{0<h\leq\delta}\left[\frac{(2+\frac{h}{2})^{\epsilon q-\frac{q}{p}+1}}{\epsilon q-\frac{q}{p}+1}-\frac{(1+\frac{h}{2})^{\epsilon q-\frac{q}{p}+1}}{\epsilon q-\frac{q}{p}+1}\right]^{\frac{1}{q}}-\left[\frac{(2)^{\epsilon q-\frac{q}{p}+1}}{\epsilon q-\frac{q}{p}+1}-\frac{(1)^{\epsilon q-\frac{q}{p}+1}}{\epsilon q-\frac{q}{p}+1}\right]^{\frac{1}{q}}$$

$$=(1-p)\ \geq\delta^{q-p}\quad\blacksquare$$

Proposition 3.3

For f is k-monotone function $L_p[-1,1]$, $0<p<q\leq 1$. then there exist a polynomial q_n of degree not exceeding n, satisfies

$$\|f-\mathrm{q}_n\|_p\leq c(p,k)n^{-m}\omega_{k-m}^{\varphi}(f,n^{-1})_p$$

Proof

$$\begin{aligned}\|f-q_n\|_p&\leq c(p,k)\ \omega_k^{\varphi}(f,n^{-1})_p\\&\leq c(p,k)n^{-1}\omega_{k-1}^{\varphi}(f,n^{-1})_p\\&\leq c(p,k)n^{-2}\omega_{k-2}^{\varphi}(f,n^{-1})_p\\&\leq c(p,k)n^{-m}\omega_{k-m}^{\varphi}(f,n^{-1})_p\quad\blacksquare\end{aligned}$$

Theorem 3.4

Let $0\leq p<q<1$, and $L_p[a,b]$ space. then there exist a polynomial q_n of degree not exceeding n, such that

$$\|f-\mathrm{q}_n\|_p\leq c(p,k)n^{-m}\gamma_{\frac{1}{n}}(f,q,p)\left\|f^{(m)}\right\|_q$$

proof

Since $p,q<1,\ 0\leq m\leq k-1$

then

$$\begin{aligned}\omega_k^{\varphi}(f,n^{-1})_p\ &\leq c(p,k)n^{-m}\omega_{k-m}^{\varphi}(f,n^{-1})_p\\&\leq c(p,k)\ n^{-(q-p)}\|f\|_p\\&\leq c(p,k)\ n^{-(q-p)}\|f\|_q\quad\blacksquare\end{aligned}$$

Now, let $0<p<q\leq 1$ and let $f\in L_p\cap\Delta^k$, where k is chosen such that $f^{(m)}\in L_p$ where $0\leq m\leq k-1$ by using Lemma 2.5 and $f^{(m)}\in\Delta^{k-m}$, we get for any $n\geq k-1$there exist a polynomial $\mathrm{q}_n$of degree not exceeding n, such that

$$\|f-\mathrm{q}_n\|_p\leq c(p,k)\ n^{-m}\ \omega_k^{\varphi}(f,n^{-1})_p.$$

Using Theorem 3.4 we get

$$\|f-\mathrm{q}_n\|_p\leq c(p,k)n^{-m}\gamma_{\frac{1}{n}}(f,q,p)\left\|f^{(m)}\right\|_q.$$

A direct consequence of the above theorem we have

Corollary 3.5

let $0 \le p < q < 1$ and f is k-monotone function in $L_p[a,b]$ space .then there is a polynomial q_n of degree not exceeding n, with

$$\|f - \mathrm{q}_n\|_p \le c(p,k) n^{-m-q+p} \left\|f^{(m)}\right\|_q, \qquad 0 \le m \le k-1 \tag{3.1}$$

Theorem 3.6

For $f \in \Delta^k$, and f is k-monotone function in $L_p\,[a,b]$, $p < 1$

Such that $f^{(m)} \in L_p[a,b], 0 \le m \le k-1$

we have

$$\|f - \mathrm{q}_n\|_p \le c(p_1,p_2,p,k)\omega_{k-m}(f^{(m)},n^{-1})_p\, n^{-m-q+p}\left\|f^{(m)}\right\|_p$$

Proof

let $\alpha = p(q-1)/q(p-1), 0 < p < q \le 1$. for $f \in \Delta^k$ with $f^{(m)} \in L_p$, $0 \le m \le k-1$

$$\|f - \mathrm{q}_n\|_p = \int_{-1}^{1} |f - \mathrm{q}_n|^p dx$$
$$\le c \int_{-1}^{1} |f - \mathrm{q}_n|^p \, dx$$

Choose $0 < p_1, p_2 < p \le 1$, such that

$$\|f - \mathrm{q}_n\|_p \le c(p_1,p_2)\|f - \mathrm{q}_n\|_{p_1}\|f - \mathrm{q}_n\|_{p_2}.$$

Then

$$\|f - \mathrm{q}_n\|_p \le c(p_1,p_2) \int_{-1}^{1} |f - \mathrm{q}_n|^{p_1} dx \int_{-1}^{1} |f - \mathrm{q}_n|^{p_2} dx$$
$$\le (p_1,p_2)\|f - \mathrm{q}_n\|_{p_1}^{\alpha}\|f - \mathrm{q}_n\|_{p_2}^{1-\alpha}$$

Using (2.2) to obtain $\|f - q_n\|_p \le$
$c(p_1,p_2,p,k) n^{-m}\omega_{k-m}^{\varphi}(f^{(m)},n^{-1})_{p_1}^{\alpha}\, \omega_{k-m}^{\varphi}(f^{(m)},n^{-1})$

Using Corollary 3.5 to get

$$\|f - \mathrm{q}_n\|_p \le c(p_1,p_2,p,k)\omega_{k-m}^{\varphi}(f^{(m)},n^{-1})_p^{\alpha}\, n^{-m-q+p}\left\|f^{(m)}\right\|_p \quad \blacksquare \tag{3.2}$$

Corollary 3.7

For $0 < p < q \le 1$ *and* $0 \le m \le k-1$ we have

$$\|f - \mathrm{q}_n\|_q = \mathrm{O}\left(n^{-m} l(n)\right) \tag{3.3}$$

where

$$l(n) = n^{-q+p}$$

Definition 3.8

Let us denote by $\Delta^k L_p$ for the set of all k-monotone function f define $(-1,1)$ and satisfying $\|f\| \leq 1$, and by Π_n the space of all algebraic polynomial of degree not exceeding n. Let,

$$E(\Delta^k L_p, \Pi_n)_p = sup_{f\in\Delta^k L_p} inf_{q_n\in\Pi_n} \|f - q_n\|_p \qquad (3,4)$$

Is the degree of best approximation of functions in $\Delta^k L_p$ a polynomial from Π_n

Theorem 3.9

For $f \in \Delta^k L_p$, we have $E(\Delta^k L_p, \Pi_n)_p \leq c(p)n^{-q+p}$, $\quad 0 < p < q \leq 1$.

Proof

Using (3.4) we get

$$E(\Delta^k L_p, \Pi_n)_p \leq \|f - q_n\|_p$$

Using Corollary 3.5 we get

$$E(\Delta^k L_p, \Pi_n)_p \leq c(p,k)n^{-m-q+p}\left\|f^{(m)}\right\|_q$$
$$\leq c(p,k)n^{p-q} \qquad \blacksquare$$

Definition 3.10

Let $E(\Delta^k L_p, \Pi_n \cap \Delta^k)_p = sup_{f\in\Delta^k L_p} inf_{q_n\in\Pi_n\cap\Delta^k} \|f - q_n\|_p$

is the degree of best approximation of function from $\Delta^k L_p$ using k-monotone polynomial in Π_n.

Theorem 3.11

For $0 < p < q \leq 1$ we have $E(\Delta^k L_p, \Pi_n \cap \Delta^k)_p \leq c(p,k)n^{p-q}$

Proof

For m=0 in Corollary 3.5 and using Lemma 2.6 and Lemma 2.7

$$E\left(\Delta^k L_p, \Pi_n \cap \Delta^k\right)_p \leq c(p)n^{-(k+1)}E_{n-k}(f^k)$$
$$\leq c(p)n^{-(k+1)}\omega_k^{\varphi}(f^k, n^{-1})_p$$
$$\leq c(p)n^{-(k+1)}n^{p-q}\left\|f^{(k)}\right\|_p$$
$$\leq c(p)n^{-(k+1)+p-q}$$

Conclusions

Many works are introduced for the direct theorem approximation using many types of moduli of smoothness, but sometimes we need to improve the degree of best approximation near the endpoints. So we need the uniform approximation. Hear we define a suitable modulus of smoothness to get uniform approximation.

CONFLICT OF INTERESTS.
There are non-conflicts of interest.

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الخلاصة

نظريات التقريب الافضل باستخدام مقياس النعومه الاعتيادي لا تعطي تقريبا منتظمة. كذلك نحتاج في بعض الاحيان لتطوير درجة التقريب الافضل قريبا من نهايات الفترة. لذا نحتاج لتطوير مقياس النعومة الاعتيادي. في هذا البحث عرفنا مقياسا جديدا للنعومة للحصول على تقريبا منتظما وتطوير درجة التقريب الافضل بدلالة هذا المقياس من النعومة. عرفنا هذا المقياس للدوال رتيبة $-k$. كذلك قدمنا بعض النظريات المباشرة بدلالة هذا المقياس. بعض تطبيقات هذا المقياس تم تقديمها في هذا البحث ايضا.

الكلمات الدالة: مقياس النعومة , الدالة $k-$ رتيبه, افضل تقريب.