Reduction for Ordinary and Partial Differential Equations by Using Lie Group

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Abstract

In this publication, We have done Lie group theory is applied to reduce the order of ordinary differential equations (ODEs) with 1-parameter and reduce a PDEs to ODEs . Also set up algorithm to solve ODEs and PDEs to obtain the exact solution .

1. Introduction

The Lie symmetries and their various generalizations have become an inseparable part of the modern physical description of wide range of phenomena of nature from quantum physics to hydrodynamics. Such success of a purely mathematical theory of continuous groups developed by Lie and Engel in 19th century[8] is explained by the remarkable fact that the overwhelming majority of mathematical models of physical, chemical, and biological processes possess nontrivial Lie symmetry. Symmetries can be applied to reduce the order of an ODE . [9] proposed a generalization of Lie's method called the non-classical method of group-invariant solutions. It is well known that the classical Lie symmetry method of point transformations is often used for reducing the number of independent variables in partial differential equation to obtain ordinary one can obtain partial solutions of the equation under study [1], [2], [3]. For partial differential equations , Lie symmetries are used to reduce the equation to an ODE via appropriate reduction variables. These can lead to the group-invariant solutions [4] of PDEs which are so important today[5],[6].

2.Reduction of Order for ODE,[1]

Consider the nth order ODE as:

$$G(x,y,y',y'',...,y^{(n)})=y^{(n)}-g(x,y,y',y'',...,y^{(n-1)})=0 \qquad ...(2.1)$$

Admits the group as

$$x^*=X(x,y,\in)=x+\in\zeta(x,y)+O(\in^2) \qquad ...(2.2)$$

$$y^*=Y(x,y,\in)=y+\in\eta(x,y)+O(\in^2) \qquad ...(2.3)$$

Where $O(\in^2)$ is acting the Tayler expansion about $\in=0$ at the second derivatives

If and only if $X^{(n)}G=0$ when $G=0$ where $X^{(n)}$ is the nth extension of the infinitesimal generator X introduced in the following:

$$X=\zeta(x,y)\frac{\partial}{\partial x}+\eta(x,y)\frac{\partial}{\partial y} \qquad ...(2.4)$$

Then G is some function of the invariants write as:

$$u(x,y),v_1(x,y,y'),...,v_n(x,y,y',...,y^{(n)}) \qquad ...(2.5)$$

Which implies:

$$X u(x,y)=0 \qquad ...(2.6)$$

Then

$$X v_j (x, y, y', ..., y^{(j)}) = 0 \, with \, \frac{\partial v_j}{\partial y^{(j)}} \neq 0, \, for \, k = 1, 2, ..., n \qquad ...(2.7)$$

For the nth extension of the group (3.2) and (3.3) we have $u^* = u$, $v_j^* = v_j$

For k = 1,2,...,n; $v_j (x, y, y', ..., y^{(j)})$ is constant of integration of the characteristic equations given as:

$$\frac{dx}{\zeta(x, y)} = \frac{dy}{\eta(x, y)} = \frac{dy'}{\eta^{(1)}(x, y, y')} = ... = \frac{dy^{(j)}}{\eta^{(j)}(x, y, y', ..., y^{(j)})} \qquad ...(2.8)$$

Where $\eta^{(j)}$ introduced by :

$$\eta^{(j)} (x, y, y', ..., y^{(j)}) = \frac{D \eta^{(j-1)}}{D x} - y^{(j)} \frac{D \zeta(x, y)}{D x}, j = 1, 2, ..., \qquad ...(2.9)$$

For any set of invariants (2.5) the ODE (2.1) turn into:

$$Z (u, v_1, v_2, ..., v_n) = 0 \qquad ...(2.10)$$

For some function $Z (u, v_1, v_2, ..., v_n)$.

3.Applications for ODEs
   In this part, we give algorithm to established some problems for ODEs how computed reduction order by using symmetries,

1-Algorithm
Step1: Write the vector field of the form:

$$X = \zeta(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y}$$

Step2: Find the nth prolongation for equation (2.1) in style:

$$X^{[n]} = X + \eta^{(1)} \frac{\partial}{\partial y'} + ... + \eta^{(n)} \frac{\partial}{\partial y^{(n)}}$$

Step3: Applying the prolongation X$^{[n]}$ on equation (2.1) found in ***Step2*** as:

$$X^{[n]} \left(y^{(n)} - g(x, y, y', y'', ..., y^{(n-1)}) \right) \Big|_{(2.1)=0} = 0$$

Step4: By using expansion of $\zeta^{(1)}, \zeta^{(2)}, \zeta^{(3)}$ from[10] and $\zeta^{(4)}, \zeta^{(5)}, ..., \zeta^{(n)}$ found by using maple package.

Step5: Replacing $y^{(n)}$ by $g(x, y, y', y'', ..., y^{(n-1)})$.

Step6: Find the general solution .

Step7: Reduce order of ODE by using new vector field found in ***Step6.***

Step8: Appling the nth prolongation for the new vector field .

Step9: Find invariance using $X^{[n]} I = 0$ whose characteristic system is :

$$\frac{dx}{\zeta(x, y)} = \frac{dy}{\eta(x, y)} = \frac{dy'}{\eta^{(1)}(x, y, y')} = ... = \frac{dy^{(j)}}{\eta^{(j)}(x, y, y', ..., y^{(j)})}$$

Step10: Solving system in ***Step9*** and substituting solution in (3.1) at end we get the general solution.

Example(3.1): reduce the ODE given in the form:

$$y^2 y'' + x y'^2 - y y' = 0 \qquad ...(3.1)$$

To solve (3.1) we must write the vector field as:

$$X = \zeta(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y} \qquad ...(3.2)$$

We need the 2-prolongation of (3.2) then:

$$X^{[2]} = X + \eta^{(1)} \frac{\partial}{\partial y'} + \eta^{(2)} \frac{\partial}{\partial y''} \qquad ...(3.3)$$

Applying equation (3.3) in equation (3.1) given as:

$$X^{[2]}\left(y^2 \, y'' + x \, y'^2 - y \, y'\right)\Big|_{(3.31)=0} = 0 \qquad \text{...(3.4)}$$

Then

$$y^2 \eta^{(2)} + 2x \, y' \eta^{(1)} - y \, \eta^{(1)} + \left(2y \, y'' - y'\right)\eta + y'^2 \, \zeta = 0 \qquad \text{...(3.5)}$$

By substituting $\eta^{(1)}$ and $\eta^{(2)}$ from definition:

$$\eta_{xx} + \left(2\eta_{xy} - \zeta_{xx}\right)y' + \left(\eta_{yy} - 2\zeta_{xy}\right)y'^2 - \zeta_{yy} \, y'^3$$
$$+ \left[(\eta_y - 2\zeta_x - 3\zeta_y \, y')y''\right]y^2 + 2x \, y'\left[\eta_x + (\eta_y - \zeta_x)y' - \zeta_y \, y'^2\right] \qquad \text{...(3.6)}$$
$$- y\left[\eta_x + (\eta_y - \zeta_x)y' - \zeta_y \, y'^2\right] + \eta\left(2y \, y'' - y'\right) + y'^2 \, \zeta = 0$$

Now , substituting y'' by $\left(\dfrac{y \, y' - x \, y'^2}{y^2}\right)$ we find:

$$\eta_{xx} + \left(2\eta_{xy} - \zeta_{xx}\right)y' + \left(\eta_{yy} - 2\zeta_{xy}\right)y'^2 - \zeta_{yy} \, y'^3 + \left[(\eta_y - 2\zeta_x - 3\zeta_y \, y')\left(\frac{y \, y' - x \, y'^2}{y^2}\right)\right]y^2$$
$$+ 2x \, y'\left[\eta_x + (\eta_y - \zeta_x)y' - \zeta_y \, y'^2\right] - y\left[\eta_x + (\eta_y - \zeta_x)y' - \zeta_y \, y'^2\right] \qquad \text{...(3.7)}$$
$$+ \eta\left[2y\left(\frac{y \, y' - x \, y'^2}{y^2}\right) - y'\right] + y'^2 \zeta = 0$$

By separation of the coefficient $y'^2 , y'^3 , y' , y \, y'^2 , y \, y' , y^2 \, y'^2$ and $y^2 \, y'$.

$$y'^2 : \zeta + 2x\left(\eta_y - \zeta_x\right) - \frac{2x}{y}\eta - x\left(\eta_y - 2\zeta_x\right) = 0 \qquad \text{...(3.8)}$$

$$y'^3 : -2x \, \zeta_y - \zeta_{yy} + 3x \, \zeta_y = 0 \qquad \text{...(3.9)}$$

$$y' : 2x \, \eta_x + \eta = 0 \qquad \text{...(3.10)}$$

$$y \, y'^2 : -4 \, \zeta_y = 0 \qquad \text{...(3.11)}$$

$$y \, y' : -\zeta_x - 2\zeta_y = 0 \qquad \text{...(3.12)}$$

$$y^2 \, y'^2 : \eta_{yy} - 2 \, \zeta_{xy} = 0 \qquad \text{...(3.13)}$$

$$y^2 \, y' : 2\eta_{xy} - \zeta_{xx} = 0 \qquad \text{...(3.14)}$$

$$1 : \eta_{xx} \, y^2 - \eta_x \, y = 0 \qquad \text{...(3.15)}$$

Then the general solution of above system is :

$$\zeta(x,y) = c_1 \, x \qquad \text{...(3.16)}$$

$$\eta(x,y) = c_1 \, y \qquad \text{...(3.17)}$$

Now , to reduce order of ODE write the vector field as follows:

$$X_1 = x \, \frac{\partial}{\partial x} + y \, \frac{\partial}{\partial y} \qquad \text{...(3.18)}$$

The 2$^{\text{nd}}$ prolongation of X_1 is :

$$X_1^{[2]} = X_1 + y \, \frac{\partial}{\partial y} + y' \, \frac{\partial}{\partial y'} + y'' \, \frac{\partial}{\partial y''} \qquad \text{...(3.19)}$$

Then

$$X_1^{[2]} = X_1 + y \, \frac{\partial}{\partial y} + 0 \, \frac{\partial}{\partial y'} + y'' \, \frac{\partial}{\partial y''} \qquad \text{...(3.20)}$$

We find invariance using $X^{[2]} I = 0$ whose characteristic system is :

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dy'}{0} = \frac{dy''}{-y''} \qquad \text{...(3.21)}$$

Solving equation (3.21) we result two invariance of $X^{[2]}$ given by:

$$\frac{y}{x} = u \qquad \text{...(3.22)}$$

$$v = y' \qquad \text{...(3.23)}$$

To write the given ODE in terms of u and v we get:

$$y^2 y'' = y^2 \frac{d\,y'}{d\,x} = y^2 \frac{d\,v}{d\,x} = y^2 \frac{d\,v}{d\,u}\frac{d\,u}{d\,x} = u\frac{d\,v}{d\,u}(x\,v - y) \qquad \text{...(3.24)}$$

Using equation ((3.22)-(3.24)) in the ODE of (1) gives:

$$u(x\,v - y)\frac{d\,v}{d\,u} + x\,v^2 - y\,v = 0 \qquad \text{...(3.25)}$$

Dividing equation (3.25) by x implies that :

$$u(v - u)\frac{d\,v}{d\,u} + v^2 - u\,v = 0 \qquad \text{...(3.26)}$$

Now , solve equation (3.26) if:

A- $v \neq u$ we get the following:

$$\frac{d\,v}{d\,u} = \frac{-v}{u} \qquad \text{...(3.27)}$$

Solve equation (3.27) we find the result:

$$v = \frac{c}{u} \rightarrow y' = \frac{d\,y}{d\,x} = \frac{c}{\frac{y}{x}} \rightarrow \frac{d\,y}{d\,x} = \frac{c\,x}{y} \qquad \text{...(3.28)}$$

Then by solving equation (3.28) we get:

$$y = \pm \sqrt{c_1 x^2 + c_2} \qquad \text{...(3.29)}$$

B- $v = u$

$$y' = \frac{y}{x} \rightarrow \frac{d\,y}{d\,x} = \frac{y}{x} \qquad \text{...(3.30)}$$

Then from equation (3.30) we obtain:

$$y = c\,x \qquad \text{...(3.31)}$$

Example(3.2): reduce the ODE write in style as:

$$x^2 y\frac{d^2 y}{d\,x^2} + x\left(\frac{d\,y}{d\,x}\right)^2 + y\left(\frac{d\,y}{d\,x}\right) - x\,y^2 = 0 \qquad \text{...(3.32)}$$

First write the vector field given as:

$$X = \zeta(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} \qquad \text{...(3.33)}$$

To compute equation (3.32) we need the 2nd prolongation given by the following:

$$X^{[2]} = X + \eta^{(1)}\frac{\partial}{\partial y'} + \eta^{(2)}\frac{\partial}{\partial y''} \qquad \text{...(3.34)}$$

Now, applying the formula of equation (3.35) in equation (3.32) we obtain:

$$X^{[2]}\left(x^2 y\,y'' + x\,y'^2 + y\,y' - x\,y\right)\Big|_{(3.32)=0} = 0 \qquad \text{...(3.35)}$$

Then

$$x^2 y\,\eta^{(2)} + 2x\,y'\eta^{(1)} + y\,\eta^{(1)} + 2x\,y\,y''\zeta + x^2 y''\eta + y'^2\zeta + y'\eta - y^2\zeta - 2x\,y\,\eta = 0 \quad \text{...(3.36)}$$

By substituting $\eta^{(1)}\,and\,\eta^{(2)}$ from definition are:

$$X^2 y\left[\eta_{xx} + \left(2\eta_{xy} - \zeta_{xx}\right)y' + \left(\eta_{yy} - 2\zeta_{xy}\right)y'^2 - \zeta_{yy}\,y'^3\right]$$
$$+\left[\left(\eta_y - 2\zeta_x - 3\zeta_y\,y'\right)y''\right] + 2x\,y'\left[\eta_x + \left(\eta_y - \zeta_x\right)y' - \zeta_y\,y'^2\right] \qquad \text{...(3.37)}$$
$$+ y\left[\eta_x + \left(\eta_y - \zeta_x\right)y' - \zeta_y\,y'^2\right] + (2x\,y\,\zeta + x^2\eta)y'' + y'^2\zeta + y'\eta$$
$$-\zeta\,y^2 - 2x\,y\,\eta = 0$$

After that , substituting y'' by $\left(x\,y^2 - x\,y'^2 - y\,y'\right)\dfrac{1}{x^2 y}$ we get:

$$X^2 y\left[\eta_{xx}+\left(2\eta_{xy}-\zeta_{xx}\right)y'+\left(\eta_{yy}-2\zeta_{xy}\right)y'^2-\zeta_{yy}\,y'^3\right]$$

$$+\left[\left(\eta_y-2\zeta_x-3\zeta_y\,y'\right)\left(x\,y^2-x\,y'^2-y\,y'\right)\frac{1}{x^2\,y}\right]+2x\,y'\left[\eta_x+\left(\eta_y-\zeta_x\right)y'-\zeta_y\,y'^2\right] \qquad ...(3.38)$$

$$+y\left[\eta_x+\left(\eta_y-\zeta_x\right)y'-\zeta_y\,y'^2\right]+\left(2x\,y\,\zeta+x^2\eta\right)\left(x\,y^2-x\,y'^2-y\,y'\right)\frac{1}{x^2\,y}+y'^2\,\zeta+y'\eta$$

$$-\zeta\,y^2-2x\,y\,\eta=0$$

Then

$$y'^2\left[2x\left(\eta_y-\zeta_x\right)-\zeta-x\left(\eta_y-2\zeta_x\right)-\frac{x}{y}\eta\right]+y'\left[2x\,\eta_x-\frac{2y}{x}\zeta\right]+y'^3\left[x\,\zeta_y-\zeta_y\right]$$

$$+y\,y'^2\left[x^2\left(\eta_{yy}-2\zeta_{xy}\right)+3y\,\zeta_y\right]+y\,y'\left[x^2\left(2\eta_{xy}-\zeta_{xx}\right)+\zeta_x\right]+y\left[x^2\eta_{xx}-2x\,\eta+\eta_x\right]$$

$$-\zeta\,y^2+x\,y^2\left(\eta_y-2\zeta_x\right)+x\,\eta_y+2y^2\,\zeta-3x\,y^2\,y'\zeta_y=0$$

$$...(3.39)$$

By separation of the coefficient of the above system:

$$y'^2: 2x\left(\eta_y-\zeta_x\right)-\zeta-x\left(\eta_y-2\zeta_x\right)-\frac{x}{y}\eta=0 \qquad ...(3.40)$$

$$y': 2x\,\eta_x-\frac{2y}{x}\zeta=0 \qquad ...(3.41)$$

$$y'^3: x\,\zeta_y-\zeta_y=0 \qquad ...(3.42)$$

$$y\,y'^2: x^2\left(\eta_{yy}-2\zeta_{xy}\right)+3y\,\zeta_y=0 \qquad ...(3.43)$$

$$y\,y': x^2\left(2\eta_{xy}-\zeta_{xx}\right)=0 \qquad ...(3.44)$$

$$y^2\,y': -3x\,\zeta_y=0 \qquad ...(3.45)$$

$$1: y\,(x^2\eta_{xx}-2x\,\eta+\eta_x)-\zeta\,y^2+x\,y^2\left(\eta_y-2\zeta_x\right)+x\,\eta_y+2y^2\,\zeta=0 \qquad ...(3.46)$$

Then the general solution of above system are:

$$\zeta(x,y)=0 \qquad ...(3.47)$$

$$\eta(x,y)=c_1\,y \qquad ...(3.48)$$

The generator given as :

$$X_1=y\,\frac{\partial}{\partial y} \qquad ...(3.49)$$

The 2nd prolongation of X_1 is :

$$X_1^{[2]}=X_1+y\,\frac{\partial}{\partial y}+y'\,\frac{\partial}{\partial y'}+y''\,\frac{\partial}{\partial y''} \qquad ...(3.50)$$

Then

$$X_1^{[2]}=0\frac{\partial}{\partial x}+y\,\frac{\partial}{\partial y}+y'\,\frac{\partial}{\partial y'}+y''\,\frac{\partial}{\partial y''} \qquad ...(3.51)$$

Solving the characteristic system $X^{[2]}I=0$ is given the differential invariants .

$$\frac{dx}{0}=\frac{dy}{y}=\frac{dy'}{y'}=\frac{dy''}{y''} \qquad ...(3.52)$$

Then

$$u=x \qquad ...(3.53)$$

$$v=\frac{y'}{y}\;\rightarrow\;y'=v\,y \qquad ...(3.54)$$

$$y''=y\frac{dv}{du}+v^2\,y \qquad ...(3.55)$$

Now , substituting equation (3.55) in equation (3.32) we obtain:

$$x^2 y^2 \left[\frac{dv}{du} + v^2 \right] + x y^2 v^2 + y^2 v - x y^2 = 0 \qquad \text{...(3.56)}$$

Dividing equation (3.56) on $x^2 y^2$ we get :

$$\left[\frac{dv}{du} + v^2 \right] + \frac{v^2}{u} + \frac{v}{u^2} - \frac{1}{u} = 0 \qquad \text{...(3.57)}$$

Multiplying equation (3.57) by u^2 we find:

$$u^2 \left[\frac{dv}{du} + v^2 \right] + v^2 u + v - u = 0 \qquad \text{...(3.58)}$$

Then

$$\frac{dv}{du} = \frac{1}{u} - \frac{v^2}{u} - \frac{v}{u} - v^2 \qquad \text{...(3.59)}$$

Then the equation (3.59) reduced 1st order ODE .

4.Reduction of Order for PDEs

In this part, established reduction order for PDEs the technique is basically the same as for ODEs
Now, consider the second order of PDE is :

$$G(t,x,u,u_x,u_t,u_{xx},u_{xt},u_{tt}) = u_t - g(t,x,u,u_x,u_t,u_{xx},u_{xt},u_{tt}) = 0 \qquad \text{...(4.1)}$$

The point symmetries of the form:

$$t^* = T(t,x,u;\in) = t + \in \tau(t,x,u) + O(\in^2)$$

$$x^* = X(t,x,u;\in) = x + \in \zeta(t,x,u) + O(\in^2) \qquad \text{...(4.2)}$$

$$u^* = U(t,x,u;\in) = u + \in \eta(t,x,u) + O(\in^2)$$

If and only if $X^{(2)} G = 0$ when $G = 0$ where $X^{(2)}$ is the 2th extension of the infinitesimal generator X introduced in the following:

$$X = \zeta(t,x,u) \frac{\partial}{\partial x} + \tau(t,x,u) \frac{\partial}{\partial t} + \eta(t,x,u) \frac{\partial}{\partial u} \qquad \text{...(4.3)}$$

Then G is some function of the invariants write as:

$$I(t,x), I_1(t,x,u),...,I_2(t,x,u,u_x,u_t,u_{xx},u_{xt},u_{tt}) \qquad \text{...(4.4)}$$

Which implies that:

$$X I = 0 \qquad \text{...(4.5)}$$

This condition (4.5) can be expressed by using the characteristic of the group which is:

$$Q = \eta - \zeta u_x - \tau u_t \qquad \text{...(4.6)}$$

From condition (4.5) , the surface u = u(t,x) is invariant provided that:

$$Q = 0 \quad \text{when } u = u(t,x) \qquad \text{...(4.7)}$$

In other words from [7], every invariant solution satisfies the invariant surface condition write as:

$$Q \equiv \eta - \zeta u_x - \tau u_t = 0 \qquad \text{...(4.8)}$$

Now , suppose that ζ and η are not both zero then the invariant surface condition is a 1-order quasilinear PDE can be solved by the method of characteristic equation are:

$$\frac{dx}{\zeta} = \frac{dt}{\tau} = \frac{du}{\eta} \qquad \text{...(4.9)}$$

5.Applications for PDEs

In the following, we confirmed algorithm for reducing order of PDE by using symmetry .Moreover calculated some examples about it.

2-Algorithm

Step1: Write the vector field of the form:

$$X = \zeta(t,x,u) \frac{\partial}{\partial x} + \tau(t,x,u) \frac{\partial}{\partial t} + \eta(t,x,u) \frac{\partial}{\partial u}$$

Step2: Find the 2th prolongation for equation (4.1) in style:

$$X^{[2]} = X + \zeta^t \frac{\partial}{\partial u_t} + \zeta^x \frac{\partial}{\partial u_x} + \zeta^{tx} \frac{\partial}{\partial u_{tx}} + \zeta^{tt} \frac{\partial}{\partial u_{tt}} + \zeta^{xx} \frac{\partial}{\partial u_{xx}}$$

Step3: Applying the prolongation $X^{[2]}$ on equation (4.1) found in **Step2** as:

$$X^{[2]}\left(u_t - g(t,x,u,u_x,u_t,u_{xx},u_{xt},u_{tt})\right)\Big|_{(4.1)=0} = 0$$

Step4: By using expansion $\zeta^t, \zeta^x, \zeta^{tx}, \zeta^{xx}$ and ζ^{tt} from[11]

Step5: Replacing u_t by $g(t,x,u,u_x,u_t,u_{xx},u_{xt},u_{tt})$.

Step6: Find the general solution .

Step7: Reduce order of PDE by using new vector field found in **Step6.**

Step8: Appling the 2th prolongation for the new vector field .

Step9: By solving the invariance $Q \equiv \eta - \zeta u_x - \tau u_t = 0$ whose characteristic equation:

$$\frac{dx}{\zeta} = \frac{dt}{\tau} = \frac{du}{\eta}$$

Step10: Solving the characteristic in **Step9** and substituting solution in (1) at end we get the general solution.

Example(5.1): reduce the PDE given in the form:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \qquad\qquad ...(5.1)$$

First write the vector field given as:

$$X = \zeta(t,x,u)\frac{\partial}{\partial x} + \tau(t,x,u)\frac{\partial}{\partial t} + \eta(t,x,u)\frac{\partial}{\partial u} \qquad ...(5.2)$$

The vector field in equation (5.2) we need the 2nd prolongation given by:

$$X^{[2]} = X + \zeta^t \frac{\partial}{\partial u_t} + \zeta^x \frac{\partial}{\partial u_x} + \zeta^{tx}\frac{\partial}{\partial u_{tx}} + \zeta^{tt}\frac{\partial}{\partial u_{tt}} + \zeta^{xx}\frac{\partial}{\partial u_{xx}} \qquad ...(5.3)$$

Now, applying formula of equation (5.3) in equation (5.1) we get :

$$X^{[2]}\left(u_t - u_{xx} - u_x\right)\Big|_{(1)=0} = 0 \qquad ...(5.4)$$

Then

$$\zeta^t - \zeta^{xx} - \zeta^x = 0 \qquad\qquad ...(5.5)$$

By using definitions of ζ^t, ζ^x and ζ^{xx} we obtain the following:

$$\eta_t + u_t\left(\eta_u - \tau_t\right) - u_t^2\tau_u - u_x\zeta_t - u_xu_t\zeta_u - \eta_{xx} - 2u_x\eta_{xu} - u_{xx}\eta_u - u_x^2\eta_{uu} + 2u_{xx}\zeta_x + u_x\zeta_{xx}$$

$$+ u_x^2\zeta_{xu} + \tau_u\left(u_tu_{xx} + 2u_xu_{xt}\right) + u_x^3\zeta_{uu} + 2u_{xt}\tau_x + u_t\tau_{xx} + 2u_xu_t\tau_{xu} + u_x^2u_t\tau_{uu} + 3u_xu_{xx}\zeta_u - \eta_x$$

$$- u_x\left(\eta_u - \zeta_x\right) + u_x^2\zeta_u + u_t\tau_x + u_xu_t\tau_u = 0$$

$$...(5.6)$$

Now , replacing u_t by $u_{xx} + u_x$ we result:

$$\eta_t + \left(u_{xx} + u_x\right)\left(\eta_u - \tau_t\right) - \left(u_{xx} + u_x\right)^2\tau_u - u_x\zeta_t - \left(u_{xx} + u_x\right)u_x\zeta_u - \eta_{xx} - 2u_x\eta_{xu}$$

$$- u_{xx}\eta_u - u_x^2\eta_{uu} + 2u_{xx}\zeta_x + u_x\zeta_{xx} + u_x^2\zeta_{xu} + \tau_u\left(\left(u_{xx} + u_x\right)u_{xx} + 2u_xu_{xt}\right)$$

$$+ u_x^3\zeta_{uu} + 2u_{xt}\tau_x + \left(u_{xx} + u_x\right)\tau_{xx} + 2\left(u_{xx} + u_x\right)u_x\tau_{xu} + \left(u_{xx} + u_x\right)u_x^2\tau_{uu}$$

$$+ 3u_xu_{xx}\zeta_u - \eta_x - u_x\left(\eta_u - \zeta_x\right) + u_x^2\zeta_u + \left(u_{xx} + u_x\right)\tau_x + \left(u_{xx} + u_x\right)u_x\tau_u = 0$$

$$...(5.7)$$

$$u_{xx}\left(\tau_{xx} + \tau_x - \tau_t + 2\zeta_x\right) + u_x\left(\zeta_{xx} + \zeta_x + \tau_{xx} + \tau_x - \tau_t - \zeta_t - 2\eta_{xu}\right)$$

$$+ u_x^2\left(\zeta_{xu} + 2\tau_{xu} - \eta_{uu}\right) + u_xu_{xx}\left(2\tau_{xu} + 2\zeta_u\right) + u_x^3\left(\zeta_{uu} + \tau_{uu}\right)$$

$$+ u_x^2u_{xx}\tau_{uu} + 2u_xu_{xt}\tau_u + 2u_{xt}\tau_x - \eta_{xx} - \eta_x + \eta_t = 0$$

$$...(5.8)$$

By using separation of the coefficient :

$$u_{xx}: \tau_{xx} + \tau_x - \tau_t + 2\zeta_x = 0 \qquad ...(5.9)$$

$$u_x \ : \zeta_{xx} + \zeta_x + \tau_{xx} + \tau_x - \tau_t - \zeta_t - 2\eta_{xu} = 0 \qquad ...(5.10)$$

$$u_x^2 \ : \zeta_{xu} + 2\tau_{xu} - \eta_{uu} = 0 \qquad ...(5.11)$$

$$u_x u_{xx} : 2\tau_{xu} + 2\zeta_u = 0 \qquad \text{...(5.12)}$$

$$u_x^3 : \zeta_{uu} + \tau_{uu} = 0 \qquad \text{...(5.13)}$$

$$u_x^2 u_{xx} : \tau_{uu} = 0 \qquad \text{...(5.14)}$$

$$u_x u_{xt} : 2\tau_u = 0 \qquad \text{...(5.15)}$$

$$u_{xt} : 2\tau_x = 0 \qquad \text{...(5.16)}$$

$$1 : -\eta_{xx} - \eta_x + \eta_t = 0 \qquad \text{...(5.17)}$$

Then the general solution of above system is :

$$\zeta(t,x,u) = \frac{1}{2}\left(c_1 x - c_2 - 4c_4\right)t + \frac{1}{2}x c_2 + c_6 \qquad \text{...(5.18)}$$

$$\tau(t,x,u) = \frac{1}{2}c_1 t^2 + c_2 t + c_3 \qquad \text{...(5.19)}$$

$$\eta(t,x,u) = -\frac{1}{8}\left[\left(t^2 + (2x+2)t + x^2\right)c_1 - 8c_4(t+x) - 8c_5\right]u + \alpha(t,x) \qquad \text{...(5.20)}$$

Then the Lie symmetries given as:

$$X_1 = \frac{1}{2}x t \frac{\partial}{\partial x} + \frac{1}{2}t^2 \frac{\partial}{\partial t} - \frac{1}{8}\left(t^2 + (2x+2)t + x^2\right)u \frac{\partial}{\partial u} \qquad \text{...(5.21)}$$

$$X_2 = \left(-t + \frac{1}{2}x\right)\frac{\partial}{\partial x} + t \frac{\partial}{\partial t} \qquad \text{...(5.22)}$$

$$X_3 = \frac{\partial}{\partial t} \qquad \text{...(5.23)}$$

$$X_4 = -4t \frac{\partial}{\partial x} - (t+x)u \frac{\partial}{\partial u} \qquad \text{...(5.24)}$$

$$X_5 = u \frac{\partial}{\partial u} \qquad \text{...(5.25)}$$

$$X_6 = \frac{\partial}{\partial x} \qquad \text{...(5.26)}$$

$$X_\alpha = \alpha(t,x) \frac{\partial}{\partial u} \qquad \text{...(5.27)}$$

Using the symmetries as:

$$X = X_1 = \frac{1}{2}x t \frac{\partial}{\partial x} + \frac{1}{2}t^2 \frac{\partial}{\partial t} - \frac{1}{8}\left(t^2 + (2x+2)t + x^2\right)u \frac{\partial}{\partial u}$$

Then

$$\frac{dx}{\frac{1}{2}x t} = \frac{dt}{\frac{1}{2}t^2} = \frac{du}{-\frac{1}{8}\left(t^2 + (2x+2)t + x^2\right)u} \qquad \text{...(5.28)}$$

By solving equation (5.28) we find the following :

$$\zeta(t,x) = \frac{x}{t} \qquad \text{...(5.29)}$$

$$v(t,x) = \varphi(\zeta)t^{(2x+2)}e^{-\frac{1}{4}\left(t - \frac{x^2}{t}\right)} \qquad \text{...(5.30)}$$

By using (5.29) and (5.30), we find $\dfrac{\partial u}{\partial t}, \dfrac{\partial u}{\partial x}$ and $\dfrac{\partial^2 u}{\partial x^2}$ we get:

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial \zeta}\cdot\frac{\partial \zeta}{\partial t} = \left[\frac{x t^{(2x+2)}}{t^2}\frac{\partial \varphi}{\partial \zeta} + \varphi\left((2x+2)t^{2x+1} - \frac{t^{(2x+2)}}{4}\left(1 + \frac{x^2}{t^2}\right)\right)\right]e^{-\frac{1}{4}\left(t - \frac{x^2}{t^2}\right)} \qquad \text{...(5.31)}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial \zeta}\cdot\frac{\partial \zeta}{\partial x} = \left[\frac{t^{(2x+2)}}{t}\frac{\partial \varphi}{\partial \zeta} + \varphi\left(2t^{2x+1} + \frac{x}{2t}\right)\right]e^{-\frac{1}{4}\left(t - \frac{x^2}{t^2}\right)} \qquad \text{...(5.32)}$$

$$\frac{\partial^2 u}{\partial x^2} = \left[\frac{\partial^2 \varphi}{\partial \zeta^2} \frac{t^{2x+2}}{t^2} + \frac{\partial \varphi}{\partial \zeta} \left(\frac{x\, t^{2x+2}}{2t^2} + \frac{4t^{2x+2}}{t} + \frac{x}{2t^2} \right) + \varphi \left(\frac{x}{2t} \left(2t^{2x+2} + \frac{x}{2t} \right) + 4t^{2x+2} + \frac{1}{2t} \right) \right] e^{-\frac{1}{4}\left(t - \frac{x^2}{t} \right)}$$

...(5.33)

Now , substituting equations ((5.31)-(5.33)) in equation (5.1) we obtain the ODE :

$$\varphi_{\zeta\zeta} - 2\,\zeta\,\varphi_\zeta + \zeta^2 \varphi = 0 \qquad\qquad\qquad \text{...(5.34)}$$

By solving equation (5.34) we find:

$$\varphi(\zeta) = c_1 e^{\frac{1}{2}\zeta^2} \cos(\zeta) + c_2 e^{\frac{1}{2}\zeta^2} \sin(\zeta) \qquad\qquad \text{...(5.35)}$$

Then

$$u(t,x) = c_1 e^{\frac{1}{2}\frac{x^2}{t^2}} \cos\left(\frac{x}{t} \right) + c_2 e^{\frac{1}{2}\frac{x^2}{t^2}} \sin\left(\frac{x}{t} \right) \qquad\qquad \text{...(5.36)}$$

Discussion and Conclusion

Lie group methods is finest for all techniques yet based on algorithm established above .In this work , introduced 2-algorithms for ODEs and PDEs, the first algorithm is applied to reduce the order of ODEs and the second algorithm is applied to reduce PDEsto ODEs . Also give some applications for ODEs and PDEs to obtain the exact solutions.

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CONFLICT OF INTERESTS
There are no conflicts of interest.

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تخفيض المعادلات التفاضلية باستخدام زمرلي

الخلاصة

في هذا المنشور ,قمنا بتطبيق نظرية زمرليلتخفيض رتبة المعادلة التفاضلية الاعتيادية مع معلمة واحدة و تخفيض المعادلات التفاضلية الجزئية لمعادلات تفاضلية اعتيادية .كذالك وضعنا خوارزمية لحل المعادلات التفاضلية الاعتيادية والمعادلات التفاضلية الجزئية للحصول على الحل الحقيقي.

الكلمات الدالة: زمرلي, التطابق,المعدلات التفاضلية الجزئية, الاستطالة,التخفيض, المعادلات التفاضلية الاعتيادية.