Improve Bounds of some Arithmetical Functions

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Abstract

We show in this article the use of the norm function to get a new lower bound of Riemann-Zeta function $\zeta(s)$ where $s = \sigma + it$. This subject has been studied deeply by Hilberdink [HIL, 12]). Getting a bound for the Riemann-Zeta function $\zeta(s)$ in the critical strip ($0 < \Re e \ \ s < 1$) is more challenging for many reasons related to the behavior of the Riemann-Zeta function in that strip. In the other words, the aim of this article is to prove that $|\zeta(\sigma + it)|$ has a strict lower bound when the real part σ is very closed to the line 1. We state this in the main theorem of this paper.

Key words: Analytic Number Theory (especially, The Riemann-Zeta function), Banach space and the norm function.

Note that: The reader must be familiar with the Analytic Number Theory concepts and advance Mathematical Analysis.

الخلاصة

نبين في هذا البحث استخدام دالة المعيار للحصول على بعض المعلومات عن القيد السفلي لدالة ريمان زيتا $\zeta(s)$ بحيث $s = \sigma + it$. هذا الموضوع درس عميقا من قبل العالم هلبيردنك (انظر المصدر الاول). الحصول على قيد لداله ريمان زيتا في شريط الفترة الحقيقية $0 < \Re e \ s < 1$ يكون من الصعب جدا وذلك للعديد من الاسباب التي تعود الى تصرفات تلك الداله في ذلك الشريط.

الهدف من هذا البحث هو اثبات $|\zeta(\sigma + it)|$ له قيد سفلي دقيق عندما الجزء الجقيقي من s يكون جدا قريب من الخط 1.

1. Introduction

Firstly, we start in this introduction by viewing some definitions of the arithmetical functions and norm function [AI, 85]. Secondly, we assert some of basic (known) theorems related to the upper and lower bounds of the Rieman-Zeta function and those theorems are without proof.

An arithmetical function is a function $f: N \to C$. Denote by A the set of all arithmetical functions. For $f, g \in A$ $\lambda \in C$ we have λf, $f + g$ and fg are also in A. More importantly, the Dirichlet convolution $f * g$ denoted by

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right)$$

is also in A. The sum here is over all divisors d of n. Dirichlet convolution is commutative. This follows from the sum

$$(f * g)(n) = \sum_{d|n} f(d) g(\tfrac{n}{d}) = \sum_{c,d>1} f(c) g(d),$$

where the sum is over all possible positive integers c, d such that $cd = n$. It is also associative since

$$f(* g * h)(n) = \sum_{bcd=n} f(b) g(c) h(d).$$

In fact $(A, +, *)$ is an algebra where $*$ is distributive with respect to $+$ and $\lambda(f * g) = (\lambda f) * g = f * (\lambda g)$ for every $\lambda \in C$. Now, we move our attention to define some arithmetical functions. We start with the divisor function $d(n)$.

Now, we move our attention to give some knowledge about the Riemann-Zeta function. We start with:

1-1 Definition:-

For $n \in N$, we define the function $d(n)$ to be the number of divisors of n. We write this as $d(n) = \sum_{d|n} 1$, where the sum ranges over all the divisors d of n.

1-2 Definition :- [Al,1985]

The Riemann zeta function is defined for $\Re e\ s > 1$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Here $\zeta(s)$ converges absolutely and locally uniformly for $\Re e\ s > 1$ [AI, 85]. Moreover, $\zeta(s)$ has an analytic continuation to the whole complex plane except for a simple pole at 1 with residue 1 and is of finite order which means that $|\zeta(s)| \leq c\, t^{\delta}$, for some $c, \delta > 0$ where δ depends on the real part of s [LHIL, 06].

The function $\zeta(s)$ is a function of complex variable has been studied by B. Riemann (1826-1866). There is an important link between the Riemann zeta function and prime numbers. This connection related to the Euler product representation for the zeta function as follows

:

$$\zeta(s) = \prod_p (1 - \frac{1}{p^s})^{-1}.$$

This product runs over all primes. Moreover, The convergence of the above product holds for the real part of s is greater than 1.

The zeros of $\zeta(s)$ inside the strip $0 \leq \Re e\ s \leq 1$ is an important subject and has a significance conjectures [AI, 85]. Riemann showed that the frequency of prime numbers is very closely related to the behavior of the zeros of $\zeta(s)$. He conjectured that all non-trivial zeros of $\zeta(s)$ lies on a line have the real part $\frac{1}{2}$.

1-3 Theorem (Basic Properties): For $s = \sigma + it$, we have

(a) $\zeta(s)$ is analytic in σ>1.

(b) $\zeta(s)$ has an analytic continuation to the half plane $\sigma > 0$ except at the simple pole $s = 1$ with residue 1. We mean by Analytic continuation that for $\sigma > 0$, we have

$$\zeta(s) = \frac{s}{s-1} - s\int_1^{\infty} \{x\} x^{-s-1}\, dx.$$

(c) $\zeta(s)$ has an Euler product representation for $\sigma > 1$

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}$$

(d) $\zeta(s)$ has no zeros for $\sigma > 1$.

For the proof of the above axioms see [Al,1976]

Remark: For more details and other (basic) information about the Riemann Zeta function see [TA, 1976], [MN,2005] and [PB, 2004].

As a part of this investigation, we need some functions from the functional analysis (which we use later on to prove the main theorem). We start with the definition of the space

$$l^2 = \left\{(a_n)_{n\in N} : \sum_{n=1}^{\infty} |a_n|^2 < \infty\right\},$$

and we call it l^2 space. Let the function $\mu_\gamma : l^2 \to l^2$ be a linear mapping defined as follows $\mu_\gamma(a_n) = (b_n)$ where $b_n = \sum_{d|n} \left(\frac{n}{d}\right)^{-\gamma} a_\gamma$. The important question here is : When is the function $\mu_\gamma : l^2 \to l^2$ bounded? From Hilberdink's work [HIL, 09], we understand that μ_γ is bounded if and only if $\gamma > 1$, in which case $\|\mu_\gamma\| = \zeta(\gamma)$. To see this, we have

$$|b_n|^2 = \left|\sum_{d|n} \left(\frac{n}{d}\right)^{-\gamma}\right|^2 = \frac{1}{n^{2\gamma}} \left|\sum_{d|n} d^\gamma a_\gamma\right|^2.$$

By Cauchy–Schwarz inequality we have the last term is

$$\le \frac{1}{n^{2\gamma}} \left(\sum_{d|n} d^\gamma\right)\left(\sum_{d|n} d^\gamma |a_d|^2\right) = \frac{1}{n^\gamma}\left(\sum_{d|n} \frac{1}{d^\gamma}\right)\left(\sum_{d|n} d^\gamma |a_d|^2\right).$$

Therefore, we have

$$|b_n|^2 \le \frac{\zeta(\gamma)}{n^\gamma} \sum_{d|n} d^\gamma |a_d|^2.$$

Hence,

$$\sum_{n=1}^{\infty} |b_n|^2 \le \zeta(\gamma) \sum_{n=1}^{\infty} \frac{1}{n^\gamma} \sum_{d|n} d^\gamma |a_d|^2 = (\zeta(\gamma))^2 \sum_{d=1}^{\infty} |a_d|^2.$$

This means that

$$\|b\|^2 \le \left(\zeta(\gamma)\right)^2 \|a\|^2.$$

Which tells us that $\|\mu_\gamma(a)\| \le \zeta(\gamma)\|a\|$, and from the knowledge of zeta function we see that the function μ_γ is bounded for $\gamma > 1$ and $\|\mu_\gamma\| \le \zeta(\gamma)$.

In this article we show that $\zeta(s)$ has a strict lower bound for the real part of s closed enough to the line 1. We illustrate this in the following main theorem:

2. The main theorem

2-1 Theorem:

Let s be a complex number. Then for $\frac{1}{2}+\varepsilon\leq \Re e\ s\leq 1-f(X)$, $(\varepsilon>0)$, we have for the imaginary part of s runs between 1 and X that the maximum of

$$|\zeta(s)|\geq e^{\left\{c\frac{e^{\frac{\log\log\log x}{2}}}{\log\log\log x}\right\}}$$

where c is a small constant and X sufficiently large ***independent*** of σ. Here $f(X)=\frac{\log\log\log X}{2\log\log X}$.

Note that: More generally, the above theorem is valid for any function $f(X)\to 0$ as $X\to\infty$.

In order to prove the above theorem we need the following **Lemma**.

2-2 Lemma:

For $1-c\geq\frac{\log\log\log T}{2\log\log T}$ we have

$$\sum_{p<P}p^{-\rho}\geq\frac{e^{\frac{\log\log\log P}{2}}}{c\ \log\log\log P}$$

For $P>P_0$.

Proof:

Using [HIL,2012], we see for $\pi(x)=\sum_{p\leq x}1$, that

$$\sum_{p\leq P}p^{-\rho}=\int_{2}^{P}u^{-\rho}d\pi(u).$$

So, by the 'Prime Number Theorem' (which means the assertion that the number of primes that $\pi(x)(=\sum_{p<x}1)\sim\frac{x}{\log x})$), we observe the last integral is greater than

$$\frac{e^{(1-\rho)\log P}}{c(1-\rho)\log P}-k,$$

For some constants $c,k>0$. The result follow immediately.

Now we could start prove the theorem.

3. Proof (of the main theorem)

For any real ρ running between $\frac{1}{2}+\delta$ and the line 1 (any $\delta>0$), we see (by [AIL,2012]) that for the imaginary part of s runs between 1 and X the maximum of

$$\zeta(s)\geq \underset{\|a\|_2}{Sup}\left(\sum_{k=1}^{X}|b_k|^2\right)^{\frac{1}{2}},$$

where a is in l^2 and $b_n=\sum_{d|n}\left(\frac{n}{d}\right)^{-\rho}a_\rho$. We note that if we take X to be as large as possible and define

$$a_n = \begin{cases} \dfrac{1}{(\sum_{d|X} 1)^{\frac{1}{2}}} & if\ n|X \\ 0 & otherwise \end{cases},$$

Then we see that

$$\|a\|^2 = \sum_{n=1}^{\infty} |a_n|^2 = \frac{\sum_{d|X} 1}{\sum_{d|X} 1} = 1.$$

$$\|b\|^2 = \sum_{n=1}^{\infty} \|b_n\|^2 \geq \sum_{n|X} b_n{}^2 \geq \frac{\sum_{d|X} d^{-\rho}}{(\sum_{d|X} 1)^{\frac{1}{2}}} = F_\rho(X).$$

Now, for

$$n = \prod_{p=2}^{P} p$$

We take X to be the largest number lying between n and $n \cdot P_0$ where P_0 is the next prime after P [WZ, 07]. Therefore, one can calculate $F_\rho(n)$ to observe

$$F_\rho(n) \geq \sum_{p<P} p^{-\rho} - \theta,$$

For some absolute constant θ. By the Prime Number Theorem [PB, 04], we see that $log\ n \sim P \sim \log X$.

Hence the above lemma finished the proof of the main theorem.

Question: What is the new in the main theorem?

We answer this question by saying that the lower bound (in the result of the main theorem) of the Remann-Zeta function is so strict in comparison with lower bounds of the Remann-Zeta function in [HIL, 09] and this is because we showed here that the constant is independent of σ which gives a strong lower bound in the region $\frac{1}{2} + \varepsilon < \ \Re e\ s\ < 1$, (any $\varepsilon > 0$).

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