Principally Pseudo-Injective Semimodule

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Abstract

In this paper, the connotation principally pseudo-injective semimodule which is a generalization of the notion principally quasi-injective semimodule will be introduced, we study some properties of principally pseudo-injective semimodule. Relationship between principally pseudo-injective and principally quasi-injective semimodule is given. A semimodule $\mathcal{B}$ is called principally pseudo-injective (shortly, P.P.-injective) if for any cyclic subsemimodule $Ų$ of $\mathcal{B}$, any monomorphism from $Ų$ into $\mathcal{B}$ can be extended to endomorphism of $\mathcal{B}$.

1.Introduction

The main purpose of this paper is to study different generalizations of injective semimodule which is principally pseudo-injective semimodule, depending on similar notion in the module theory, a module $\mathcal{B}'$ is called pseudo-$\mathcal{B}$-injective (or pseudo-injective relative to $\mathcal{B}$)if for every submodule $Ų$ of $\mathcal{B}$, any monomorphism α: $Ų \to \mathcal{B}'$, then there is a homomorphism $\theta: \mathcal{B} \to \mathcal{B}'$ which is extending to α [1]. Any two modules $\mathcal{B}$ and $\mathcal{B}'$ are called relatively (pseudo-injective) if $\mathcal{B}$ is pseudo-$\mathcal{B}'$-injective and $\mathcal{B}'$ is pseudo-$\mathcal{B}$-injective[1]. A module $\mathcal{B}$ is called pseudo -injective if $\mathcal{B}$ is pseudo-$\mathcal{B}$-injective. An $\mathcal{R}$-module $\mathcal{B}$ is called Principally-Ç-injective module if for any cyclic submodule $Ų$ of Ç, any homomorphism γ: $Ų \longrightarrow \mathcal{B}$ can be extended to homomorphism from Ç to $\mathcal{B}$[2]. An $\mathcal{R}$-module $\mathcal{B}$ is called principally pseudo-Ç- injective (shortly, P.P.-Ç- injective) if for any cyclic submodule $Ų$ of Ç, any monomorphism γ: $Ų \longrightarrow \mathcal{B}$, there exists homomorphism from Ç to $\mathcal{B}$ extending to γ [2]. An $\mathcal{R}$-module $\mathcal{B}$ is called principally pseudo-injective(in short, P.P.-injective) if $\mathcal{B}$ is principally pseudo- $\mathcal{B}$-injective[2]. A ring is called principally pseudo-injective if $\mathcal{R}$ is a principally pseudo-injective $\mathcal{R}$-module[2]. In the present work, these concepts for semimodules will be discussed which are generalizations of principally quasi-injective semimodule [3] and some interesting results on these semimodules will be given. Also, the property PC_2 is defined of module where, a module M satisfies PC_2, if each cyclic submodule of M which is isomorphic to a direct summand is a direct summand of M. Analogously this concept will be defined for semimodule. Also it is explained that any P.P.-injective semimodule satisfies this property. A semiring is a nonempty set $\mathcal{R}$ together with two operations, addition and multiplication, where these two operations are associative, addition is commutative operation, the distribution law holds, there is $0 \in \mathcal{R}$ (additive identity element) such that $t+0 = t = 0+t$, $t0 = 0t = 0$ for each t in $\mathcal{R}$ and there is a multiplicative identity element (denoted 1) where $1 \neq 0$. It is commutative if the second operation is commutative. For instance the set of natural number $\mathbb{N}$ is a commutative semiring under usual addition and multiplication, but it is not ring[4]. A (left) semimodule $\mathcal{B}$ over a semiring $\mathcal{R}$ is defined similarly in module over a ring[4]. A subset $Ų$ of the $\mathcal{R}$ -semimodule $\mathcal{B}$ is called a subsemimodule of $\mathcal{B}$ if b , $c \in Ų$ and $r \in \mathcal{R}$ implies that $b+ c \in Ų$ and $rb \in Ų$ and denoted by $Ų \hookrightarrow \mathcal{B}$[4], in this case $Ų$ itself is an $\mathcal{R}$ -semimodule. The concepts of homomorphism, kernel, image are defined similar to the case in modules[4].

Throughout this paper, $\mathcal{R}$ stands for a commutative semiring with identity and a semimodule means a unitary left $\mathcal{R}$ -semimodule.

2.Preliminaries

Some definitions that needed in this paper, will be inserted.

Definition 2.1 [4]. A nonempty subset Į of a semiring $\mathcal{R}$ is a right (resp. left) **ideal** of $\mathcal{R}$ if for $i, j \in$ Į and $t \in \mathcal{R}$ then $i+j \in$ Į and $i\,t$ (resp. $t\,i$) $\in$ Į . Į is (two- sided) ideal of $\mathcal{R}$ if it is both a left and a right ideal of $\mathcal{R}$.

Definition 2.2 [5]. A subsemimodule Ų of $\mathcal{B}$ is called a **subtractive** subsemimodule, if for each $b, c \in \mathcal{B}$, that $b + c, b \in$ Ų leads to $c \in$ Ų. It is clear that $\{0\}$ and $\mathcal{B}$ are subtractive subsemimodules of $\mathcal{B}$. A semimodule $\mathcal{B}$ is called subtractive semimodule if it has only subtractive subsemimodules.

Definition 2.3 [6]. A semimodule $\mathcal{B}$ is called a **semisubtractive**, if for any $b, c \in \mathcal{B}$ there is always some $h \in \mathcal{B}$ satisfying $b+h=c$ or some $k \in \mathcal{B}$ satisfying $c+k=b$.

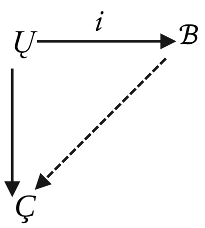
Definition 2.4 [4]. An element a' of a left $\mathcal{R}$-semimodule $\mathcal{B}$ is cancellable if $a'+n=a'+k$ implies that $n=k$. The $\mathcal{R}$-semimodule $\mathcal{B}$ is **cancellative** if and only if every element of $\mathcal{B}$ is cancellable.

Definition 2.5[4]. An $\mathcal{R}$-semimodule $\mathcal{B}$ is said to be a **direct sum** of subsemimodules $Ų_1, Ų_2, \ldots, Ų_k$ of $\mathcal{B}$, if each $b \in \mathcal{B}$ can be uniquely written as $b=u_1+u_2+\ldots+u_k$ where $u_i \in Ų_i$, $1 \leq i \leq k$. It is denoted by $\mathcal{B} = Ų_1 \oplus Ų_2 \oplus \ldots \oplus Ų_k$. In this case each $Ų_i$ is called a direct summand of $\mathcal{B}$.

Definition 2.6 [7]. A left $\mathcal{R}$-semimodule $\mathcal{B}$ is called **cyclic** if $\mathcal{B}$ can be generated by a single element, that is $\mathcal{B} = \langle b \rangle = \mathcal{R}b = \{t\,b \mid t \in \mathcal{R}\}$ for some $b \in \mathcal{B}$.

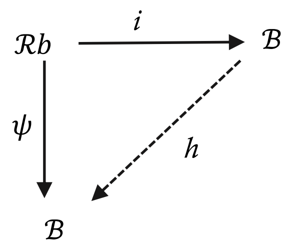
Definition 2.7[7]. A nonzero $\mathcal{R}$-semimodule $\mathcal{B}$ is called **simple** if $\mathcal{B}$ has no nonzero proper $\mathcal{R}$-subsemimodule.

Definition 2.8 [3]. An $\mathcal{R}$-semimodule Ç is $\mathcal{B}$**-injective** (Ç is injective relative to $\mathcal{B}$) if, for each subsemimodule Ų of $\mathcal{B}$, any $\mathcal{R}$-homomorphism from Ų to Ç can be extended to an $\mathcal{R}$-homomorphism from $\mathcal{B}$ to Ç. (where i is the inclusion map) as the following diagram:



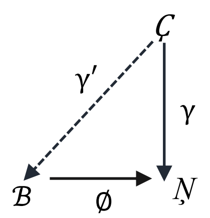
A left $\mathcal{R}$-semimodule Ç is **injective** if it is injective relative to every left $\mathcal{R}$-semimodule

Definition 2.9[3]. An $\mathcal{R}$-semimodule is called **principally quasi-injective** if each $\mathcal{R}$-homomorphism from cyclic subsemimodule of $\mathcal{B}$ to $\mathcal{B}$ can be extended to an endomorphism of $\mathcal{B}$. In other words, the following diagram is commutative. i.e., $hi=\psi$.



Definition 2.10 [3]. An $\mathcal{R}$-semimodule $\mathcal{B}$ is called **P-injective** if for any principal ideal Į of $\mathcal{R}$ and each $\mathcal{R}$-homomorphism ϑ : Į $\rightarrow \mathcal{B}$, there exists an $\mathcal{R}$-homomorphism $\alpha : \mathcal{R} \rightarrow \mathcal{B}$, which extends ϑ.

Definition 2.11 [3]. A left $\mathcal{R}$-semimodule Ç is said to be $\mathcal{B}$**-projective** if for every epimorphism $\phi: \mathcal{B} \rightarrow$ Ņ and for every homomorphism γ: Ç $\rightarrow$ Ņ there is a homomorphism γ': Ç $\rightarrow \mathcal{B}$ such that the diagram



commutes.

A semimodule $Ç$ is **projective** if it is projective relative to every left $\mathcal{R}$-emimodule.

Definition 2.12 [7]. A semimodule $\mathcal{B}$ is called **regular** if every cyclic subsemimodule of $\mathcal{B}$ is a direct summand.

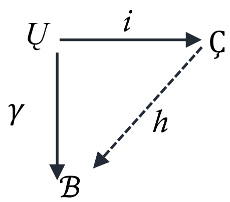
Definition 2.13[3]. A semimodule $\mathcal{B}$ is called ***Z*-regular** if every cyclic subsemimodule of $\mathcal{B}$ is projective and direct summand.

Definition 2.14[4]. A nonzero element a of a semiring $\mathcal{R}$ is a zero-divisor iff $a\ b = 0$ for some non-zero element b of $\mathcal{R}$. A semiring $\mathcal{R}$ having no zero divisors is entire.

3.Principally pseudo-Ç-Injective Semimodule

In this section we introduce the concept principally pseudo injective-Ç- semimodule, this concept was discuss for modules in [2] where an $\mathcal{R}$-module $\mathcal{B}$ is called pseudo-Ç-injective if for any subsemimodule $Ų$ of Ç and every $\mathcal{R}$-monomorphism from $Ų$ into $\mathcal{B}$ can be extended to an $\mathcal{B}$-homomorphism from Ç into $\mathcal{B}$. An $\mathcal{R}$-module $\mathcal{B}$ is called pseudo -injective if $\mathcal{B}$ is pseudo-$\mathcal{B}$-injective [2]. We discuss this concept for semimodule and some generalizations of it. Also explain the relationship between P.P.-injective and P.Q.-injective semimodules in Proposition(3.23) and Proposition(3.16) shows that every P.P.-injective semimodule has PC_2.

Definition 3.1. Let $\mathcal{B}$ and Ç be two $\mathcal{R}$-semimodules. $\mathcal{B}$ is said to be principally pseudo- Ç-injective (shortly, P.P.-Ç-injective) if for any cyclic subsemimodule $Ų$ of Ç and any monomorphism γ: $Ų\longrightarrow \mathcal{B}$ can be extended to homomorphism from Ç to $\mathcal{B}$. i.e., the following diagram commutes.



Definition 3.2. An $\mathcal{R}$-semimodule $\mathcal{B}$ is called principally pseudo-injective(shortly, P.P.-injective)if $\mathcal{B}$ is principally pseudo- $\mathcal{B}$-injective. A semiring is called principally pseudo-injective if $\mathcal{R}$ is a principally pseudo-injective $\mathcal{R}$-semimodule.

Examples 3.3.

(1) Every P.Q.-injective semimodule is P.P.-injective.

(2) Every $\mathcal{Z}$-regular semimodule is P.P.-injective.

We offer some properties about P.P. -injective semimodule which are mentioned in [2] and [8]for modules.

Proposition 3.4. Every direct summand of P.P.- Ç-injective semimodules is also P.P.-Ç-injective.

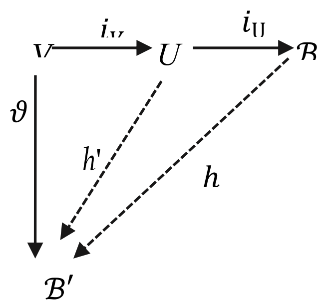
Proof: Let $\mathcal{B}$ be any P.P.- Ç-injective semimodule and $Ų$ be any direct summand of $\mathcal{B}$. Then there is $V \hookrightarrow \mathcal{B}$ such that $\mathcal{B}$= $Ų\oplus V$. Let K be any cyclic subsemimodule of Ç and α: $K\longrightarrow Ų$ be any monomorphism. Define γ: $K\longrightarrow \mathcal{B}=Ų\oplus V$ by $\gamma(k) = (\alpha(k), 0)$ for all $k \in K$, it is clear that γ is monomorphism, since $\mathcal{B}$ is any P.P.- Ç-injective semimodule, thus there exists homomorphism δ: Ç$\longrightarrow \mathcal{B}$ such that $\delta(k) = \gamma(k)$. Let $\pi_{Ų}: \mathcal{B} \longrightarrow Ų$ be the projection homomorphism. Put $\gamma_1 = \pi_{Ų}\,\gamma$, thus γ_1 is homomorphism and . $\gamma_1(k) = \pi_{Ų}\,\gamma(k) = \pi_{Ų}\big((\alpha(k), 0)\big) = \alpha(k)$. Therefore $Ų$ is P.P.-injective semimodule. ////

Corollary 3.5. Every direct summand of P.P.- injective semimodules is P.P.- injective.

Proposition 3.6. Let $\mathcal{B}$ and $\mathcal{B}'$ be semimodules. If $\mathcal{B}'$ is P.P.-$\mathcal{B}$-injective, then $\mathcal{B}'$ is P.P.- $Ų$- injective for any subsemimodule $Ų$ of $\mathcal{B}$.

Proof: Let V be cyclic subsemimodule of $Ų$, ϑ:$V\longrightarrow \mathcal{B}'$ be any monomorphism, i_v: $V\longrightarrow Ų$ and

$i_{Ų}$: $Ų \longrightarrow \mathcal{B}$ be the inclusion maps of V into $Ų$ and $Ų$ into $\mathcal{B}$ respectively. Consider the following diagram:



Since $\mathcal{B}'$ is P.P.- $\mathcal{B}$-injective, then there exists homomorphism $h: \mathcal{B} \longrightarrow \mathcal{B}'$ such that $h\ i_{Ų}\ i_v = \vartheta$. Define $h' = h\ i_{Ų}$. Note that h' extends ϑ, that is $h'\ i_v(v) = h'(i_v(v)) = h'(v) = h\ i_{Ų}(v) = h\ i_{Ų}(\ i_v(v)) = h\ i_{Ų}\ i_v(v) = \vartheta(v)$. ////

Lemma 3.7[3]. Let $\mathcal{B}$ be a subtractive, semisubtractive, cancellative semimodule, then $\mathcal{B} = Ų \oplus Ç$ if and only if $\mathcal{B} = Ų + Ç$ and $Ų \cap Ç = \{0\}$.

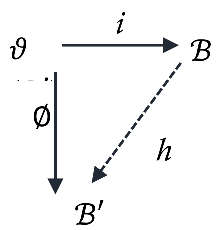
Note. We use the notation SSC for a subtractive, semisubtractive, cancellative semimodule.

Lemma 3.8[9]. If $\mathcal{B}$ is SSC semimodule, then a subsemimodule $Ų$ is direct summand of $\mathcal{B}$ if and only if the inclusion map $i: Ų \rightarrow \mathcal{B}$ has left inverse.

Remark 3.9. If $\mathcal{B}$ is SSC semimodule**,** then $\alpha: Ų \longrightarrow \mathcal{B}$ is said to be split monomorphism if there exists a homomorphism $\gamma: \mathcal{B} \longrightarrow Ų$ such that $\gamma\alpha = 1_{Ų}$.

Proposition 3.10. Let $\mathcal{B}$ be SSC semimodule and $\mathcal{B}'$ be cyclic semimodule. If $\mathcal{B}'$ is P.P.- $\mathcal{B}$-injective, then any monomorphism $\vartheta: \mathcal{B}' \longrightarrow \mathcal{B}$ splits.

Proof: Let $\vartheta: \mathcal{B}' \longrightarrow \mathcal{B}$ be a monomorphism and $\vartheta': \mathcal{B}' \longrightarrow \vartheta(\mathcal{B}')$ such that $\vartheta'(x) = \vartheta(x)$ it is clear that ϑ' is an isomorphism, let $\emptyset$ be the inverse of ϑ'. It is clear that $\vartheta(\mathcal{B}')$ is cyclic subsemimodule of $\mathcal{B}$. Consider the following diagram:-



Where $i: \vartheta(\mathcal{B}') \longrightarrow \mathcal{B}$ is the inclusion map. Since $\mathcal{B}'$ is P.P.-$\mathcal{B}$-injective, then there exists homomorphism h: $\mathcal{B} \longrightarrow \mathcal{B}'$ such that $hi = \emptyset$. Put $\gamma = h\ \vartheta$ it is clear that γ is an identity map on $\mathcal{B}'$. Hence $\vartheta(\mathcal{B}')$ is a direct summand of $\mathcal{B}$. ////

Proposition 3.11. Let $\mathcal{B}$ and $\mathcal{B}'$ be two semimodules and Ş= End($\mathcal{B}$), then the following statements are equivalent:

(1)$\mathcal{B}$ is P.P.- $\mathcal{B}'$-injective.

(2)If $ann_{\mathcal{R}}(b)$= $ann_{\mathcal{R}}(c)$ for each $b \in \mathcal{B}$, $c \in \mathcal{B}'$, there exists homomorphism $\gamma: \mathcal{B}' \longrightarrow \mathcal{B}$ such that $\gamma(c) = b$.

(3) If $ann_{\mathcal{R}}(b)$= $ann_{\mathcal{R}}(c)$, for each $b \in \mathcal{B}$, $c \in \mathcal{B}'$, then bŞ $\subseteq c$ Hom($\mathcal{B}', \mathcal{B}$) .

(4) For each monomorphism $\alpha: Ų \longrightarrow \mathcal{B}$(where $Ų \hookrightarrow \mathcal{B}'$) and $u \in Ų$, there exists a homomorphism $\gamma: \mathcal{B}' \longrightarrow \mathcal{B}$ such that $\alpha(u) = \gamma(u)$.

Proof: (1)⟹(2) Let $\mathcal{B}$ be P.P.- $\mathcal{B}'$-injective such that $ann_{\mathcal{R}}(b)$= $ann_{\mathcal{R}}(c)$, for each $b \in \mathcal{B}$, $c \in \mathcal{B}'$, define $\gamma: \mathcal{R}c \longrightarrow \mathcal{B}$ by $\gamma(rc) = rb$, for all r in $\mathcal{R}$. It is clear that γ is well-defined monomorphism, since $\mathcal{B}$ is P.P.- $\mathcal{B}'$-injective, thus there is homomorphism $\alpha: \mathcal{B}' \longrightarrow \mathcal{B}$ such that $\alpha(x) = \gamma(x)$, for each $x \in \mathcal{R}c$. Therefore $\alpha(c) = \gamma(c) = b$.

(2) ⟹(3)Let $ann_{\mathcal{R}}(b)$= $ann_{\mathcal{R}}(c)$, for each $b \in \mathcal{B}$, $c \in \mathcal{B}'$ by hypothesis, there exists homomorphism $\alpha: \mathcal{B}' \longrightarrow \mathcal{B}$ such that $\alpha(c) = b$. Let $\theta \in$ Ş, thus $\theta(b) = \theta\ (\alpha(c)) = \theta\ \alpha(c) \in$ Hom($\mathcal{B}', \mathcal{B}$).Therefore bŞ $\subseteq c$ Hom($\mathcal{B}', \mathcal{B}$) , since θ is an arbitrary element of Ş.

(3) ⟹(4) Let $\alpha: Ų \longrightarrow \mathcal{B}$ be any monomorphism where $Ų$ is any subsemimodule of $\mathcal{B}'$ and let $u \in Ų$. Put $b = \alpha(u)$, since $b \in \mathcal{B}$ and $ann_{\mathcal{R}}(b)$= $ann_{\mathcal{R}}(u)$, thus by hypothesis we have b Ş $\subseteq u$ Hom($\mathcal{B}', \mathcal{B}$). Let I:$\mathcal{B} \longrightarrow \mathcal{B}$ be the identity map. Since I ∈ Ş , there is homomorphism $\alpha' \in$ Hom($\mathcal{B}', \mathcal{B}$) such that I($b$) = α' (b) hence α' (u) = $b = \alpha(u)$.

(4) ⟹(1) Let $Ų = \mathcal{R}u$ be any cyclic subsemimodule of $\mathcal{B}'$ and $\alpha: Ų \longrightarrow \mathcal{B}$ be any monomorphism. Since $u \in Ų$, then by hypothesis there is homomorphism $\gamma: \mathcal{B}' \longrightarrow \mathcal{B}$ such that $\gamma(u) = \alpha(u)$ for each $x \in Ų$, $x = ru$ for some $r \in \mathcal{R}$ we have that $\gamma(x) = \gamma(ru) = r\gamma(u) = r\ \alpha(u) = \alpha(ru) = \alpha(x)$. Therefore $\mathcal{B}$ is P.P.- $\mathcal{B}'$-injective. ////

As a direct consequence of proposition (3.11) we get the following corollary

Corollary 3. 12. The following statements are equivalent for a semimodule $\mathcal{B}$

(1)$\mathcal{B}$ is P.P.- injective.

(2)If $ann_{\mathcal{R}}(b)$= $ann_{\mathcal{R}}(b')$, for each $b, b' \in \mathcal{B}$, there exists homomorphism $\gamma: \mathcal{B} \longrightarrow \mathcal{B}$ such that $\gamma(b') = b$.

(3) If $ann_{\mathcal{R}}(b)$= $ann_{\mathcal{R}}(b')$, for each $b, b' \in \mathcal{B}$, we have b'Ş $\subseteq b$Ş .

(4) For each monomorphism $\alpha: \mathcal{U} \longrightarrow \mathcal{B}$ ($\mathcal{U} \hookrightarrow \mathcal{B}$) and $u \in \mathcal{U}$, there exists homomorphism $\gamma: \mathcal{B} \longrightarrow \mathcal{B}$ such that $\alpha(u) = \gamma(u)$.

Proposition 3.13. Let $\mathcal{U}$ be cyclic subsemimodule of $\mathcal{B}$ where $\mathcal{B}$ is SSC semimodule**.** If $\mathcal{U}$ is P.P.- $\mathcal{B}$- injective, then $\mathcal{U}$ is a direct summand of $\mathcal{B}$.

Proof: Let i: $\mathcal{U} \longrightarrow \mathcal{B}$ be the inclusion map. Since $\mathcal{U}$ is P.P.-$\mathcal{B}$- injective semimodule, then there exists homomorphism $\gamma: \mathcal{B} \longrightarrow \mathcal{B}$ such that $\gamma(u) = i(u)$ for each u in $\mathcal{U}$. Therefore $\gamma i(u) = u$, then i has left inverse (by Lemma(3.8)) and hence $\mathcal{U}$ is a direct summand of $\mathcal{B}$. ////

By Proposition (3.13) and definition (2.12) we have the following corollary:

Corollary 3.14. If every cyclic subsemimodule of $\mathcal{B}$ is P.P.- $\mathcal{B}$- injective, then $\mathcal{B}$ is regular semimodule.

Lemma 3.15. Every semimodule which is isomorphic to P.P.-injective semimodule, is P.P.-injective.

Proof: Let $\mathcal{B}$ be P.P.-injective semimodule and $\phi: \mathcal{B}' \to \mathcal{B}$ be an isomorphism where $\mathcal{B}'$ is any semimodule, assume that $\alpha: \mathcal{R}b \to \mathcal{B}'$ is a monomorphism where b in $\mathcal{B}'$, let $\gamma = \phi^{-1}|_{\mathcal{R}c}$, where $c = \phi(b)$, so $\gamma: \mathcal{R}c \longrightarrow \mathcal{R}b$ and $\delta = \phi\ \alpha\ \gamma: \mathcal{R}c \longrightarrow \mathcal{B}$ are monomorphism. Since $\mathcal{B}$ is P.P.-injective, then there exists $g \in End(\mathcal{B})$ such that $g|_{\mathcal{R}c} = \delta$. Now, put $\theta = \phi^{-1} g\ \phi: \mathcal{B}' \to \mathcal{B}'$, $\theta(b) = \phi^{-1} g\ \phi(b) = \phi^{-1}(g\ \phi(b)) = \phi^{-1}(g(c)) = \phi^{-1}(\delta(c)) = \phi^{-1}(\phi\ \alpha\ \gamma(c)) = \alpha\ \gamma(c) = \alpha(b)$, then θ is an extension of α to $\mathcal{B}'$. ////

In [2] the property PC_2 was defined for modules, analogously we define this property for semimodule and explain the relationship between this property and P.P.-injective semimodule. An R-semimodule has PC_2, if every cyclic subsemimodule of $\mathcal{B}$ which is isomorphic to direct summand of $\mathcal{B}$ is a direct summand of $\mathcal{B}$.

Proposition 3.16. Any P.P.-injective SSC semimodule has PC_2.

Proof: Let $\mathcal{B}$ be a P.P.-injective semimodule and $\mathcal{U}$ be any cyclic subsemimodule of $\mathcal{B}$ which is isomorphic to direct summand V of $\mathcal{B}$. since $\mathcal{B}$ is P.P.-injective, this means $\mathcal{B}$ is P.P.-$\mathcal{B}$-injective, since each direct summand of P.P.-injective semimodule is also P.P.-$\mathcal{B}$-injective semimodule by Proposition (3.4) we have V is P.P.-$\mathcal{B}$-injective semimodule. Since $\mathcal{U}$ is isomorphic to V then by Lemma (3.15)we get $\mathcal{U}$ is P.P.-$\mathcal{B}$-injective V, since $\mathcal{U}$ is cyclic subsemimodule of $\mathcal{B}$by Proposition(3.13), then $\mathcal{U}$ is direct summand of $\mathcal{B}$. ////

Definition 3.17[7]. A semiring $\mathcal{R}$ is said to be **(Von Neumann) regular** if for each $t \in \mathcal{R}$, there is some $s \in \mathcal{R}$, such that $t=tst$.

Example 3.18 [10].

Let $\mathcal{R} = \{0, r, t\}$ be a semiring with the following Cayley tables

+	0	r	t
0	0	r	t
r	0	r	t
t	0	r	t

.	0	r	t
0	0	r	t
r	0	r	t
t	0	r	t

$\mathcal{R}$ is regular (Von Neumann) semiriing.

Proposition 3.19. For a semiring $\mathcal{R}$, the following statements are equivalent:

(1) $\mathcal{R}$ is (Von Neumann) regular semiring.
(2) Every $\mathcal{R}$-semimodule is P.P.- $\mathcal{R}$-injective.

(3) Every ideal of $\mathcal{R}$ is P.P.- $\mathcal{R}$-injective semimodule.

(4) Every cyclic ideal of $\mathcal{R}$ is P.P.- $\mathcal{R}$-injective semimodule. (we need to add the condition $\mathcal{R}$ is an entire, i.e. $\mathcal{R}$ has no zero divisor).

Proof: (1)⟹(2) Let $\mathcal{R}$ be a regular semiring and $\mathcal{B}$ be any $\mathcal{R}$-semimodule . Let $\vartheta: \mathcal{R}a \longrightarrow \mathcal{B}$ be any monomorphism where $\mathcal{R}a$ be any cyclic ideal of $\mathcal{R}$. Since $\mathcal{R}$ is a regular semiring and $a \in \mathcal{R}$, then there is $t \in \mathcal{R}$ such that $a = a\ ta$. Put $b = \vartheta(ta)$ and defined $\alpha: \mathcal{R} \longrightarrow \mathcal{B}$ by $\alpha(x) = x\ b$ for all $x \in \mathcal{R}$. It is clear that α is homomorphism. For each $y \in \mathcal{R}a$, $y = sa$ for some $s \in \mathcal{R}$, then $\alpha(y) = \alpha(sa) = s\ \alpha(a) = s(ab) = sa\ \vartheta(ta) = s\ \vartheta(at\ a) = s\ \vartheta(a) = \vartheta(sa) = \vartheta(y)$. Then $\mathcal{B}$ is P.P.- $\mathcal{R}$-injective.

(2)⟹(3) and **(3)⟹(4)** are obvious. **(4)⟹(1)** Let $a \in \mathcal{R}$, provided that a is not zero divisor element of $\mathcal{R}$, by (4) $\mathcal{R}a$ is P.P.- $\mathcal{R}$-injective. Let $\vartheta: \mathcal{R}a \longrightarrow \mathcal{R}$ defined by $ra \mapsto r$, then ϑ is a monomorphism, and so there is $\beta: \mathcal{R} \longrightarrow \mathcal{R}$ extendeing ϑ, that is $\beta(r\, a) = \vartheta(ra) = r$, let $\beta(1) = b \Rightarrow \beta(a) = a\, \beta(1) = a\, b$, but $\beta(a) = \vartheta(a) = 1$, so $a\, b = 1 \Rightarrow aba = a$ and hence $\mathcal{R}$ is regular semiring. ////

Definition 3.20 [11]. The **singular** subsemimodule of $\mathcal{B}$ is defined by $\mathcal{Z}(b)$={$b \in \mathcal{B}$| $ann_{\mathcal{R}}$ (b) is essential ideal in $\mathcal{R}$}. If $\mathcal{Z}(\mathcal{B})= \mathcal{B}$, $\mathcal{B}$ is called singular and if $\mathcal{Z}(\mathcal{B})$=0, then $\mathcal{B}$ is called nonsingular semimodule.

Definition 3.21[7]. A semimodule $\mathcal{B}$ is called uniform semimodule if the intersection of any nonzero two subtractive subsemimodules of $\mathcal{B}$ is nonzero subsemimodule of $\mathcal{B}$.

Lemma.3.22. Let $\theta: \mathcal{R}b \longrightarrow \mathcal{B}$, $0 \neq b \in \mathcal{B}$ be a homomorphism and $\mathcal{B}$ be a nonsingular, uniform semimodule, if $\theta \neq 0$, then θ is monomorphism.

Proof: If ker $(\theta) \neq 0$, then ker $(\theta) = I\, b$ where $I = ann_{\mathcal{R}}\ \theta(b)$, since $\mathcal{B}$ is nonsingular, then I is not essential in $\mathcal{R}$ hence $I\, b$ is not essential in $\mathcal{R}b$ which contradiction with the condition that $\mathcal{B}$ is uniform, then ker(θ) = 0 and hence θ is monomorphism. ////

We know that, every P.Q.-injective semimodule is P. pseudo-injective but the converse is not true.

The following proposition explain the conditions to be added to achieve the opposite.

Proposition 3.23. Every uniform nonsingular P.P.-injective semimodule is P.Q.-injective.

Proof: Let V be cyclic subsemimodule of a uniform nonsingular P.P.-injective semimodule $\mathcal{B}$ and let $\vartheta: V \longrightarrow \mathcal{B}$ be a homomorphism. ker(ϑ) is subsemimodule of V, either ker$(\vartheta) = 0$, then ϑ is monomorphism and hence ϑ can be extended to a homomorphism from $\mathcal{B}$ to $\mathcal{B}$, since $\mathcal{B}$ is P.P.-injective, then $\mathcal{B}$ is P.Q.-injective. Or ker$(\vartheta) \neq 0$, since $\mathcal{B}$ is nonsingular, then $\mathcal{B}$ has no proper essential subsemimodule, by Lemma(3.22) we have ϑ is monomorphism and hence can be extended to a homomorphism of $\mathcal{B}$ to $\mathcal{B}$. Hence $\mathcal{B}$ is P.Q.-injective. ////

CONFLICT OF INTERESTS

There are no conflicts of interest.

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شبه المقاسات الاغمارية الكاذبة رئيسيا

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الخلاصة

نقدم في هذا البحث, مفهوم شبه المقاس الرئيس الاغماري الكاذب الذي هو تعميم لمفهوم شبه المقاس الرئيس شبه الاغماري رئيسا, وندرس بعض خواص شبه المقاس الرئيس الاغماري الكاذب والعلاقة بين شبه المقاس الرئيس الاغماري الكاذب و شبه المقاس الرئيس شبه الاغماري. يسمى شبه المقاس اغماري كاذب رئيسيا اذا كان لكل احاد تشاكل من اي شبه مقاس جزئي دوري من $\mathcal{B}$ الى $\mathcal{B}$ يمكن توسيعه الى تشاكل في شبه حلقة التشاكلات في $\mathcal{B}$.

الكلمات الدالة: شبه المقاس ,شبه المقاس الاغماري, شبه المقاس الاغماري رئيسا, شبه المقاس الاغماري الكاذب رئيسيا.