Bounds on Real and Imaginary Parts of Non-Real Eigenvalues of a Non-Definite Sturm-Liouville Differential Equation with Cauchy Boundary Conditions

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Abstract

We are concerned in the paper with spectral properties of indefinite Sturm – Liouville type problems with Cauchy boundary conditions where differential equations coefficients are real valued integrable functions, and the spectral parameter is also purely complex number. And also we get bounds on the real and imaginary parts of indefinite Sturm-Liouville problem with cauchy conditions.

1. Introduction

In this study, we derived the real and imaginary parts for an eigenvalue corresponding to the spectral problem

$$-y'' + q(x)y = \lambda\rho(x)y \quad (1)$$

$$y(a) = 0, y'(b) = 0 \quad (2)$$

where $x \in [a, b]$, $\lambda \in \mathbb{C}$.

The functions q and ρ are assumed to be real valued integrable functions, and ρ takes on positive and negative values on subsets of $[a, b]$ with the positive Lebesgue measure. Thus, indefinite Sturm-Liouville problems are used to define such a problem.

Many authors have obtained bounds of non-real eigenvalues for the different type of spectral problems [1],[3],[6], and [7].

Jussi, Shaozhou, Friedrich, and Jaingang obtained boundaries in 2013 on non-real proper values of indefinite Sturm-Liouville Problems with Dirichlet boundary conditions where the Sturm-Liouville equation coefficients are real integrable functions [1].

The main purpose of this paper is to prove boundaries on the unreal spectrum of second-order differential equation with Cauchy boundary conditions (1)-(2), and we get boundaries of the eigenvalue, and the authors [1]and [7] find boundaries of the eigenvalues of Sturm-Liouville's boundary conditions with Dirichlet.

2. Fundamental Results

In this section, we look at the issue (1)-(2) and progress on the boundaries obtained in [1].

Theorem 2.1 Suppose that there exists a function $g \in H'(a, b)$ such that $g(x)\rho(x) > 0$ on $(a, b)$and let $\epsilon > 0$ be such that

$$\alpha = |\{x \in (a, b): g(x)\rho(x) < \epsilon\}| \leq \frac{1}{8(b-a)\|q_-\|_1^2}.$$

Then for any non- real eigenvalue $\lambda \in \mathbb{C}\backslash\mathbb{R}$ of problem (1)-(2), we have

$$|\mathrm{Im}\,\lambda| \leq \frac{2}{\epsilon}\sqrt{b-a}\ \ \|g'\|_2\left(\|q-\|_1+\sqrt{\|q-\|_1^2+Q(b)|y(b)|^2}\right)^2$$

and

$$|Re\lambda| \leq \frac{2}{\epsilon}\left(\|q-\|_1+\sqrt{\|q-\|_1^2+Q(b)|y(b)|^2}\right)^2\left(\sqrt{b-a}\ \ \|g'\|_2+2(b-a)\|g\|_\infty\|q-\|_1\right)$$

Proof. Let $y(x)$ be an eigenfunction corresponding toλ. Without loss of generality we can assume that $\|y\|_2=1$. Multiplication of the differential equation in (1) by $\bar{y}$, and integration from x to b $\int_x^b -y''\bar{y}dx+\int_x^b q(x)y\bar{y}dx=\int_x^b \lambda\rho(x)y\bar{y}dx$

$$\lambda\int_x^b \rho|y|^2dx = y'(x)\,\bar{y}(x)+\int_x^b(|y'|^2+q(x)|y|^2)\,dx \qquad (3)$$

Taking the real and imaginary part of (3) gives

$$(Re\,\lambda)\int_x^b \rho|y|^2dx = Re[y'(x)\bar{y}(x)]+\int_x^b(|y'|^2+q(x)|y|^2)\,dx \qquad (4)$$

$$(Im\,\lambda)\int_x^b \rho|y|^2dx = Im[y'(x)\bar{y}(x)] \qquad (5)$$

Hence by setting $x=a$ in (4) and (5) we obtain

$$(Re\,\lambda)\int_a^b \rho|y|^2dx = Re[y'(a)\bar{y}(a)]+\int_a^b(|y'|^2+q(x)|y|^2)\,dx$$

Since $y(a)=0$, then the conjucate of $y(a)$ that is $\bar{y}(a)=0$

$$(Re\,\lambda)\int_a^b \rho|y|^2dx = \int_a^b(|y'|^2+q(x)|y|^2),$$

and

$$(Im\,\lambda)\int_a^b \rho|y|^2dx = Im[y'(a)\bar{y}(a)].$$

Then

$$(Im\,\lambda)\int_a^b \rho|y|^2dx = 0\,,$$

$$\int_a^b \rho|y|^2dx = \int_a^b(|y'|^2+q(x)|y|^2)=0. \qquad (6)$$

Now, for each $x\in[a,b]$ we have that

$$|y| = \left|\int_a^x y'\right| \leq \int_a^x|y'| \leq \sqrt{b-a}\|y'\|_2 \qquad (7)$$

Moreover, putting $Q(x)=\int_a^x q_-(t)\,dt$ for $x\in[a,b]$, the relation (6) yields

$$\int_a^b(|y'|^2+q(x)|y|^2)=0$$

$$\int_a^b|y'|^2 = -\int_a^b q|y|^2$$

$$\|y'\|_2^2 = -\int_a^b q|y|^2 \leq \int_a^b q_-|y|^2 = \int_a^b Q'|y|^2$$

And integrating by parts

$$\int_a^b Q'|y|^2 = Q(b)|y(b)|^2-Q(a)|y(a)|^2-2\int_a^b Q\,|y||y'|$$

$$= Q(b)|y(b)|^2-2\int_a^b Q\,Re(y\bar{y'})$$

$$\leq Q(b)|y(b)|^2+2\|q_-\|_1\|y'\|_2$$

This implies $\|y'\|_2^2-2\|q_-\|_1\|y'\|_2 \leq Q(b)|y(b)|^2$

$$\|y'\|_2^2-2\|q_-\|_1\|y'\|_2+\|q_-\|_1^2 \leq \|q_-\|_1^2+Q(b)|y(b)|^2$$

$$(\|y'\|_2-\|q_-\|_1)^2 \leq \|q_-\|_1^2+Q(b)|y(b)|^2$$

$$\|y'\|_2-\|q_-\|_1 \leq \sqrt{\|q_-\|_1^2+Q(b)|y(b)|^2}$$

$$\|y'\|_2 \leq \|q_-\|_1+\sqrt{\|q_-\|_1^2+Q(b)|y(b)|^2}$$

And $|y| \leq \sqrt{b-a}\|y'\|_2$,then

$$\|y\|_\infty \leq \sqrt{b-a}\left(\|q_-\|_1+\sqrt{\|q_-\|_1^2+Q(b)|y(b)|^2}\right) \qquad (8)$$

And

$$\|y'\|_2 \leq \|q_-\|_1+\sqrt{\|q_-\|_1^2+Q(b)|y(b)|^2} \qquad (9)$$

Now, let $\Delta=\{x\in(a,b): g(x)\rho(x)<\epsilon\}$.

Then we obtain from equations (7), (8) and (9) that

$$\int_a^b g'(x)\int_x^b \rho(t)\,|y(t)|^2dtdx = \int_a^b g\rho|y|^2 \geq \epsilon\int_{\Delta^c}|y|^2$$

$$=\epsilon\left(1-\int_\Delta|y|^2\right)$$

$$\geq \epsilon(1 - \|y\|_\infty^2 |\Delta|)$$
$$\geq \frac{\epsilon}{2}.$$

Hence, the relations (5),(8) and (9) imply

$$|Im\,\lambda| \int_a^b \int_x^b \rho\, |y|^2 g' = \int_a^b g'\, Im(y'\bar{y}) \qquad (10)$$

$$\frac{\epsilon}{2}|Im\,\lambda| \leq \left|\int_a^b g'\, Im(y'\bar{y})\right|$$
$$\leq \int_a^b |g' y\, y'|$$
$$\leq \|y\|_\infty \|g'\|_2 \|y'\|_2$$
$$\leq \sqrt{b-a}\|g'\|_2 \left(\|q-\|_1 + \sqrt{\|q-\|_1^2 + Q(b)|y(b)|^2}\right)^2$$
$$|Im\,\lambda| \leq \frac{2}{\epsilon}\sqrt{b-a}\|g'\|_2 \left(\|q-\|_1 + \sqrt{\|q-\|_1^2 + Q(b)|y(b)|^2}\right)^2$$

So that the estimates on imaginary λ is established.
For the real part we exploit (4) and (6) to derive

$$\frac{\epsilon}{2}|Re\,\lambda| \leq \left|\int_a^b g'(x)(Re(y'(x)\bar{y}(x)) + \int_x^b (|y'|^2 + q|y|^2))dx\right| \qquad (11)$$
$$\leq \|y\|_\infty \|g'\|_2 \|y'\|_2 + \left|\int_a^b g(|y'|^2 + q|y|^2)\right|$$

Now, setting $D_+ = |y'|^2 + q_+|y|^2$, $D_- = q_-|y|^2$ and
$D = D_+ - D_- = |y'|^2 + q|y|^2$, we obtain

$$\left|\int_a^b gD\right| \leq \int_a^b |g_\mp D_+ + g_\mp D_-|$$
$$\leq \|g\|_\infty \int_a^b (D + 2D_-)$$
$$= 2\|g\|_\infty \int_a^b q_-|y|^2$$
$$\leq 2\|g\|_\infty \|y\|_\infty^2 \|q_-\|_1$$
$$\frac{\epsilon}{2}|Re\,\lambda| \leq \|y\|_\infty \|g'\|_2 \|y'\|_2 + 2\|g\|_\infty \|y\|_\infty^2 \|q_-\|_1$$

$$= \sqrt{b-a}\|g'\|_2 \left(\|q-\|_1 + \sqrt{\|q-\|_1^2 + Q(b)|y(b)|^2}\right)^2 + 2\|g\|_\infty(\sqrt{b-a})^2 \left(\|q-\|_1 + \sqrt{\|q-\|_1^2 + Q(b)|y(b)|^2}\right)^2 \|q-\|_1$$

$$= \sqrt{b-a}\|g'\|_2 \left(\|q-\|_1 + \sqrt{\|q-\|_1^2 + Q(b)|y(b)|^2}\right)^2 + 2(b-a)\|g\|_\infty \left(\|q-\|_1 + \sqrt{\|q-\|_1^2 + Q(b)|y(b)|^2}\right)^2 \|q-\|_1$$

$$|Re\,\lambda| \leq \frac{2}{\epsilon}\left(\|q-\|_1 + \sqrt{\|q-\|_1^2 + Q(b)|y(b)|^2}\right)^2 \left(\sqrt{b-a}\|g'\|_2 + 2(b-a)\|g\|_\infty \|q-\|_1\right)$$

Definition Let σ be a real-valued function defined on the closed bounded interval $[a,b]$ and $p = \{x_0, x_1, \dots, x_k\}$ be a partition of $[a,b]$. We define the variation of σ with respect to P by $V(\sigma, P) = \sum_{i=1}^{k} |\sigma(x_i) - \sigma(x_{i-1})|$
And the total variation of σ on $[a,b]$ by
$TV(\sigma) = sup\{V(\sigma,P)/\ P\ \ a\ partion\ \ of\ [a,b]\}$.

Definition A real valued function σ on the closed, bounded interval $[a,b]$ is said to be of bounded variation on $[a,b]$ if $TV(\sigma) < \infty$.

Lemma 2.1[4] Let $f \geq 0$ and g be a functions of bounded variation on the closed interval J, then

$$\int_J f dg \leq \left(\inf_J f + \operatorname*{Var}_J f\right)\left(\sup_{K\subset J} \int_K dg\right)$$

Where $\mathrm{Var}_J f = \int_J |df(x)|$ and sup is taken over all compact subset of J.

Lemma 2.2 Let σ be of bounded variation over all of $[a,b]$, that is σ satisfies the inequality $\int_x^b |d\sigma(x)| < \infty$. Then for all $x \in (a,b]$ and for every $\delta > 0$ there exist a $\rho = \rho(\delta, x) > 0$ such that

$$\int_x^b |f(t)|^2 |d\sigma(t)| \leq \rho(\delta,x) \int_x^b |f(t)|^2 dt + \delta \int_x^b |f'(t)|^2 dt \qquad (12)$$

Where $\rho(\delta,x) = \frac{1}{b-x} + \frac{c}{\delta}$, $c = \int_a^b |d\sigma(x)|$

Proof. We assume Lemma 1 with f and g replaced by $|f|^2$ and the variation of σ satisfy the assumptions of Lemma 1 we have that

$$\int_x^b |f(t)|^2 |d\sigma(t)| \leq \left(\inf_{[x,b]} |f(t)|^2 + \mathrm{Var}_{[x,b]} |f(t)|^2\right)\left(\int_x^b |d\sigma(t)|\right) \qquad (13)$$

For $x \in (a,b]$

$\inf_{[x,b]}|f(t)|^2 \leq \frac{1}{b-x}\int_x^b |f(t)|^2 dt$ (14)

$$\underset{[x,b]}{\mathrm{Var}}|f(t)|^2 = \int_x^b |d|f(t)|^2| = \int_x^b 2\,|f(t)||f(t)|' dt$$
$$= \int_x^b \left|2\,Re(f(t))\overline{(f'(t))}\right| dt$$

And by Cauchy Schwarz inequality

$$\int_x^b \left|2\,Re(f(t))\overline{(f'(t))}\right| dt \leq 2\left(\int_x^b |f(t)|^2\right)^{\frac{1}{2}}\left(\int_x^b |(f'(t)|^2\right)^{\frac{1}{2}}$$

Hence, $\mathrm{Var}_{[x,b]}|f(t)|^2 \leq 2\left(\int_x^b |f(t)|^2\right)^{\frac{1}{2}}\left(\int_x^b |(f'(t)|^2\right)^{\frac{1}{2}}$ (15)

Let $\alpha(x) = \left(\int_x^b |f(t)|^2\right)^{\frac{1}{2}}$ and $\beta(x) = \left(\int_x^b |(f'(t)|^2\right)^{\frac{1}{2}}$

Then inserting (14), (15) in to (13) yields

$$\int_x^b |f(t)|^2 |d\sigma(t)| \leq \left(\frac{1}{b-x}\alpha^2(x) + 2\alpha(x)\beta(x)\right)\int_x^b |d\sigma(t)|$$

For some $\delta > 0$, we see that

$$\left(\frac{1}{\sqrt{\delta}}\alpha(x) - \sqrt{\delta}\beta(x)\right)^{\mathbf{2}} \geq 0$$

And so $2\alpha(x)\beta(x) \leq \frac{1}{\delta}\alpha^2(x) + \delta\beta^2(x)$

Thus

$$\int_x^b |f(t)|^2 |d\sigma(t)| \leq \left(\frac{1}{b-x}\alpha^2(x) + \frac{1}{\delta}\alpha^2(x) + \delta\beta^2(x)\right)\int_x^b |d\sigma(t)|$$
$$\leq \left(\left(\frac{1}{b-x} + \frac{1}{\delta}\right)\int_x^b |f(t)|^2 + \delta\int_x^b |(f'(t)|^2\right)\int_x^b |d\sigma(t)|$$

Replacing δ with $\frac{\delta}{c}$ where $c = \int_a^b |d\sigma(t)|$, we have

$\int_x^b |f(t)|^2 |d\sigma(t)| \leq \left(\left(\frac{1}{b-x} + \frac{c}{\delta}\right)\int_x^b |f(t)|^2 dt + \delta\int_x^b |(f'(t)|^2 dt\right)$ (16)

Hence we obtain equation (12).

Lemma 2.3 Let $q_- \in L^1(a,b)$ and $y \in D$. Then for all $x \in (a,b]$

$\int_x^b |y(t)|^2 q_-(t) dt \leq \left(\frac{c}{b-x} + \frac{c^2}{\delta}\right)\int_x^b |y(t)|^2 dt + \delta\int_x^b |(y'(t)|^2 dt$ (17)

Where $c = \|q_-\|, D = \{y \in L^2(a,b) : -y'' + q(x)y \in L^2(a,b), y(a) = y'(b) = 0\}$.

Proof. This follows from Lemma 2.2 with $f(t)$ and $\sigma(t)$ replaced by $y(t)$ and $\int_t^b q_- dx$, respectively, so that $\int_x^b |d\sigma(t)| = \int_x^b \left|d(\int_t^b q_-(x)dx)\right| = \int_x^b q_-(t)dt$.

Using this result in (12), we have (17).

Theorem 2.2 Suppose that there exists a function $g \in H'(a,b)$ such that $g\rho > 0$ on (a,b) and let $\epsilon > 0$ be such that $\Delta = \{x \in (a,b) : g(x)\rho(x) < \epsilon\}$, where $\epsilon > 0$ is choosen such that $\Delta^c \neq \emptyset$ and

$$|\Delta| \leq \frac{1}{8(b-a)\|q_-\|_1^2}$$

Then for any non- real eigenvalue $\lambda \in \mathbb{C}\backslash\mathbb{R}$ of problem (1)-(2), we have

$$|\mathrm{Im}\,\lambda| \leq \frac{2}{\epsilon}\|g'\|_2\sqrt{2 + 4(b-a)\|q-\|_1}\ \left(\|q-\|_1 + \sqrt{\|q-\|_1^2 + Q(b)|y(b)|^2}\right)$$

And

$$|Re\lambda| \leq \frac{2}{\epsilon}\left(\|q-\|_1 + \sqrt{\|q-\|_1^2 + Q(b)|y(b)|^2}\right)\left[\|g'\|_2\sqrt{2 + 4(b-a)\|q-\|_1} + 2(b - a)\left(\|q-\|_1 + \sqrt{\|q-\|_1^2 + Q(b)|y(b)|^2}\right)\|q-\|_1\|g\|_\infty\right]$$

Proof. From Equation (6)

$$\int_a^b |y'|^2 dt = -\int_a^b |y|^2\,q(t)dt \leq \int_a^b |y|^2\,q_-(t)dt$$

Which yields

$\|y'\|_2^2 \leq \int_a^b |y|^2\,q_-(t)dt$ (18)

We set $x = a$ in equation (17) and insert the result in to the right hand side of the inequality in (18) to get

$\int_a^b |y|^2\,q_-(t)dt \leq \left(\frac{1}{b-a} + \frac{c}{\delta}\right)\int_a^b |y|^2 + \delta\int_a^b |y'|^2 \qquad c = \|q_-\|$ (19)

Hence

$$\|y'\|_2^2 \le \left(\frac{1}{b-a}+\frac{c}{\delta}\right)\|y\|_2^2+\delta\|y'\|_2^2$$

Such as in the proof of Theorem 2.1 we assume without loss of generality $\|y\|_2 = 1$, then

$$\|y'\|_2^2(1-\delta) \le \frac{1}{b-a}+\frac{c}{\delta}$$

$$\|y'\|_2 \le \sqrt{\frac{1}{(1-\delta)(b-a)}+\frac{c}{\delta(1-\delta)}}$$

Setting $\delta = \frac{1}{2}$, we have that

$$\|y'\|_2 \le \sqrt{\frac{2}{b-a}+4\|q_-\|} \tag{20}$$

By inserting (20) in Equation (10) and (11) we get boundaries as shown in (21) and (22) below on the imaginary and real parts of non-real eigenvalues.

$$|\mathrm{Im}\,\lambda| \le \frac{2}{\epsilon}\|g'\|_2\sqrt{2+4(b-a)\|q_-\|_1}\ \left(\|q_-\|_1+\sqrt{\|q_-\|_1^2+Q(b)|y(b)|^2}\right) \tag{21}$$

And

$$|Re\lambda| \le \frac{2}{\epsilon}\left(\|q_-\|_1+\sqrt{\|q_-\|_1^2+Q(b)|y(b)|^2}\right)\Big[\|g'\|_2\sqrt{2+4(b-a)\|q_-\|_1}+2(b-a)\left(\|q_-\|_1+\sqrt{\|q_-\|_1^2+Q(b)|y(b)|^2}\right)\|q_-\|_1\|g\|_\infty\Big] \tag{22}$$

Remark

We note that the boundaries in Theorem 2.2 are an improvement on the boundaries in Theorem 2.1 as long as $\|q_-\|_1 \ge \frac{1+\sqrt{1+\frac{2}{b-a}}}{2} \ge 1$.

3. Comparing the bounds

In this section, by approximating the imaginary eigenvalues of a special indefinite Sturm-Liouville problem and then comparing their volume with our result, we satisfy the above remark.

Example:

$$-y''+q(x)y = \lambda\rho(x)y$$

$$y(-1) = y'(1) = 0 \qquad \text{where } x \in [-1,1] .$$

$$y(x) = c_1e^{i\sqrt{\lambda\rho-q}x}+c_2e^{-i\sqrt{\lambda\rho-q}x}$$

$$y(1) = c_1e^{i\sqrt{\lambda\rho-q}}+c_2e^{-i\sqrt{\lambda\rho-q}}$$

And if $c_1 = c_2$ then $y(1) = 2\cos\sqrt{\lambda\rho-q}$

And since $|\cos x| \le 1$ then $|y(1)| \le 2$

$$\text{Let } g(x) = \begin{cases} -1 & if\ x \in (-1,\xi) \\ \frac{1}{\xi^2}x^2+\frac{2}{\xi}x & if\ x \in (-\xi,0) \\ -\frac{1}{\xi^2}x^2+\frac{2}{\xi}x & if\ x \in (0,\xi) \\ 1 & if\ x \in (\xi,1) \end{cases}$$

$$\text{And } \rho(x) = \begin{cases} -1 & if\ x \in (-1,0) \\ 1 & if\ x \in (0,1) \end{cases}$$

For $x \in (-1,1)$

$$g'(x) = \begin{cases} \frac{2}{\xi^2}x+\frac{2}{\xi} & if\ x \in (-\xi,0) \\ -\frac{2}{\xi^2}x+\frac{2}{\xi} & if\ x \in (0,\xi) \\ 0 & otherwise \end{cases}$$

Clearly, $\|g\|_\infty = 1$, $\|q_-\|_1 = \sqrt{2|q_0|}$, $\|g'\|_2 = \left(\int_{-1}^{1}|g'|^2dx\right)^{\frac{1}{2}} = \sqrt{\frac{8}{3\xi}}$

We are comparing the bounds of imaginary and real part of eigenvalue λ in Theorem 2.1 and Theorem 2.2.

By theorem 2.1

$$|\mathrm{Im}\,\lambda| \le \frac{2}{\epsilon}\sqrt{b-a}\ \|g'\|_2\left(\|q_-\|_1+\sqrt{\|q_-\|_1^2+Q(b)|y(b)|^2}\right)^2$$

And

$$|Re\lambda| \le \frac{2}{\epsilon}\left(\|q_-\|_1+\sqrt{\|q_-\|_1^2+Q(b)|y(b)|^2}\right)^2\left(\sqrt{b-a}\ \|g'\|_2+2(b-a)\|g\|_\infty\|q_-\|_1\right)$$

And by theorem 2.2

$$|\mathrm{Im}\,\lambda| \le \frac{8\sqrt{2}}{\epsilon\sqrt{3\xi}}\ \left(\|q_-\|_1+\sqrt{\|q_-\|_1^2+Q(b)|y(b)|^2}\right)$$

$$|\mathrm{Re}\,\lambda| \leq \frac{2}{\epsilon}\left(\frac{4}{\sqrt{3\xi}} + 4\|q-\|_1\right)\ \left(\|q-\|_1 + \sqrt{\|q-\|_1^2 + Q(b)|y(b)|^2}\right)$$

Let $\epsilon = 1$, then according to the given data, we have

$$|\Delta| = \frac{1}{32|q_0|}$$

Where $|\Delta|$ is the length between of the sub-interval $(-1,1)$ on $g(x)\rho(x) < 1$. In this case let $|\Delta| = 2\xi$ so that $\xi \leq \frac{1}{64|q_0|}$. For the particular case when
$q_0 = -6\pi^2$,we have that $\xi \leq \frac{1}{384\pi^2}$, thus we can set $\xi = \frac{1}{384\pi^2}$. So the bounds of real and imaginary part of eigenvalue by theorem 1 and theorem 2 respectively we get

$|\mathrm{Im}\,\lambda| \leq 180575.1406$,
$|\mathrm{Re}\,\lambda| \leq 235864.7599$,
$|\mathrm{Im}\,\lambda| \leq 47793.16334$,
$|\mathrm{Re}\,\lambda| \leq 103082.7827$.

Conclusion

In conclusion, we realized that the real and imaginary parts of the complex eigenvalue of an indefinite Sturm-Liouville problem with Cauchy boundary conditions on the finite interval [a,b].

CONFLICT OF INTERESTS

There are no conflicts of interest.

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حدود على أجزاء حقيقية ومتخيلة من القيم الذاتية غير الحقيقية لمعادلة ستيورم – ليوفل التفاضلية غير المحددة مع شروط حدود كوشي

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الخلاصة

نحن مهتمون في الورقة ذات الخصائص الطيفية لمشكلات النوع إلى أجل غير مسمى ستيورم – ليوفل في ظروف حدود كوشي حيث تكون معاملات المعادلات التفاضلية وظائف قابلة للتكامل ذات قيمة حقيقية ، والمعلمة الطيفية هي أيضًا رقم معقد. **وأيضا نحصل على حدود حقيقية وخيالية لمشكلة** ستيورم – ليوفل **لأجل غير مسمى في ظروف الكوشي.**

الكلمات الدالة:– ستيورم، ليوفل مشكلة، حدود القيم الذاتية، شروط حدود كوشي، مشكلة طيفية .