Approximation of Functions in L_P Spaces for $p < 1$,Using Radial Basis Function Neural Networks

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Abstract

Many papers introduced about the approximation of continuous functions using neural networks. In this paper we use neural network to approximate functions in L_p spaces for $p < 1$. We study the approximation for functions defined on the complete real line using neural networks with radial basis having constant weight. We also use convolution to approximate functions in L_p spaces for $p < 1$ on a compact interval using radial basis function neural networks of constant weights.

Keywords: L_p spaces, constructive approximation, neural networks, convolution.

1. Introduction

Many papers (see [1],[2],[3]and[4]) showed that we can consider the multilayered neural networks with radial basis function, as a universal approximation of a continuous function. Then in [5] the authors proved theorems for the approximation of functions in C^2 define on $[0,1]^k$,$k \in N$, using the multilayered neural networks with radial basis functions. In [6] Li introduced a result for the simultaneous approximation using neural networks with radial basis functions, for a function and all it's derivatives. In[7] we can find a numerical approach for the results in [6].

The papers above used various weights in neural networks with radial basis, which make approximation theorems difficult in applications for engineering and any other science field.

Now there is a natural question: Can we use a fixed weight in radial basis function neural network, in order to have an easy usage for general usage in life?

Here let us answer the above question, and approximate any function in $L_p$spaces for $p < 1$, using neural networks with radial basis with constant weight.We also construct an approximation for any function belongs to L_p spaces for $p < 1$ defined on a closed and bounded interval using neural networks with radial basis with constant weight. We introduce a direct estimation in L_p spaces for $P < 1$, and we define the convolution of two functions and prove a version of direct theorem by using it .We introduce a direct theorem for the approximation using radial basis function defined on bounded closed interval subset of the real line R.

There are various contributions to knowledge for the neural networks approximation, such as geophysics, metrology and graph rendering. In those applications, the data are all over sphere, and we need to represent them by a functional model.

2. Main Results

2-1 A Jackson Type Theorem

We can approximate any function in L_p , $p < 1$ using radial basis function neural networks. If we want to make that possible we need some limits on the functions in L_p, $p < 1$. As we see in our first result.

Theorem 1.

If $f \in L_p, p < 1$ with $\lim_{|x|\to\infty} |f(x)| \to 0$ and σ be a radial basis function. Then there are real constants c_i, θ_i and positive integers W *and* M with

$$\|f - N\|_p \leq c(p)\omega_k(f,\delta)_p,$$

$$\text{where } N(x) = \sum_{i=1}^{M} c_i\sigma(Wx + \theta_i).$$

Proof.

By our hypotheses $\lim_{|x|\to\infty} |f(x)| = 0, h > 0, \exists k \in Z^+$ such that

$$\|\Delta_h^m f(x)\|_{Lp(J)} \leq c(p)\omega_m(f,\delta)_{p(J)} \quad where |h| < \delta$$

$when\, J = (k,\infty) \cup (-\infty,-k)$

$$\|\Delta_h^m f\|_{Lp(J)} \leq c(p)\omega_m(f,\delta)_{Lp(J)}$$

Let $M = \left[\frac{1}{\omega_m(f,\delta)_p}\right] > k \;\; and\; M > \left[\frac{1}{\delta}\right],$

Where $[y] is (the\ integer\ part\ of\ y) + 1$.

Let $I =$ $[-M, M]$. If we devided I in to $2M^2$ equidistance points , the distance between any two points equal to $1/M$

$$I_1 = [x_o, x_1], I_2 = [x_1, x_2], \cdots I_{2M^2} = [x_{2M^2-1}, x_{2M^2}]$$

Let $\theta_i = \frac{x_i + x_{i+1}}{2}$, $i = 0,1,2,\cdots,2M^2 - 1$

Since σ is radial basis function, we can find $L > 0$, such that $\|\sigma\|_{Lp(J)} \leq c(p)\omega_m(f,\delta)_{Lp(J)}$

For $I = (L,\infty) \cup (-\infty, L)$.Define $W \in Z^+, with\ W/\omega_m(f,\delta)_{Lp(I)} > L$.

So we can define the following neural network

$$N(x) = \sum_{i=0}^{2M^2-1} \Delta_h^m f(x_i) - \sigma\big(W(x-\theta_i)\big).$$

Then on $C = (-\infty, -M) \cup (M, \infty)$

$$\|f-N\|_{Lp(C)} = \|f\|_{Lp(C)} + \|N\|_{Lp(C)}$$

$$\leq c(p)\omega_m(f,\delta)_{Lp(C)} + c(p) \sum_{i=0}^{2M^2-1} \|\Delta_h^m f(x_i)\|_{Lp(C)} + \|\sigma\|_{Lp(C)}$$

$$\leq c(p)\omega_m(f,\delta)_{Lp(C)}$$

From now on $c(p)$ means a positive constant depends on p only and may vary from one step to another.

Then on $C = [-M, M]$, there is $j = 0,1,2,\cdots,2M^2-1$, with $C = \cup_{i=1}^{2M^2-1} I_i$

$$\|f-N\|_{Lp(C)} = \|f-N\|_{Lp(\cup_{i=1}^{2M^2-1} I_i)}$$

$$\leq c(p)\left\|f - \sum_{i=0}^{j-1} \Delta_h^m f(x_i)\sigma\big(W(x-\theta_i)\big)\right\|_{Lp\left(\cup_{i=1}^{2M^2-1} I_i\right)}$$

$$+ \left\|\sum_{i=j+1}^{2M^2-1} \Delta_h^m f(x_i)\sigma\big(W(x-\theta_i)\big)\right\|_{Lp(\cup_{i=1}^{2M^2-1} I_i)}$$

$$\leq C(p)\ \omega_m(f,\delta)_{Lp(C)}.$$

This completes the proof.

2-2 Convolution and its Approximation

Let us recall the definition of the convolution of two functions f_1, f_2.

$$(f_1 * f_2)(z) = \int_R f_1(y) f_2(z-y)dy.$$

Where z is a real. Also consider the map

$$F_k(x) = \begin{cases} d\Delta_h^k(f(x))e^{-\frac{1}{1-x^2}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

$d\ is$ a constant that make $\int_R F_k$ equal to one. It is clear $F_k(x) \in L_P^0(R)$,then we use the definition of convolution to introduce a version of direct theorem.

Now let us define the space $L_P^0(R) = L_P(R) \cap \left\{f: R \to R\ sit\ \lim_{|x|\to 0} |f(x)| = 0\right\}$, we have the following result.

Theorem2.

$If\ f \in L_P^0(R)$, then $\|((F_K * f) - f\|_p \leq c(p)\omega_k(f,\delta)_p$

Proof.

$$\|((F_K * f) - f\|_p = \left\|\int_R F_K(y)f(x-y)dy - \int_R F_K(y)f(x)dy\right\|_p$$

$$= \left\|\int_{|y|<1} F_K(y)f(x-y)dy + \int_{|y|>1} F_K(y)f(x-y)dy - \int_R F_K(y)f(x)dy\right\|_p$$

$$\leq (\int_R (\int_{|y|<1} \left|\Delta_h^k f(y) e^{-\frac{1}{1-y^2}} f(x-y)\right| dy + \int_{|y|>1} |F_K(y)f(x-y)|dy + \int_R |F_K(y)||f(x)|dy)^p\, dx$$

$$\leq c(p)\left(\int_{|y|<1}(\int_{|y|<1}|\Delta_h^k f(y)|dy + 0 + (\int_{|y|<1}(\int_{|y|<1}|\Delta_h^k f(y)|)dy)\right)^p dx$$

$$\leq c(p)\left(\int_R |\Delta_h^k f(y)|^p\, dx\right)^{1/p}$$

$$\leq c(p)\omega_k(f,\delta)_p$$

The proof is complete .

2-3 Approximation By Using Radial Basis Function In L_p Spaces For $p < 1$ With A Fixed Weight

In this section we introduce a result by using radial basis function neural networks with fixed weight. as we see in the following theorem .

Theorem3.

Let $f \in L_P^0([a_1, a_2])$, then If σ is a measurable radial basis function on R then we can find real constants c_i and θ_i and psitive integer $W\ and\ M$ with

$$\|f - N\|_p \leq c(p)\omega_k(f,\delta)_p$$

Proof.

Consider the real function:

$$\tilde{f}(x) = \begin{cases} f(a_1)x + (a_1 - 1)f(a_1) & if x \in [a_1 - 1, a_1] \\ f(x), & if\ x \in [a_1, a_2] \\ -f(a_2)x + (a_2 + 1)f(a_2)\,, & if\ x \in [a_2, a_2 + 1] \\ 0 & if\ x \in (-\infty, a_1 - 1] \cup [a_2 + 1, \infty) \end{cases}$$

Using theorem 2 to get $\left\|F_k * \tilde{f} - \tilde{f}\right\|_p \leq c(p)\omega_k(\tilde{f},\delta)_p$,so

$$\left\|F_k * \tilde{f} - \tilde{f}\right\|_p \leq c(p)\omega_k(\tilde{f},\delta)_p \text{ on } [a_1, a_2] \qquad (2-1)$$

Since $\int_R F_k(x-y)\tilde{f}(y)\ dy$ is finite for any k in $\mathbb{Z}^+$. And there exists a Riemann sum for the approximation of any convolution.

If k is a positive integer we can find M_k in $\mathbb{Z}^+$. And z_i , C_i for $i = 1,2,\cdots,M_k$ satisfy

$$\left\| (F_k * f)_{(x)} - \sum_{i=1}^{M_k} C_i F_k(x - z_i)\tilde{f}(z_i) \right\|_p \leq c(p)\omega_k(f, \delta)_p \qquad (2-2)$$

Using theorem 1 there exist real constants $\alpha_{j,k}, \beta_{j,k}$ and $K \in \mathbb{Z}^+$ satisfy

$$\left\| F_k(x - z_i) - \sum_{j,k} \beta_{j,k}\sigma(k(x - z_i)) + \alpha_{j,k} \right\|_p \leq c(p)\omega_k(F_k, \delta)_p \qquad (2-3)$$

Using eq. 2-1 to choose k $\in \mathbb{Z}^+$ that satisfy

$$\|f(x) - (F_k * f)(x)\|_p \leq c(p)\omega_k(f, \delta)_p \qquad (2-4)$$

From eq. (2-2), (2-3) and (2-4) we obtain

$$\left\| f(x) - \sum_{i=1}^{M_k} C_i f^{\sim}(z_i) \sum_{j,k} \beta_{j,k}\sigma(k(x - z_i) + \alpha_{j,k}) \right\|_p$$

$$\leq \|f(x) - (F_k * f)(x)\|_p + \left\| (F_k * f)(x) - \sum_{i=1}^{M_k} C_i F_k(x - z_i) f^{\sim}(z_i) \right\|_p$$

$$+ \left\| \sum_{i=1}^{M_k} C_i F_k(x - z_i) f^{\sim}(z_i) - \sum_{i=1}^{M_k} C_i f^{\sim}(z_i) \sum_{j,k} \beta_{j,k}\sigma(k(x - z_i) + \alpha_{j,k}) \right\|_p$$

$$\leq c(p)\omega_k(f, \delta)_p.$$

This completes the proof.

3. Conclusion

There are main two problems in the field of studying approximation using neural networks, these problems are density and complexity. In our work here we consider the density problem. We proved a direct theorem for any function in L_p spaces for $p < 1$, satisfying $\lim_{|x|\to\infty} |f(x)| \to 0$, using neural networks with radial basis function. What distinguishes our work is the ability of neural networks approximation in approximation of functions. These thoughts lead to thoughts in the studying complexity approximation problem using fixed weight neural networks, as a future work.

Conflict of Interests.
There are non-conflicts of interest .

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الخلاصة

تناولت العديد من البحوث تقريب الدوال المستمرة باستخدام الشبكات العصبية. في هذا البحث استخدمنا الشبكات العصبية لتقريب الدوال في الفضاء L_p عندما $1 < p$. اذ قمنا بدراسة تقريب الدوال المعرفة على الخط الحقيقي الكامل باستخدام الشبكات العصبية التي لها وزن ثابت. كما استخدمنا الالتفاف لتقريب الدوال في الفضاء L_p عندما $1 < p$ معرفة على فترة مرصوصة باستخدام الشبكات العصبية التي لها اوزان ثابتة.

الكلمات الدالة: الفضاء L_p ، التقريب المقيد، الشبكات العصبية، الالتفاف.