Solving Some Types of the Second and Third Order Spectral Linear Ordinary Differential Equations

Aryan Ali Mohammed
Department of Mathematical Sciences, College of Basic Education, University of Sulaimani, Sulaimaniyah, Iraq
aryan.mohammed@univsul.edu.iq, aryanmath76@yahoo.com.

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Abstract

Our article is for finding complete primitive for two different kinds of linear ordinary differential equations in a spectral type. First kind is consisting of two different types of coefficients of order two; one of them is polynomials, where as the other type is continuous functions and both are of real types. The second kind is for a third order, and here the coefficients of this kind are also several real polynomials, and in both kinds of spectral linear ordinary differential equations, the coefficients are converted to the constants via varying the independent variable to a new one. We gave examples to explain our mechanism.

Keywords: Order of differential equations, real functions and polynomials, complete primitive, variation of variables.

1. Introduction

The second and third order linear ordinary differential equations (L.O.D.E) are widely used of many problems in the distinct fields as: Engineering, mechanic, and ... etc. The known methods like operator method, undetermined coefficients, reduction of order and variation of parameters are studied to find the complete primitive of second and third linear differential equations with constant and variable coefficients. The third and second-order differential equations have been accurately inspected by many authors. In [12] Moore, studied the behavior of their solutions, and their waverly feature studied via Coppel [18], Marini [7], Hochtadt [3] and Hartman [9, 10]; and in [6, 13, 16, 17] the study of the of limited and asymptotic behavior were considered, finally in [15], Richard calculated the solutions numerically.

A lot of writers examined the asymptotic behavior for eigenvalues and identical eigenfunctions to various kinds of cases for the spectral type [5, 14].

In [4], Karwan and Aryan studied the boundedness of non-zero solutions to the spectral L.O.D.E. of second order. In [8], Marini and Zezza they studied asymptotically formula of eigenfunctions L.D.E. of order two, further they specified needful and appropriate terms to the integrals of their defined D.E.

In [11], Johnson, Busawon, and Barbot presented a substitutional procedure for resolving the public non-homogeneous second order L.O.D.E.

In [2], Arficho derived a modern procedure for finding particular integral of L.O.D.E. of order two by means of one given integral to the related homogeneous D.E., further, he constructed another integral of the related homogeneous D.E. of this modern procedure. Furthermore, he found complete integral of the defined L.O.D.E of order two without using the known procedure for finding the particular integral.

In [1], Aryan constructed a complete integral to the L.O.D.E. of order n by the procedure for variation of parameter, as well as, he determined the non-zero solutions of second order spectral L.O.D.E. with eigenparameter dependent boundary conditions. Moreover, he specified the limitedness of non-zero solutions. Finally, they studied the asymptotically formula for eigenvalues to the defined D.E.

In this paper, we study the solution of the second and third order spectral L.O.D.E. with variable coefficients in which the coefficients of the first one consist of two distinct kinds, one of them are polynomials, where as the other kind are continuous functions and both of them are of real types. However, the coefficients of third order are polynomials of real type.

Firstly, we examine two types of spectral L.O.D.E. of order two of the forms:

$$a_o(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)y(x)=\lambda^2a_0(x)y(x), x\in(a,b) \qquad (1)$$

and

$$a_o(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}=k\lambda^2y(x), x\in(a,b) \qquad (2)$$

where $\lambda\neq 0$ is a spectral parameter, $a>0$, and $a<b$, and $a_i(x), i=0,1,2,$ are some real polynomials and $a_o(x)\neq 0$, and k is non-zero arbitrary constant, provided that $y(x)$ and their derivatives are defined and continuous on the given interval. We reduce equations (1) to an equation with constant coefficients by the substitution $y(x)=u(x)g(x)$, where

$g(x)=e^{\left(b\frac{1}{2}x-\frac{1}{2}\int\frac{a_1(x)}{a_0(x)}dx\right)}$,

and $u(x)$ is a new unknown function should be specified, and b is a constant.
Finally, we convert equation (2) to the spectral linear differential equation by making change of variable $t=\int e^{-\int\frac{a_1(x)}{a_0(x)}dx}dx$, to a form

$$\frac{d^2y}{dt^2}+k\lambda^2y(t)=0.$$

Secondly, we investigate spectral L.O.D.E. of order three of the type

$$a_o(x)\frac{d^3y}{dx^3}+a_1(x)\frac{d^2y}{dx^2}+a_2(x)\frac{dy}{dx}=-\lambda^3a_3(x)y(x), x\in(a,b) \qquad (3)$$

where $\lambda\neq 0$ is a spectral parameter, $a>0$ and $a<b$, where $a_i(x), i=0,1,2,3$ are some real polynomials and $a_o(x)\neq 0, \forall x\in(a,b)$. By making change of variable $s=k\int\left(\frac{a_3(x)}{a_0(x)}\right)^{\frac{1}{3}}dx$, equation (3) may reduce to the constant coefficients.

Note: Throughout the paper, L.O.D.E. stands for linear ordinary differential equations.

2. Studying the different kinds of spectral L.O.D.E. of order two

In this section, we consider the spectral L.O.D.E. of order two with distinct types of coefficients through two kinds of transformations, and examples are given to explain our technique.

Theorem 1. The spectral differential equation of the form

$$a_o(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)y(x)=\lambda^2a_0(x)y(x), x\in(a,b) \qquad (1)$$

where $\lambda, a_0(x), a_2(x) \neq 0$ and $a_i(x), i = 0,1,2$ are real polynomial functions, reduced to an equation with constant coefficients by the substitution $y(x) = u(x)\ g(x)$, where $g(x) = e^{\left(b\frac{1}{2}x - \frac{1}{2}\int \frac{a_1(x)}{a_0(x)} dx\right)}$, if

$$\left(\frac{b^2}{4} - \frac{1}{4}\left(\frac{a_1^2(x)}{a_0^2(x)}\right)\right) - \frac{1}{2}\left(\frac{a_1'(x)a_0(x) - a_1(x)a_0'(x)}{a_0^2(x)}\right) + \frac{a_2(x)}{a_0(x)} = A = \text{constant},$$

where b is an arbitrary constant, and $u(x)$ is a new unknown function to be determined.

Proof: At the beginning we find the first and second derivative by above substitutions.

$$\frac{dy}{dx} = u(x)g'(x) + u'(x)g(x)$$

$$\frac{dy}{dx} = u(x)\left(b\frac{1}{2} - \frac{1}{2}\left(\frac{a_1(x)}{a_0(x)}\right)\right) e^{\left(b\frac{1}{2}x - \frac{1}{2}\int \frac{a_1(x)}{a_0(x)} dx\right)} + u'(x)\, e^{\left(b\frac{1}{2}x - \frac{1}{2}\int \frac{a_1(x)}{a_0(x)} dx\right)},$$

$$\begin{aligned}\frac{d^2y}{dx^2} = {}& u(x)\left(b\frac{1}{2} - \frac{1}{2}\left(\frac{a_1(x)}{a_0(x)}\right)\right)^2 e^{\left(b\frac{1}{2}x - \frac{1}{2}\int \frac{a_1(x)}{a_0(x)} dx\right)} \\ &+u(x)\left(-\frac{1}{2}\frac{a_1'(x)a_0(x) - a_1(x)a_0'(x)}{a_0^2(x)}\right) e^{\left(b\frac{1}{2}x - \frac{1}{2}\int \frac{a_1(x)}{a_0(x)} dx\right)} \\ &+u'(x)\left(b\frac{1}{2} - \frac{1}{2}\left(\frac{a_1(x)}{a_0(x)}\right)\right) e^{\left(b\frac{1}{2}x - \frac{1}{2}\int \frac{a_1(x)}{a_0(x)} dx\right)} + u''(x) e^{\left(b\frac{1}{2}x - \frac{1}{2}\int \frac{a_1(x)}{a_0(x)} dx\right)} \\ &+ u'(x)\left(b\frac{1}{2} - \frac{1}{2}\left(\frac{a_1(x)}{a_0(x)}\right)\right) e^{\left(b\frac{1}{2}x - \frac{1}{2}\int \frac{a_1(x)}{a_0(x)} dx\right)},\end{aligned}$$

$$\begin{aligned}\frac{d^2y}{dx^2} = {}& u''(x)\, e^{\left(b\frac{1}{2}x - \frac{1}{2}\int \frac{a_1(x)}{a_0(x)} dx\right)} + 2u'(x)\left(b\frac{1}{2} - \frac{1}{2}\left(\frac{a_1(x)}{a_0(x)}\right)\right) e^{\left(b\frac{1}{2}x - \frac{1}{2}\int \frac{a_1(x)}{a_0(x)} dx\right)} + \\ & u(x) e^{\left(b\frac{1}{2}x - \frac{1}{2}\int \frac{a_1(x)}{a_0(x)} dx\right)}\left(\left(b\frac{1}{2} - \frac{1}{2}\left(\frac{a_1(x)}{a_0(x)}\right)\right)^2 \right. \\ & \left. - \frac{1}{2}\left(\frac{a_1'(x)a_0(x) - a_1(x)a_0'(x)}{a_0^2(x)}\right)\right).\end{aligned}$$

Setting $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given differential equation, yields

$$\begin{aligned} & a_0(x)\ (\ u''(x)\, e^{\left(b\frac{1}{2}x - \frac{1}{2}\int \frac{a_1(x)}{a_0(x)} dx\right)} + 2u'(x)\left(b\frac{1}{2} - \frac{1}{2}\left(\frac{a_1(x)}{a_0(x)}\right)\right) e^{\left(b\frac{x}{2} - \frac{1}{2}\int \frac{a_1(x)}{a_0(x)} dx\right)} + \\ & u(x) e^{\left(b\frac{1}{2}x - \frac{1}{2}\int \frac{a_1(x)}{a_0(x)} dx\right)}\left(\left(b\frac{1}{2} - \frac{1}{2}\left(\frac{a_1(x)}{a_0(x)}\right)\right)^2 - \frac{1}{2}\left(\frac{a_1'(x)a_0(x) - a_1(x)a_0'(x)}{a_0^2(x)}\right)\right)) \\ & + a_1(x)\left(u(x)\left(b\frac{1}{2} - \frac{1}{2}\left(\frac{a_1(x)}{a_0(x)}\right)\right) e^{\left(b\frac{1}{2}x - \frac{1}{2}\int \frac{a_1(x)}{a_0(x)} dx\right)} + u'(x)\, e^{\left(b\frac{1}{2}x - \frac{1}{2}\int \frac{a_1(x)}{a_0(x)} dx\right)}\right) \\ & +a_2(x)\, u(x)\, e^{\left(b\frac{1}{2}x - \frac{1}{2}\int \frac{a_1(x)}{a_0(x)} dx\right)} = \lambda^2\, a_0(x)u(x)\, e^{\left(b\frac{1}{2}x - \frac{1}{2}\int \frac{a_1(x)}{a_0(x)} dx\right)}.\end{aligned}$$

Dividing both sides by $a_0(x)$ and $e^{\left(b\frac{1}{2}x-\frac{1}{2}\int\frac{a_1(x)}{a_0(x)}dx\right)}$ and after simplification we deduce

$$u''(x)+b\,u'(x)+\left(\left(\frac{b^2}{4}-\frac{1}{4}\left(\frac{a_1^2(x)}{a_0^2(x)}\right)\right)-\frac{1}{2}\left(\frac{a_1'(x)a_0(x)-a(x)a_0'(x)}{a_0^2(x)}\right)+\frac{a_2(x)}{a_0(x)}\right)$$

$u(x)=\lambda^2\,u(x)$.

If $\left(\frac{b^2}{4}-\frac{1}{4}\left(\frac{a_1^2(x)}{a_0^2(x)}\right)\right)-\frac{1}{2}\left(\frac{a_1'(x)a_0(x)-a(x)a_0'(x)}{a_0^2(x)}\right)+\frac{a_2(x)}{a_0(x)}=A=$ constant, then the last equation reduces to

$u''(x)+b\,u'(x)+A\,u(x)=\lambda^2\,u(x)$,

is a spectral L.O.D.E. with constant coefficients in variable u and x, therefore $u(x)$ can be found with known methods and hence the solution $y(x)$ is specified from $y(x)=u(x)g(x)$.

Example: Solve the spectral differential equation of the form

$$x^2\frac{d^2y}{dx^2}+2x\,(x+2\,)\frac{dy}{dx}+2\,(x+1)^2\,y=x^2\lambda^2\,y.$$

Solution: Comparing the given equation with our differential form

$a_0(x)\frac{d^2y}{dx^2}\ +\ a_1(x)\frac{dy}{dx}+a_2(x)y=\lambda^2\,a_0(x)y$, we get that:

$a_0(x)=x^2, a_1(x)=2x(x+2), a_2(x)=2(x+1)^2$,

$a_0'(x)=2x, a_1'(x)=4x+4, a_1^2(x)=(2x^2+4x)^2, a_0^2(x)=x^4$.

Now

$$g(x)=e^{\left(b\frac{1}{2}x-\frac{1}{2}\int\frac{a_1(x)}{a_0(x)}dx\right)}=e^{\left(b\frac{x}{2}x-\frac{1}{2}\int\frac{2x^2+4x}{x^2}dx\right)}=e^{(b\frac{x}{2}-\int dx-2\int x^{-1}dx)}$$

$$=e^{(b\frac{x}{2}-\frac{x}{2}-2Lnx)}.$$

Let $b=2$, so $g(x)=\frac{1}{x^2}$.

$$\left(\frac{b^2}{4}-\frac{1}{4}\left(\frac{a_1^2(x)}{a_0^2(x)}\right)\right)-\frac{1}{2}\left(\frac{a_1'(x)a_0(x)-a_1(x)a_0'(x)}{a_0^2(x)}\right)+\frac{a_2(x)}{a_0(x)}$$

$$=\left(1-\frac{1}{4}\left(\frac{4x^4+16x^3+16x^2}{x^4}\right)\right)-\frac{1}{2}\left(\frac{\left((4x+4)x^2-(2x^2+4x)2x\right)}{x^4}\right)$$

$$+\frac{(2x^2+4x+2)}{x^2}$$

$$=\left(\frac{x^4-x^4-4x^3-4x^2}{x^4}\right)-\frac{1}{2}\left(\frac{-4x^2}{x^4}\right)+\frac{(2x^2+4x+2)}{x^2}$$

$$=\frac{-4x-4}{x^2}+\frac{2x^2+4x+4}{x^2}=\frac{2x^2}{x^2}=2=A=\text{constant}.$$

Thus the equation $u''(x)+d\,u'(x)+A\,u(x)=\lambda^2\,u(x)$ becomes

$u''(x)+2\,u'(x)+2\,u(x)=\lambda^2\,u(x)$,

or

$u''(x)+2\,u'(x)+(2\,-\lambda^2)\,u(x)=0$,

is a spectral L.O.D.E. of order two, so the complete integral with respect to the variables u and x can be found as follows:

$\left(D^2+2D-(2-\lambda^2)\right)z(x)=0$, where $D=\frac{d}{dx}$.

or

$m^2+2m+(2-\lambda^2)=0, m=\dfrac{-2\mp\sqrt{4-4(2-\lambda^2)}}{2}=\dfrac{-2\mp 2\sqrt{\lambda^2-1}}{2}$

$m=-1\mp\sqrt{\lambda^2-1}$.

$u(x)=a_1\, e^{(\sqrt{\lambda^2-1}-1)x}+\ b_1 e^{(-\sqrt{\lambda^2-1}-1)x}$.

Thereby we find $y(x)$ from $y(x)=u(x)g(x)$

$y(x)=x^{-2}\ \left(a_1\, e^{(\sqrt{\lambda^2-1}-1)x}+\ b_1 e^{(-\sqrt{\lambda^2-1}-1)x}\right)$,

where a_1 and b_1 are constants.

Example: Solve the following spectral D.E.

$$x^2\frac{d^2y}{dx^2}-x(3x+2)\frac{dy}{dx}+(2x^2+3x+2)y=\ \lambda^2x^2y.$$

Solution: $a_0=x^2, a_1=-3x^2-2x, a_2=2x^2+3x+2$,

$a_0'=2x, a_1'=-6x-2, a_1^2=(-3x^2-2x)^2, a_0^2(x)=x^4$.

Now,

$g(x)=e^{\left(b\frac{1}{2}x-\frac{1}{2}\int\frac{a_1(x)}{a_0(x)}dx\right)}=e^{(b\frac{x}{2}-\frac{1}{2}\int\frac{-3x^2-2x}{x^2}dx)}=e^{(b\frac{x}{2}+\frac{3}{2}\int dx+\int x^{-1}\,dx)}$

$g(x)=e^{(d\frac{x}{2}+\frac{3}{2}x+lnx)}$.

At $b=-3$, we have $g(x)=e^{lnx}=x$.

Therefore

$$\left(\frac{b^2}{4}-\frac{1}{4}\left(\frac{a_1^2(x)}{a_0^2(x)}\right)\right)-\frac{1}{2}\left(\frac{a_1'(x)a_0(x)-a_1(x)a_0'(x)}{a_0^2(x)}\right)+\frac{a_2(x)}{a_0(x)}=$$

$$\left(\frac{9}{4}-\frac{1}{4}\left(\frac{9x^4+12x^3+4x^2}{x^4}\right)\right)-\frac{1}{2}\left(\frac{((-6x-2)x^2-(-3x^2-2x)2x)}{x^4}\right)$$

$$+\frac{2x^2+3x+2}{x^2}$$

$$=\frac{-12x^3-4x^2}{4x^4}-\frac{1}{2}\left(\frac{2x^2}{x^4}\right)+\frac{2x^2+3x+2}{x^2}=\frac{8x^4}{4x^4}=2=A=\text{constant}.$$

Thus, the equation $u''(x)+b\,u'(x)+A\,u(x)=\lambda^2\,u(x)$ reduces to

$u''(x)-3u'(x)+2\,u(x)=\ \lambda^2\,u(x)$,

or

$u''(x)-3u'(x)+(2-\ \lambda^2)\,u(x)=0$.

Thus the complete integral of the resultant equation can be found by:

$(D^2-3D+(2-\ \lambda^2)\,z(x)=0$, where $D=\dfrac{d}{dx}$

$m^2-3m+(2-\ \lambda^2)\ =0$,

$m=\dfrac{3\mp\sqrt{9-4(2-\lambda^2)}}{2}=\dfrac{3}{2}\mp\dfrac{\sqrt{1+4\lambda^2}}{2}$,

$z(x)=c_3\, e^{\left(\frac{3}{2}+\frac{\sqrt{1+4\lambda^2}}{2}\right)x}+\ c_4 e^{\left(\frac{3}{2}-\frac{\sqrt{1+4\lambda^2}}{2}\right)x}$,

where c_3 and c_4 are arbitrary constants.

And since $y=g(x)u(x)$, hence a complete integral to the given differential equation is

$$y(x)=x\left(\ c_3\, e^{\left(\frac{3}{2}+\frac{\sqrt{1+4\lambda^2}}{2}\right)x}+\ c_4 e^{\left(\frac{3}{2}-\frac{\sqrt{1+4\lambda^2}}{2}\right)x}\right).$$

Theorem 2. The spectral differential equation

$$a_0(x)\frac{d^2y}{dx^2}+ a_1(x)\frac{dy}{dx} = k\,\lambda^2\, y, x \in (a,b)$$

where λ , $a_0(x) \neq 0$ and $a_i(x), i = 0,1$ are real valued continuous functions, can be reduced to a form $y''(t) - K\lambda^2 y(t) = 0$ by making the change of variable

$$t = \int e^{-\int \frac{a_1(x)}{a_0(x)}dx}\, dx,$$

$$if\ \frac{1}{a_0(x)} = k_1\, e^{-2\int \frac{a_1(x)}{a_0(x)}dx}$$, where k_1 and k are arbitrary non zero constants, and $K = k_1\, k.$

Proof: Consider the differential equation

$$a_0(x)\frac{d^2y}{dx^2}+ a_1(x)\frac{dy}{dx} = k\,\lambda^2\, y,$$

now,

$$t = \int e^{-\int \frac{a_1(x)}{a_0(x)}dx}\, dx,\ \frac{dt}{dx} = e^{-\int \frac{a_1(x)}{a_0(x)}dx}.$$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = e^{-\int \frac{a_1(x)}{a_0(x)}dx}\ \frac{dy}{dt},$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(e^{-\int \frac{a_1}{a_0}dx}\ \frac{dy}{dt}\right) = \frac{d}{dx}\left(e^{-\int \frac{a_1}{a_0}dx}\right)\frac{dy}{dt} + e^{-\int \frac{a_1}{a_0}dx}\ \frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx}$$

$$= -\frac{a_1(x)}{a_0(x)}\ e^{-\int \frac{a_1}{a_0}dx}\ \frac{dy}{dt} + e^{-\int \frac{a_1}{a_0}dx}\ \frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = e^{-2\int \frac{a_1}{a_0}dx}\ \frac{d^2y}{dt^2} - \frac{a_1(x)}{a_0(x)}\ e^{-\int \frac{a_1}{a_0}dx}\ \frac{dy}{dt}.$$

Substituting $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given differential equation, yields

$$a_0(x)\, e^{-2\int \frac{a_1}{a_0}dx}\ \frac{d^2y}{dt^2} - a_1(x)\, e^{-\int \frac{a_1}{a_0}dx}\ \frac{dy}{dt} + a_1(x)\, e^{-\int \frac{a_1}{a_0}dx}\ \frac{dy}{dt} = k\,\lambda^2\, y,$$

or

$$\frac{d^2y}{dt^2} - \frac{1}{a_0(x)}\ e^{-2\int \frac{a_1}{a_0}dx}\ k\,\lambda^2\, y = 0.$$

If $\frac{1}{a_0(x)} = k_1 e^{-2\int \frac{a_1}{a_0}dx}$, where k_1 is a non zero arbitrary constant, so the last equation reduces to

$$\frac{d^2y}{dt^2} - K\,\lambda^2\, y(x) = 0, \text{where } K = k_1\, k,$$

Thus, the proof is finished.

Example: Consider the spectral Ordinary differential equation

$$\mathrm{Sin}^2\, 4x\frac{d^2y}{dx^2} + 2\sin 8x\frac{dy}{dx} = \lambda^2\, y.$$

Solution: $a_0 = \sin^2 4x,\ a_1 = 2\sin 8x$

$$t = \int e^{-\int \frac{a_1}{a_0}dx}\, dx = \int e^{-\int \frac{\sin^2 4x}{2\sin 8x}dx}\, dx = \int e^{-\int \frac{4\sin 4x\cos 4x}{\sin^2 4x}dx}\, dx$$

$$t = \int e^{-\int (sin4x)^{-1}\,(4\cos 4x)\,dx}\,dx = \int e^{-ln\, sin4x}\,dx = \int \frac{1}{\sin 4x}$$
$$= \int \frac{1}{2sin2xcos2x}\,dx$$
$$t = \frac{1}{2}\int \frac{\sec^2 2x}{tan2x}\,dx = \frac{1}{4}\,ln\,|tan2x|,$$
or
$$t = \frac{1}{4}\,ln\,tan2x, \frac{dt}{dx} = \frac{1}{sin4x}.$$
$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{sin4x}\frac{dy}{dt},$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{sin4x}\frac{dy}{dt}\right) = \frac{-4cos4x}{\sin^2 4x}\frac{dy}{dt} + \frac{1}{sin4x}\frac{d^2y}{dt^2}\frac{1}{sin4x},$$
$$\frac{d^2y}{dx^2} = \frac{1}{\sin^2 4x}\frac{d^2y}{dt^2} - \frac{4cos4x}{\sin^2 4x}\frac{dy}{dt}.$$
Since
$$\sin^2 4x\;\; y''(x) + 2\sin 8x\;\; y'(x) = \lambda^2\, y(x),$$
then
$$\sin^2 4x \frac{1}{\sin^2 4x}\frac{d^2y}{dt^2} - \sin^2 4x\;\frac{4cos4x}{\sin^2 4x}\frac{dy}{dt} + 2sin8x\;\frac{1}{sin4x}\frac{dy}{dt} = \lambda^2\, y(t),$$
$$\frac{d^2y}{dt^2} - 4cos4x\frac{dy}{dt} + 4sin4xcos4x\frac{1}{sin4x}\frac{dy}{dt} = \lambda^2\, y(t),$$
$$\frac{d^2y}{dt^2} - \lambda^2\, y(t) = 0,$$
Thus, the complete integral with respect to the variables y and t is:
$$m^2 - \lambda^2 = 0, m = \mp\,\lambda.$$
$$y(t) = c_5 e^{\lambda\, t} + c_6\, e^{-\lambda\, t},$$
or
$$y(x) = c_5\,(tan2x)^{\frac{\lambda}{4}} + c_6 (tan2x)^{-\frac{\lambda}{4}}.$$

3. Solving spectral L.O.D.E. of order three

In the following theorem, we determine the complete integral of a defined spectral L.O.D.E. of order three where its coefficients are some real polynomial.

Theorem 3. Consider the spectral ordinary differential equation

$$a_o(x)\frac{d^3y}{dx^3} + a_1(x)\frac{d^2y}{dx^2} + a_2(x)\frac{dy}{dx} = -\lambda^3 a_3(x)y, x \in (a,b) \qquad (3)$$

where $\lambda \neq 0$, $a > 0$ and $a < b$, where $a_i(x), i = 0,1,2,3$ are some real polynomials and $a_o(x) \neq 0, \forall x \in (a,b)$. By making change of variable $s = k\,\int \left(\frac{a_3(x)}{a_0(x)}\right)^{\frac{1}{3}} dx$, equation (3) may be reduced to the constant coefficients if

$$h_1(s) = \frac{1}{k}\left(\frac{a_1(s)}{a_0^{\frac{2}{3}}(s)a_3^{\frac{1}{3}}(s)} + \frac{a_0^{\frac{4}{3}}(s)}{a_3^{\frac{4}{3}}(s)}\left(\frac{a_3(s)}{a_o(s)}\right)'\right) = h_2(s)$$

$$= \frac{1}{3k^2}\ (\frac{1}{a_3^{\frac{5}{3}}(s)}\left(\frac{a_3''(s)a_o(s) - a_3'(s)\,a_o'(s)}{a_o^{\frac{1}{3}}(s)}\right)$$

$$- \frac{1}{a_3^{\frac{5}{3}}(s)}\left(\frac{a_3'(s)a_o'(s) + a_3(s)\,a_o''(s)}{a_o^{\frac{1}{3}}(s)}\right)$$

$$-2\frac{a_0^{\frac{4}{3}}(s)\big(a_o'(s)\big)^2}{a_3^{\frac{2}{3}}(s)} - \frac{2}{3}\left(\frac{a_0^{\frac{8}{3}}(s)}{a_3^{\frac{8}{3}}(s)}\ \left(\left(\frac{a_3(s)}{a_o(s)}\right)'\right)^2\right) + \frac{a_1(s)a_0^{\frac{2}{3}}(s)}{a_3^{\frac{5}{3}}(s)}\ \left(\frac{a_3(s)}{a_o(s)}\right)'$$

$$+\frac{1}{3}\left(\frac{a_2(s)}{a_0^{\frac{1}{3}}(s)\,a_3^{\frac{2}{3}}(s)}\right)\ ) = \text{constant, where } k \text{ is a non zero constant.}$$

Proof: We rewrite equation (3) as follows

$$a_o(x)\frac{d^3y}{dx^3} + a_1(x)\frac{d^2y}{dx^2} + a_2(x)\frac{dy}{dx} + \lambda^3 a_3(x)y = 0. \qquad (3)$$

We change the independent variable in equation (3) from x to s by using the chain rule.

$$\text{Now, since } s = k\int\left(\frac{a_3(x)}{a_0(x)}\right)^{\frac{1}{3}}dx, \qquad (4)$$

then

$$\frac{dy}{dx} = \frac{dy}{ds}\frac{ds}{dx} = k\left(\frac{a_3}{a_o}\right)^{\frac{1}{3}}\frac{dy}{ds},$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(k\left(\frac{a_3}{a_o}\right)^{\frac{1}{3}}\frac{dy}{ds}\right),$$

$$\frac{d^2y}{dx^2} = k\frac{d}{dx}\left(k\left(\frac{a_3}{a_o}\right)^{\frac{1}{3}}\right)\ \frac{dy}{ds} + k\left(\frac{a_3}{a_o}\right)^{\frac{1}{3}}\ \frac{d}{dx}\left(\frac{dy}{ds}\right),$$

$$= \frac{k}{3}\left(\frac{a_3}{a_o}\right)^{-\frac{2}{3}}\ \left(\frac{a_3'\,a_o - a_3\,a_o'}{(a_o)^2}\right)\frac{dy}{ds} + k\left(\frac{a_3}{a_o}\right)^{\frac{1}{3}}\ \frac{d}{ds}\left(\frac{dy}{ds}\right)\frac{ds}{dx},$$

or

$$\frac{d^2y}{dx^2} = \frac{k}{3}\left(\frac{a_3}{a_o}\right)^{-\frac{2}{3}}\ \left(\frac{a_3'\,a_o - a_3\,a_o'}{(a_o)^2}\right)\frac{dy}{ds} + k^2\left(\frac{a_3}{a_o}\right)^{\frac{2}{3}}\ \frac{d^2y}{ds^2}.$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

$$= \frac{d}{dx}\left(\frac{k}{3}\left(\frac{a_3}{a_o}\right)^{-\frac{2}{3}}\ \left(\frac{a_3'\,a_o - a_3\,a_o'}{(a_o)^2}\right)\ \frac{dy}{ds} + k^2\left(\frac{a_3}{a_o}\right)^{\frac{2}{3}}\ \frac{d^2y}{ds^2}\right)$$

$$= \frac{d}{dx}\left(\frac{k}{3}\left(\frac{a_3}{a_o}\right)^{-\frac{2}{3}}\ \left(\frac{a_3'\,a_o - a_3\,a_o'}{(a_o)^2}\right)\frac{dy}{ds}\right) + \frac{d}{dx}\left(k^2\left(\frac{a_3}{a_o}\right)^{\frac{2}{3}}\ \frac{d^2y}{ds^2}\right)$$

$$= \left(\frac{-2k}{9}\left(\frac{a_3}{a_o}\right)^{-\frac{5}{3}}\left(\frac{a_3'\,a_o - a_3\,a_o'}{(a_o)^2}\right)\frac{dy}{ds} + \frac{k}{3}\left(\frac{a_3}{a_o}\right)^{-\frac{2}{3}}\frac{d}{dx}\left(\frac{dy}{ds}\right)\right)$$

$$\left(\frac{a_3'\, a_o - a_3\, a_o'}{(a_o)^2}\right) + \frac{k}{3}\left(\frac{a_3}{a_o}\right)^{-\frac{2}{3}}\left(\frac{dy}{ds}\right)\ \frac{d}{dx}\left(\frac{a_3'}{a_0} - \frac{a_3\, a_o'}{{a_0}^2}\right) + k^2\frac{d^2y}{ds^2}$$
$$\left(\frac{2}{3}\left(\frac{a_3}{a_o}\right)^{-\frac{1}{3}}\left(\frac{a_3'\, a_o - a_3\, a_o'}{(a_o)^2}\right)\right) + k^2\left(\frac{a_3}{a_o}\right)^{\frac{2}{3}}\ \frac{d}{ds}\left(\frac{d^2y}{ds^2}\right)\frac{ds}{dx},$$

$$\frac{d^3y}{dx^3} = \left(\frac{-2k}{9}\left(\frac{a_3}{a_o}\right)^{-\frac{5}{3}}\left(\frac{a_3'\, a_o - a_3\, a_o'}{(a_o)^2}\right)\ \frac{dy}{ds} + \frac{k}{3}\left(\frac{a_3}{a_o}\right)^{-\frac{2}{3}}\ \frac{ds}{dx}\left(\frac{d^2y}{ds^2}\right)\right)$$
$$\left(\frac{a_3'\, a_o - a_3\, a_o'}{(a_o)^2}\right) + \frac{k}{3}\left(\frac{a_3}{a_o}\right)^{-\frac{2}{3}}\left(\frac{dy}{ds}\right)\ (\frac{a_3''\ a_o - a_3'\, a_o'}{(a_o)^2}$$
$$-\frac{(a_3'\, a_o' - a_3\, a_o'')(a_o)^2 - 2a_3\, a_o'\, a_o\, a_o'}{(a_o)^4}) + k^2\frac{d^2y}{ds^2}\left(\frac{2}{3}\left(\frac{a_3}{a_o}\right)^{-\frac{1}{3}}\frac{a_3'\, a_o - a_3\, a_o'}{(a_o)^2}\right)$$
$$+k^2\left(\frac{a_3}{a_o}\right)^{\frac{2}{3}}\frac{d^3y}{ds^3}\ k\ \left(\frac{a_3}{a_o}\right)^{\frac{1}{3}}.$$

After some minor algebraic operations, we get

$$\frac{d^3y}{dx^3} = k^3\left(\frac{a_3}{a_o}\right)\left(\frac{d^3y}{ds^3}\right) + k^2\left(\frac{a_3}{a_o}\right)^{-\frac{1}{3}}\left(\frac{a_3'\, a_o - a_3\, a_o'}{(a_o)^2}\right)\frac{d^2y}{ds^2}$$
$$-\frac{2k}{9}\left(\frac{a_3}{a_o}\right)^{-\frac{5}{3}}\left(\frac{a_3'\, a_o - a_3\, a_o'}{(a_o)^2}\right)\frac{dy}{ds} + \frac{k}{3}\left(\frac{a_3}{a_o}\right)^{-\frac{2}{3}}\left(\frac{a_3''\, a_o - a_3'\, a_o'}{(a_o)^2}\right)\frac{dy}{ds}$$
$$-\frac{k}{3}\left(\frac{a_3}{a_o}\right)^{-\frac{2}{3}}\left(\frac{(a_3'\, a_o' + a_3\, a_o'')}{(a_o)^2}\right)\frac{dy}{ds} - \frac{2k}{3}\left(\frac{a_3}{a_o}\right)^{-\frac{2}{3}}\left(\frac{a_3\,(a_o')^2}{(a_o)^3}\right)\frac{dy}{ds}.$$

putting the expressions for $\frac{\mathrm{dy}}{\mathrm{dx}}$, $\frac{\mathrm{d^2y}}{\mathrm{dx^2}}$ and $\frac{d^3y}{dx^3}$ in equation (3), we get:

$$a_o\, k^3\left(\frac{a_3}{a_o}\right)\left(\frac{d^3y}{ds^3}\right) + k^2 a_o\left(\frac{a_3}{a_o}\right)^{-\frac{1}{3}}\left(\frac{a_3'\, a_o - a_3\, a_o'}{(a_o)^2}\right)\frac{d^2y}{ds^2}$$
$$-\frac{2k}{9}\left(\frac{a_3}{a_o}\right)^{-\frac{5}{3}}\left(\frac{a_3'\, a_o - a_3\, a_o'}{(a_o)^2}\right)\frac{dy}{ds} + \frac{k}{3}\ a_o\left(\frac{a_3}{a_o}\right)^{-\frac{2}{3}}\left(\frac{a_3''\, a_o - a_3'\, a_o'}{(a_o)^2}\right)\frac{dy}{ds}$$
$$-\frac{k}{3}\ a_o\left(\frac{a_3}{a_o}\right)^{-\frac{2}{3}}\left(\frac{(a_3'\, a_o' + a_3\, a_o'')}{(a_o)^2}\right)\frac{dy}{ds} - \frac{2k}{3}\ a_o\left(\frac{a_3}{a_o}\right)^{-\frac{2}{3}}\left(\frac{a_3\,(a_o')^2}{(a_o)^3}\right)\frac{dy}{ds}$$
$$+a_1\frac{k}{3}\left(\frac{a_3}{a_o}\right)^{-\frac{2}{3}}\left(\frac{a_3'\, a_o - a_3\, a_o'}{(a_o)^2}\right)\ \frac{dy}{ds} + k^2 a_1\left(\frac{a_3}{a_o}\right)^{\frac{2}{3}}\ \frac{d^2y}{ds^2} + k\, a_2\left(\frac{a_3}{a_o}\right)^{\frac{1}{3}}\frac{dy}{ds}$$
$$+\lambda^3 a_3\, y = 0,$$

or

$$k^3 a_3\ \frac{d^3y}{ds^3} + k^2\left(a_1\left(\frac{a_3}{a_o}\right)^{\frac{2}{3}} + \frac{a_o^{\frac{4}{3}}}{a_3^{\frac{1}{3}}}\left(\frac{a_3}{a_o}\right)'\right)\frac{d^2y}{ds^2} + \frac{k}{3}\ \left(\ \left(\frac{a_0}{p_3}\right)^{\frac{2}{3}}\left(\frac{a_3''\ a_o - a_3'\ a_o'}{a_o}\right)\right.$$
$$-\left(\frac{a_0}{a_3}\right)^{\frac{2}{3}}\left(\frac{a_3'\, a_o' + a_3\, a_o''}{a_o}\right) - 2\left(\frac{a_0}{a_3}\right)^{\frac{2}{3}}\left(\frac{a_3\,(a_o')^2}{(a_o)^2}\right) - \frac{2}{3}\left(\frac{a_o^{\frac{8}{3}}}{a_3^{\frac{5}{3}}}\right)\left(\left(\frac{a_3}{a_o}\right)'\right)^2$$

$$+a_1\left(\frac{a_0}{a_3}\right)^{\frac{2}{3}}\left(\frac{a_3}{a_o}\right)'+\frac{1}{3}\,a_2\left(\frac{a_3}{a_o}\right)^{\frac{1}{3}}\Big)\,\frac{dy}{ds}+\,\lambda^3 a_3\,y=0.$$

Multiplying the last equation by $\dfrac{1}{k^3 a_3}$, we obtain

$$\frac{d^3y}{ds^3}+\frac{1}{k}\left(\frac{a_1}{a_0^{\frac{2}{3}}\,a_3^{\frac{1}{3}}}+\frac{a_0^{\frac{4}{3}}}{a_3^{\frac{4}{3}}}\left(\frac{a_3}{a_o}\right)'\right)\frac{d^2y}{ds^2}+\frac{1}{3k^2}\,\Bigg(\,\frac{1}{a_3^{\frac{5}{3}}}\left(\frac{a_3''\;a_o-a_3'\,a_o'}{a_o^{\frac{1}{3}}(s)}\right)$$

$$-\frac{1}{a_3^{\frac{5}{3}}}\left(\frac{a_3'\,a_o'+a_3\,a_o''}{a_o^{\frac{1}{3}}}\right)-2\left(\frac{a_0^{\frac{4}{3}}\,(\,a_o')^2}{a_3^{\frac{2}{3}}}\right)-\frac{2}{3}\left(\frac{a_0^{\frac{8}{3}}}{a_3^{\frac{8}{3}}}\right)\left(\left(\frac{a_3}{a_o}\right)'\right)^2$$

$$+\left(\frac{a_1\,a_0^{\frac{2}{3}}}{a_3^{\frac{5}{3}}}\right)\left(\frac{a_3}{a_o}\right)'+\frac{1}{3}\left(\frac{a_2}{a_0^{\frac{1}{3}}\;a_3^{\frac{2}{3}}}\right)\Bigg)\,\frac{dy}{ds}+\frac{\lambda^3}{k^3}\;y=0.$$

The last equation becomes

$$\frac{d^3y}{ds^3}+h_1(s)\frac{d^2y}{ds^2}+h_2(s)\frac{dy}{ds}+\frac{1}{k^3}\,\lambda^3\;y(s)=0,$$

where

$$h_1(s)=\frac{1}{k}\left(\frac{a_1}{a_0^{\frac{2}{3}}\,a_3^{\frac{1}{3}}}+\frac{a_0^{\frac{4}{3}}}{a_3^{\frac{4}{3}}}\left(\frac{a_3}{a_o}\right)'\right),$$

$$h_2(s)=\frac{1}{3k^2}\,\Bigg(\,\frac{1}{a_3^{\frac{5}{3}}}\left(\frac{a_3''\;a_o-a_3'\,a_o'}{a_o^{\frac{1}{3}}(s)}\right)-\frac{1}{a_3^{\frac{5}{3}}}\left(\frac{a_3'\,a_o'+a_3\,a_o''}{a_o^{\frac{1}{3}}}\right)-\,2\frac{a_0^{\frac{4}{3}}\,(\,a_o')^2}{a_3^{\frac{2}{3}}}$$

$$-\frac{2}{3}\left(\frac{a_0^{\frac{8}{3}}}{a_3^{\frac{8}{3}}}\right)\left(\left(\frac{a_3}{a_o}\right)'\right)^2+\frac{a_1\,a_0^{\frac{2}{3}}}{a_3^{\frac{5}{3}}}\;\left(\frac{a_3}{a_o}\right)'+\frac{1}{3}\left(\frac{a_2}{a_0^{\frac{1}{3}}\;a_3^{\frac{2}{3}}}\right)\Bigg).$$

If $h_1(s)=h_2(s)=$ constant, then the resultant equation become L.O.D.E. with constant coefficients.

Example: Determine the complete integral to the following spectral D.E.

$$3x^3\frac{d^3y}{dx^3}+9x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}=-24\lambda^3 y.$$

Solution: $3x^3\frac{d^3y}{dx^3}+9x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+24\lambda^3 y=0.$ (5)

If we compare equation (5) with equation (3), we get that:
$a_0=3x^3$, $a_1=9x^2$, $a_2=3x$, and $a_3=24$.

Now

$$s=k\int\left(\frac{a_3}{a_0}\right)^{\frac{1}{3}}dx=k\int\left(\frac{24}{3x^3}\right)^{\frac{1}{3}}dx=2k\int\frac{1}{x}\,dx=2k\,ln|x|+c.$$

Let $x>0$, thus $s=2k\,lnx+c$, and setting $2k=1$, and $c=0$,
$s=lnx$, then $x=e^s$ and $\dfrac{dx}{ds}=e^s$.

$$\frac{dy}{dx}=\frac{dy}{ds}\frac{ds}{dx}=\left(\frac{1}{e^s}\right)\frac{dy}{ds},$$
$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{ds}\left(\left(\frac{1}{e^s}\right)\frac{dy}{ds}\right)\frac{ds}{dx}=\frac{1}{e^s}\left(-e^{-s}\frac{dy}{ds}+\left(\frac{1}{e^s}\right)\frac{d^2y}{ds^2}\right),$$
$$\frac{d^2y}{dx^2}=\left(\frac{1}{e^{2s}}\right)\frac{d^2y}{ds^2}-\left(\frac{1}{e^{2s}}\right)\frac{dy}{ds},$$
$$\frac{d^3y}{dx^3}=\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=\frac{d}{ds}\left(\left(\frac{1}{e^{2s}}\right)\frac{d^2y}{ds^2}-\left(\frac{1}{e^{2s}}\right)\frac{dy}{ds}\right)\frac{ds}{dx}$$
$$=\left(\frac{d}{ds}\left(\left(\frac{1}{e^{2s}}\right)\frac{d^2y}{ds^2}\right)+\frac{d}{ds}\left(\left(-\frac{1}{e^{2s}}\right)\frac{dy}{ds}\right)\right)\frac{1}{e^s},$$
$$\frac{d^3y}{dx^3}=\left(\left(\frac{1}{e^{2s}}\right)\frac{d^3y}{ds^3}-2e^{-2s}\frac{d^2y}{ds^2}+2e^{-2s}\frac{dy}{ds}-e^{-2s}\frac{d^2y}{ds^2}\right)\frac{1}{e^s},$$
$$\frac{d^3y}{dx^3}=\left(\frac{1}{e^{3s}}\right)\frac{d^3y}{ds^3}-\left(\frac{3}{e^{3s}}\right)\frac{d^2y}{ds^2}+\left(\frac{2}{e^{3s}}\right)\frac{dy}{ds}.$$

By placing the first, second, and third derivatives in the given differential equations, we obtain:

$$3\,e^{3s}\left(\left(\frac{1}{e^{3s}}\right)\frac{d^3y}{ds^3}-\left(\frac{3}{e^{3s}}\right)\frac{d^2y}{ds^2}+\left(\frac{2}{e^{3s}}\right)\frac{dy}{ds}\right)+9e^{2s}\left(\left(\frac{1}{e^{2s}}\right)\frac{d^2y}{ds^2}-\left(\frac{1}{e^{2s}}\right)\frac{dy}{ds}\right)+$$
$$3e^s\left(\left(\frac{1}{e^s}\right)\frac{dy}{ds}\right)+24\lambda^3\,y=0,$$
$$\frac{d^3y}{ds^3}-3\frac{d^2y}{ds^2}+2\frac{dy}{ds}-3\frac{dy}{ds}+3\frac{d^2y}{ds^2}+\frac{dy}{ds}+8\lambda^3\,y=0,$$
$$\frac{d^3y}{ds^3}+8\lambda^3\,y=0,$$

which is a linear spectral ordinary equation of third order with constant equations in variable y and s, hence the general solution is:

$$y(s)=c_1e^{-2\lambda s}+e^{\lambda s}\,(c_2\cos\sqrt{3}\,\lambda s+c_3\sin\sqrt{s}\,\lambda s),$$

or

$$y(x)=c_1e^{-2\lambda\, lnx}+e^{\lambda\, lnx}\,(c_2\cos\sqrt{3}\,\lambda\, lnx+c_3\sin\sqrt{s}\,\lambda\, lnx),$$
$$y(x)=\frac{c_1}{x^{2\lambda}}+x^{\lambda}\left(c_2\cos\sqrt{3}\,\lambda\, lnx+c_3\sin\sqrt{s}\,\lambda\, lnx\right),$$

where $c_i, i=1,2,3$ are constants.

4. Conclusion

In this paper, we have presented some techniques for solving two types of spectral linear ordinary differential equations which are second and third orders including two distinct kinds of coefficients. The first one is several polynomials, while the other kind is continuous functions and all these kinds are of real type. We have shown how these kinds are converted to the constant coefficients by modifying the independent variable to another one via the specified transformations. We brought examples to explain the usefulness and applicability of our technique. The importance of this work is to opening the way to use another techniques and ideas to solve other types of spectral linear ordinary differential equations of higher order.

Conflict of Interests.
There are non-conflicts of interest .

References

[1] Aryan Ali Mohammed, Eigenfunctions and Asymptotic Behavior of Eigenvalues to The Given Boundary Value Problem with Eigenparameter in the Boundary Conditions, Journal of Zankoy Sulaimani – Part A, vol. 18, no. 1, pp. 179-190, 2016.

[2] D. Arficho, Method for Solving Particular Solution of Linear Second Order Ordinary Differential Equations, J Appl Computat Math, vol. 4, no. 2, pp.1-3, 2015.

[3] H. Hochtadt, On The Stability of Certain Second-Order Differential Equations, Sot. Ind. Appl. Math 12, pp. 58-59, 1964.

[4] Karwan H. M. Jwamer and Aryan Ali M., Boundedness of Normalized Eigenfunctions of The Spectral Problem in The Case of Weight Function Satisfying The Lipschitz Condition, Journal of Zankoy Sulaimani – Part A, vol. 15, no. 1, pp. 79-94, 2013.

[5] Karwan H.M. Jwamer and Aryan Ali M., Study the Behavior of The Solution and Asymptotic Behaviors of Eigenvalues of a Six Order Boundary Value Problem,International Journal of Research and Reviews in Applied Sciences, vol. 13, no. 3, pp. 790-799, 2012.

[6] L. Cesari, Asymptotic Behavior and Stability Problems in Ordinary Differential Equations, Springer-Verlag, Berlin, 1963.

[7] M. Marini, Criteri di Limitatezza per le Soluzioni dell' Equazione Lineare de1 Second Ordine, Boll. U.M.I 4, pp. 154-165, 1975.

[8] M. Marini and P. Zezza, On the Asymptotic Behavior of the Solutions of a Class of Second Order Linear Differential Equations, Journal of Differential Equations 28, pp. 1- 17, 1978.

[9] P. Hartman P and A. Wintner, On The Laplace-Fourier Transcendent, Amer. J. Math71**,** pp. 367-372, 1949.

[10] P. Hartman, Ordinary Differential Equations, Wiley, New York, 1964.

[11] P. Johnson, K. Busawon and J. Barbot, Alternative Solution of The Inhomogeneous Linear Differential Equation of order two, J. Math. Anal. Appl. 339, pp. 582–589, 2008.

[12] R. A. Moore, The Behavior of Solutions of a Linear Differential Equation of Second Order, Pacific J. Math **5**, pp. 125-145, 1955.

[13] R. Bellman, Stability Theory of Differential Equations, McGraw-Hill, New York, 1953.

[14] T. Yu. Gadzhieva, Analysis of Spectral Characteristics of One Nonself Adjoint Problem with Smooth Coefficients, PhD thesis, Dagestan State University, South of Russian, 2010.

[15] U. Richard, Serie Asintotiche per Una Classe di Equazioni Differenziali non Oscillanti de1 Second Ordine, Uniw. Polit. Torino, Rend. Sent. Mat 23, pp. 171-217, 1964.

[16] V. V. Nemytskii and V. V. Stepanov, Qualitative Theory of Differential Equations, Princeton Univ. Press, Princeton, New Jersey, 1960.

[17] W. A. Coppel, Stability and Asymptotic Behavior of Differential Equations, Heat, Boston, Mass, 1965.
[18] W. A. Coppel, Disconjugacy, Springer-Verlag, Berlin, 1971.

الخلاصة

عملنا هي عبارةعن إيجاد الحل العام لنوعين مختلفين من المعادلات التفاضلية العادية الخطية من النوع الطيفي. النوع الأول يتكون من نوعين مختلفين من المعاملات من الرتبة الثانية، واحد منهم هو متعددات الحدود، حيث أن النوع الآخر هو دوال مستمرة وكلاهما من أنواع حقيقية. النوع الثاني للمعادلات هى من الرتبة الثالثة، وهنا المعاملات من هذا النوع أيضًا للعديد من المتعددات الحدود الحقيقية، وفي كلا النوعين من المعادلات التفاضلية العادية الخطية الطيفية ، يتم تحويل المعاملات إلى الثوابت عبر تغيير المتغير المستقل إلى متغير جديد. قدمنا أمثلة لشرح آليتنا.

الكلمات الدالة: رتبة المعادلات التفاضلية، متعددات الحدود والدوال الحقيقية، الحل العام، تبديل المتغيرات.