New Fractional Spline Polynomial for Computing Fractional Differential Equations

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Abstract

This paper illustrates how fractional lacunary for quartic polynomial using spline function has been constructed. On the other hand the existence, the uniqueness and error bounded for these functions have been discussed and proved. The goal of this method is to solve fractional differential equations. The efficiency and pertinence of these new methods are illustrated by giving numerically computing examples.

Keywords: Spline function; fractional derivatives; error bounded; convergence analysis.

1. Introduction

Fractional order differential equations are used in many fields such as engineering, chemistry and physics [1]. Some equations are required to be solved by an easy method. A big number of developed models in the different areas such as diffusion processes, viscoelasticity, electrochemistry, etc are modeled in terms of derivatives of non-integer (fractional) order. There are some numerical methods which are used to solve of various derivatives of fractional order problems [2] and [3].

One of the most useful tools for the numerical approximation of functions is spline functions with fractional order. Researchers also recommend a new rewarding problems for the other challenging [3] and [4]. Spline functions are a great tool that can be used for the numerical approximation of functions on the one hand. On the other hand, researchers can recommend new, rewarding problems on the other and challenging [4] and [5]. Spline lacunary interpolation is appeared when the problem related to a function and its derivative [4-6] and [9-10]

This paper describes the spline interpolation in which spline is referred to as fractional spline, error estimation for the fractional spline function and convergence analysis are derived.

2. Preliminaries and Basic definition

2.1. The Caputo Fractional Derivative

Suppose that $\alpha > 0, t > a, \alpha, a, t \in \mathbb{N}$. The fractional operator

$$D_*^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, & n-1 < \alpha < n, n \in \mathbb{N} \\ \frac{d^n}{dt^n} f(t), & \alpha = n \in \mathbb{N} \end{cases}$$

is called the Caputo fractional differential operator of order $\boldsymbol{\alpha}$ [7].

3. Description of the Method

The interpolation of fractional lacunary polynomial of degree four have been constructed, and their error estimates are illustrated. Throughout this paper, we let

$0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1$ be a uniform partition of [0,1] and the step size

$h = x_{i+1} - x_i \ (i = 0,1,\ldots,n)$.

Theorem 1. Let $g \in C^{2m}[0,h]$ and p_{2m-1} be the unique Hermite interpolation polynomial of degree $2m-1$ that matches g and its first $m-1$ derivatives g^r at 0 and h. Then,

$$\left|e^{(r)}\right| \leq \frac{h^r[x(h-x)]^{m-r}G}{r!(2m-2r)!}, r = 0,1,\ldots,m, 0 \leq x \leq h, \qquad (1)$$

Where

$$\left|e^{(r)}\right| = \left|g^r(x) - p_{2m-1}{}^{(r)}(x)\right| \ and\ G = \max_{0 \leq x \leq h} |g^{(2m)}(x)| \qquad (2)$$

The error estimation (6) are the best possible for only $r = 0$ [8], [9] and [11].

3.3. Existence and Uniqueness

We suppose that $u^{(\frac{1}{2})}(x) \in C^8[0,1]$ and $u(x)$ in each subinterval $[x_i, x_{i+1}]$ has a form:

$$u(x) = a_8(x-x_i)^4 + a_7(x-x_i)^{\frac{7}{2}} + a_6(x-x_i)^3 + a_5(x-x_i)^{\frac{5}{2}} + a_4(x-x_i)^2 + a_3(x-x_i)^{\frac{3}{2}} + a_2(x-x_i) + a_1(x-x_i)^{\frac{1}{2}} + a_0. \quad (3)$$

Where $a_8, a_7, a_6, a_5, a_4, a_3, a_2, a_1$ and a_0 are constants.

Theorem 2. Suppose that $u^{(\frac{1}{2})}(x) \in C^8[0,1]$ and $u(x)$ in each subinterval $[x_i, x_{i+1}]$ have the form (3). Given the real numbers $u_i^{(\frac{1}{2})} = f_i^{(\frac{1}{2})}, u_i^{(\frac{3}{2})} = f_i^{(\frac{3}{2})}, u_i^{(\frac{5}{2})} = f_i^{(\frac{5}{2})}, u_i^{(\frac{7}{2})} = f_i^{(\frac{7}{2})}$ $(i = 0,1,\dots,n)$ and f_0. There exists a unique $u(x)$ such that

$$u_i^{(\frac{1}{2})} = f_i^{(\frac{1}{2})}, u_i^{(\frac{3}{2})} = f_i^{(\frac{3}{2})}, u_i^{(\frac{5}{2})} = f_i^{(\frac{5}{2})}, u_i^{(\frac{7}{2})} = f_i^{(\frac{7}{2})} \ (i = 0,1,\dots,n) \ u_0 = f_0 \quad (4)$$

The fractional spline which satisfies eq. (4) in $[x_i, x_{i+1}]$ is of the form:

$$\begin{aligned} u(x) = u_i A_0(t) + u_{i+1} A_1(t) + h^{\frac{1}{2}}\left[u_i^{\frac{1}{2}} A_2(t) + u_{i+1}^{\frac{1}{2}} A_3(t)\right] \\ + h^{\frac{3}{2}}\left[u_i^{\frac{3}{2}} A_4(t) + u_{i+1}^{\frac{3}{2}} A_5(t)\right] + h^{\frac{5}{2}}\left[u_i^{\frac{5}{2}} A_6(t) + u_{i+1}^{\frac{5}{2}} A_7(t)\right] \\ + h^{\frac{7}{2}} u_i^{\frac{7}{2}} A_8(t). \quad (5) \end{aligned}$$

Where

$$A_0(t) = 1 - \frac{224}{61} t^{\frac{15}{2}} - \frac{144}{671} t^{\frac{13}{2}} + \frac{12948}{671} t^{\frac{11}{2}} - \frac{1001}{61} t^{\frac{9}{2}}$$

$$A_1(t) = \frac{224}{61} t^{\frac{15}{2}} + \frac{144}{671} t^{\frac{13}{2}} - \frac{12948}{671} t^{\frac{11}{2}} + \frac{1001}{61} t^{\frac{9}{2}}$$

$$A_2(t) = \frac{-2\sqrt{t}}{287859\sqrt{\pi}}[696608t^7 + 22864t^6 - 3544580t^5 + 31112967t^4 - 287859]$$

$$A_3(t) = \frac{-2048t^{\frac{9}{2}}}{287859\sqrt{\pi}}[352t^3 + 38t^2 - 1963t + 1573]$$

$$A_4(t) = \frac{-4t^{\frac{3}{2}}}{2747745\sqrt{\pi}}[1399328t^6 + 30672t^5 - 7325604t^4 + 6811519t^3 - 915915]$$

$$A_5(t) = \frac{-512t^{\frac{9}{2}}}{30225195\sqrt{\pi}}[20944t^3 - 1704t^2 - 64857t + 45617]$$

(6)

$$A_6(t) = \frac{8t^{\frac{5}{2}}}{30225195\sqrt{\pi}}[1623776t^5 + 24624t^4 - 9192508t^3 + 9559121t^2 - 2015013]$$

$$A_7(t) = \frac{256t^{\frac{9}{2}}}{30225195\sqrt{\pi}}[9240t^3 - 192t^2 - 26351t + 17303]$$

and

$$A_8(t) = \frac{-16t^{\frac{7}{2}}}{30225195\sqrt{\pi}}[81312t^4 + 848t^3 - 541476t^2 + 747175t - 287859]$$

where $x = x_i + th, t \in [0,1]$, with a similar expression for $u(x)$ in $[x_{i-1}, x_i]$. The coefficients in eq. (5) are given by the following recurrence formula:

$$\frac{651105\sqrt{\pi}}{3904}[u_i - u_{i-1}] = h^{\frac{1}{2}}\left[\frac{31840187375}{40684072}f_{i-1}{}^{\frac{1}{2}} + \frac{47040}{671}f_i{}^{\frac{1}{2}}\right] + h^{\frac{3}{2}}\left[\frac{104139}{976}f_{i-1}{}^{\frac{3}{2}} + \frac{30376}{671}f_i{}^{\frac{3}{2}}\right] + h^{\frac{5}{2}}\left[\frac{107709}{5368}f_{i-1}{}^{\frac{5}{2}} - \frac{7875}{671}f_i{}^{\frac{5}{2}}\right] + h^{\frac{7}{2}}\left[\frac{4515}{2684}f_{i-1}{}^{\frac{7}{2}} + f_i{}^{\frac{7}{2}}\right]$$
(7)

Proof:

We can write $p(t)$ in [0,1] as the following form:

$$p(x) = p_0A_0(t) + p_1A_1(t) + p_0{}^{\frac{1}{2}}A_2(t) + p_1{}^{\frac{1}{2}}A_3(t) + p_0{}^{\frac{3}{2}}A_4(t) + p_0{}^{\frac{5}{2}}A_6(t) + p_1{}^{\frac{5}{2}}A_7(t) + p_0{}^{\frac{7}{2}}A_8(t)$$

to determine $A_0, A_1, A_2, A_3, A_4, A_5, A_6, A_7$ *and* A_8 , we write the above equality for

$$p(x) = 1, t^{\frac{1}{2}}, t^{\frac{3}{2}}, t^{\frac{5}{2}}, t^{\frac{7}{2}}, t^{\frac{9}{2}}, t^{\frac{11}{2}}, t^{\frac{13}{2}}, t^{\frac{15}{2}}.$$

$$A_0(t) + A_1(t) = 1$$

$$A_1(t) + \frac{\sqrt{\pi}}{2}A_2(t) + \frac{\sqrt{\pi}}{2}A_3(t) = t^{\frac{1}{2}}$$

$$A_1(t) + \frac{3\sqrt{\pi}}{4}A_3(t) + \frac{3\sqrt{\pi}}{4}A_4(t) + \frac{3\sqrt{\pi}}{4}A_5(t) = t^{\frac{3}{2}}$$

$$A_1(t) + \frac{15\sqrt{\pi}}{16}A_3(t) + \frac{15\sqrt{\pi}}{8}A_5(t) + \frac{15\sqrt{\pi}}{8}A_6(t) + \frac{15\sqrt{\pi}}{8}A_{73}(t) = t^{\frac{5}{2}}$$

$$A_1(t) + \frac{35\sqrt{\pi}}{32}A_3(t) + \frac{105\sqrt{\pi}}{32}A_5(t) + \frac{105\sqrt{\pi}}{16}A_7(t) + \frac{105\sqrt{\pi}}{16}A_8(t) = t^{\frac{7}{2}}$$

$$A_1(t)+\frac{315\sqrt{\pi}}{256}A_3(t)+\frac{315\sqrt{\pi}}{64}A_5(t)+\frac{945\sqrt{\pi}}{64}A_7(t)=t^{\frac{9}{2}}$$

$$A_1(t)+\frac{693\sqrt{\pi}}{512}A_3(t)+\frac{3465\sqrt{\pi}}{512}A_5(t)+\frac{3465\sqrt{\pi}}{128}A_7(t)=t^{\frac{11}{2}}$$

$$A_1(t)+\frac{9009\sqrt{\pi}}{1536}A_3(t)+\frac{45045\sqrt{\pi}}{1024}A_5(t)+\frac{45045\sqrt{\pi}}{256}A_7(t)=t^{\frac{13}{2}}$$

$$A_1(t)+\frac{19305\sqrt{\pi}}{12288}A_3(t)+\frac{1351355\sqrt{\pi}}{12288}A_5(t)+\frac{1351355\sqrt{\pi}}{2048}A_7(t)=t^{\frac{15}{2}}$$

Solving the above system, we obtain

$$A_0(t)=1-\frac{224}{61}t^{\frac{15}{2}}-\frac{144}{671}t^{\frac{13}{2}}+\frac{12948}{671}t^{\frac{11}{2}}-\frac{1001}{61}t^{\frac{9}{2}}$$

$$A_1(t)=\frac{224}{61}t^{\frac{15}{2}}+\frac{144}{671}t^{\frac{13}{2}}-\frac{12948}{671}t^{\frac{11}{2}}+\frac{1001}{61}t^{\frac{9}{2}}$$

$$A_2(t)=\frac{-2\sqrt{t}}{287859\sqrt{\pi}}[696608t^7+22864t^6-3544580t^5+31112967t^4-287859]$$

$$A_3(t)=\frac{-2048t^{\frac{9}{2}}}{287859\sqrt{\pi}}[352t^3+38t^2-1963t+1573]$$

$$A_4(t)=\frac{-4t^{\frac{3}{2}}}{2747745\sqrt{\pi}}[1399328t^6+30672t^5-7325604t^4+6811519t^3-915915]$$

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Now, for a fixed $i \in \{0,1,\dots,n\}$, set $x = x_i + th, t \in [0,1]$. From the continuity condition of

$u^{\frac{7}{2}}(x_i^{-}) = u^{\frac{7}{2}}(x_i^{+})$ we arrive the above recurrence formula eq. (7). This completes the proof.

3.4. Error Estimation for the fractional lacunary polynomial of Degree 4:

Theorem 3. Suppose that $u(x)$ is the fractional lacunary polynomial defined in eq. (3) , $f^{(\frac{1}{2})}(x) \in C^8[0,1]$ and that $f^{(p)}(0) = 0, p = 1,2,3,4,5,6,7,8$. Thus for any $x \in [0,1]$ we have

$|s(x) - f(x)| \leq \frac{h^8}{128(8!)\sqrt{\pi}} \parallel f^{(\frac{17}{2})} \parallel$
(9)

Proof:

Since $u^{(\frac{1}{2})}(x)$ is the fractional Hermite polynomial of degree 4 interpolating

$f^{(\frac{1}{2})}, f^{(\frac{3}{2})}, f^{(\frac{5}{2})}, f^{(\frac{7}{2})}$ at $x = x_i, x_{i+1}$. Therefore, for any $x \in [x_i, x_{i+1}]$ and using eq. (1) with setting

$$m = 4, \qquad g = f^{(\frac{1}{2})}, \qquad and\ p_4 = u^{(\frac{1}{2})},$$

$|s(x) - f(x)| \leq \frac{h^8}{128(8!)\sqrt{\pi}} D^8 D^{\frac{1}{2}} f$

we have

$D^8 D^{\frac{1}{2}} f = D^{\frac{17}{2}} f = f^{(\frac{17}{2})}$ pp. 20 , [13]

this leads to

$$|s(x) - f(x)| \leq \frac{h^8}{128(8!)\sqrt{\pi}} \parallel f^{(\frac{17}{2})} \parallel$$

As required.

4. Numerical Results

Example 1: Consider the semi differential equation of order 6

$$y'''(t) + D^{\frac{1}{2}}y(t) + 2y(t) = 10e^{2t} + \sqrt{2}\, e^{2t}\, \mathrm{erf}\left(\sqrt{2t}\right),$$

With the initial conditions $y(0) = 1, y'(0) = 2$ and $y''(0) = 4$, where erf is the error function defined by $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2} dt$ and the exact solution is $y(t) = e^{2t}$ [12].

Example 2: Consider a linear fractional order differential equation

$$D^{\frac{1}{2}}y(x) = \cos(x) + x^{\frac{-1}{2}} Cos_{1,\frac{1}{2}}(x) - y(x)$$

with the condition $y(0) = 1$, the exact solution is $y(x) = cos(x)$ [13].

Table 1: Absolute error for Example 1

h	Exact solution	Degree 4	Method of [12]
0.05	1.105170918	6.5505×10^{-16}	0.11308×10^{-3}
0.1	1.221402758	8.9299×10^{-16}	0.70160×10^{-5}
0.15	1.349858808	1.2269×10^{-15}	0.43789×10^{-5}

Table 2: Absolute error for Example 2

h	Exact solution	Degree 4
0.05	0.9987502604	4.7191×10^{-20}
0.1	0.9950041653	2.9960×10^{-13}
0.15	0.9887710779	1.5116×10^{-12}

Table 3: Exact and Approximate solution for Example 2

x	Exact solution	Approximate solution
0	1	1
0.1	0.9950041	0.994795
0.2	0.9800665	0.979497
0.3	0.95533	0.954272

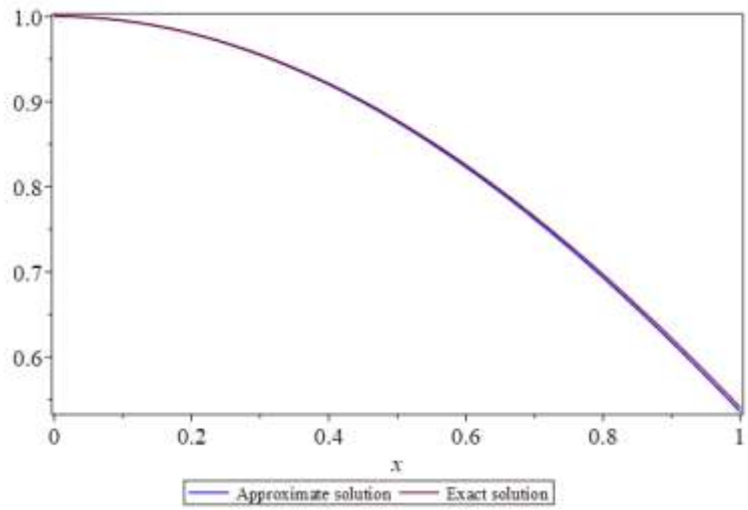

Figure 1. Exact and approximate results of Example 2

5. Conclusions

In this paper, we constructed a new fractional Lacunary polynomial spline function to solve a fractional differential equation. The uniqueness and error bounded are discussed. We applied this method on two examples and our results are very close to the analytic solution which are shown in the three tables and illustrated in a figure.

Conflict of Interests.
There are non-conflicts of interest .

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الخلاصة

هذا البحث يوضح كيفية تركيب لاكونري كسري لمتعددة حدود من الدرجة الرابعة بأستخدام دالة سبلاين. من ناحية أخرى, تمت مناقشة و اثبات الوحدوية و حدود الخطأ لهذه الدوال. الهدف من هذه الطريقة هو حل المعادلات التفاضلية الكسرية. تم توضيح كفاءة وملاءمة هذه الطرق الجديدة من خلال اعطاء أمثلة حسابية عددية