# π-Projective Semimodule Over Semiring

**Muna M.T. Altaee** **Asaad A. M. Alhossaini**
*College of Education for Pure Science, University of Babylon*

anc012.t3@gmail.com asaad_hosain@itnet.uobabylon.edu

**ARTICLE INFO**
**Submission date**: 25 /11/ 2019
**Acceptance date**: 11 / 3/ 2020
**Publication date**: 31/ 3 / 2020

## Abstract

Previously the concept of π-projective modules over ring was studied by some authors. The aim of this research is to give a comprehensive study of π-projective semimodule and access to some new properties and characterizations for this class of semimodules.

Let $S$ be a commutative semiring with identity $1\neq0$ and $T$ a unital left semimodule, then we say that $T$ is π-projective if for every two subsemimodules $M$ and $L$ of $T$ with $T=M+L$, there exist $f$ and $g$ $\epsilon\text{End}(T)$, such that $f+g=1_T$, $f(T)\subseteq M$ and $g(T)\subseteq L$.

**Key wards:** semisubtractive semimodule, subtractive subsemimodule, π-projective semimodule, quasi-projective semimodule, , dividing semimodule.

## 1. Introduction.

The concept of π- projective modules was studied by many authors, one of them is **[14]**.The definition π-projective modules was given by **[14, p.359]** **(**An $S$-module $T$. is π-projective if for every two submodules $C$ and $D$ of T with $T=C+D$, there exists a homomorphism $h\in \text{End}(T)$ with $h(T)\subseteq C$ and$(1-h)(T)\subseteq D$. Also some characterizations of this concept and some propositions related to this concept were appeared in **[1, p.359]** and by **[2]** the detail proofs were given.

Now in this research, $S$ denotes a commutative semiring with identity $1\neq0$ and $T$ a unitary left $S$-semimodule. Now the concept of π-projective will be for semimodule as follows:

An $S$-semimodule $T$ is said to be π-projective if for every two subsemimodules $M$ and $L$ of $T$ where $M+L=T$, there exist $f$ and $g$ $\epsilon\text{End}(T)$ such that $f+g=1_T$, $f(T)\subseteq M$ and $g(T)\subseteq L$.

**Section 2 consists the primitives related to the work.**

By section 3 the concept of π-projective semimodule will be introduced and investigated. Some interesting results, analogous to that in modules, also, obtained.

In section 4, other properties will be explained for the concept π-projective semimodule. In addition some related concepts will be introduced.

Some conditions have been added for some of the results in the modules to apply to semimodules.

## 2. Preliminaries

This section contains the primitives related to the research.

**Definition 2.1. [3 ]** Let $S$ be a semiring. A left $S$-semimodule $T$ is a commutative monoid $(T,+,0)$ such that a function $S \times T \rightarrow T$ defined by$(s, t) \rightarrow st$ ($s \in S$ and $t \in T$) such that for all $s$, $s' \in S$ and t, $t'' \in T$, the next conditions must be satisfied: (a) $s(t + t'')=st + st''$. (b) $(s+ s')\, t=st + s't$. (c) $ss'(t)=s(s't)$. (d) $0t=0$.Note: When $1t = t$ holds for each $t \in T$ implies that a left $S$-semimodule is said to be unitary, in this work $S$-semimodule means left unitary $S$-semimodule.

**Definition 2.2.[4]**Let $K$ be a subset of an $S$-semimodule $T$. If $K$ is closed under addition and scalar multiplication, then $K$ is said to be subsemimodule of $T$ (denoted by $K \subseteq T$).

**Definition 2.3. [4]**An $S$-subsemimodule $K$ is called subtractive if for every $c$, $d \in$ semimodule $T$,

$c$, $c + d \in K$ then $d \in K$.$\{0\}$ and $T$ are subtractive.

A semimodule $T$ is a subtractive if every subsemimodule of it is subtractive.

**Definition 2.4. [4]**A semimodule $T$ is called semisubtractive if for every $c$, $k \in T$ there exists $d \in$ T implies that $c=k + d$ or $k=c + d$.

**Definition 2.5.[5]** A semimodule $T$ is additively cancellative if $m + l= d + l$ then $m=d$ for all $m$, $l$, $d \in T$.

(CSS) denote to the semimodule that satisfy the three conditions, cancellative, semisubtractive and subtractive.

**Definition 2.6.[4]** let $M$ and $L$ be subsemimodules of a semimodule $T$. $T$ is said to be a direct sum of $M$ and $L$, denoted by $T=M \oplus L$if each $t \in T$ uniquely written as $t=m+l$ where $m \in M$ and $l \in L$ ,then we can say that $M$ (similarly $L$) is a direct summand of $T$.

**Remark 2.7.[6]** Let $T$ be (CSS) semimodule, then $T=L \oplus M$ if and only if $T=M+L$ and $M \cap L=0$.

**Definition 2.8.[4]** If $H$ and $K$ are semimodules, then a map $\beta$: $H \rightarrow K$ is said to be homomorphism if for all $d$, $d' \in H$ and $s \in S$ where $S$ is a semiring, the two cases are satisfy:

1. $\beta(d + d')=\beta(d)+\beta(d')$.
2. $\beta(sd)=s\beta(d)$.

For a homomorphism $\beta:H\rightarrow K$ of $S$-semimodules we define:

1. $\ker(\beta)=\{d \in H \mid \beta(d)=0\}$
2. monomorphism, If $\beta$ is one-one.
3. epimorphism, $\beta$ is onto.
4. isomorphism , if $\beta$ is one-one and onto.

For any $S$-semimodule $T$, End($T$) means the set of all endomorphisms of $T$ . In fact End($T$) is a semiring with usual addition and composition of maps in $T$[7].

**Definition 2.9.[3]** A subsemimodule $K$ is a small in a semimodule $T$ if for each subsemimodule $H$ of $T$, $T=K+H$ implies $H=T$.(denoted by $K \ll T$).

**Definition 2.10. [3]** A semimodule $T$ is said to be hollow if all its proper subsemimodules of $T$ are small.

**Definition 2.11. [8]** A subsemimodule $H$ of a semimodule $T$ is large in $T$ if for each subsemimodule $K$ of $T$, $H\cap K=0$, implies $K=0$.

**Definition 2.12. [7]** A semimodule $T$ is said to be uniform if all its non-zero subsemimodules $H$ of $T$ are large in $T$.

**Definition 2.13**. **[8]** A semimodule $T$ is called local if it has a largest proper subsemimodule.

**Definition 2.14.[5]** If $H$ is a subsemimodule of a semimodule $T$, then $T/H$ is called quotient (factor) semimodule of $T$ by $H$ , defined by $T/H=\{[t]|, t \in T\}$.

**Definition 2.17 [9, p.71]** A semimodule $T$ is said to be injective if for any monomorphism $h: C\rightarrow B$ and for every homomorphism $g: C\rightarrow T$, there is a homomorphism $\phi: B \rightarrow T$ such that $\phi h=g$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "h"] \arrow[d, "g"'] & B \arrow[dl, dashed, "\phi"] \\
 & T &
\end{tikzcd}$$

**Definition 2.18.[10]** A semimodule $T$ is said to be quasi-injective if for any monomorphism $\beta:C\rightarrow T$ and for any homomorphism $h: C\rightarrow T$, then there exists a homomorphism $\phi: T\rightarrow T$ such that $\phi\beta=h$.

$$\begin{tikzcd}
{} \arrow[r] & C \arrow[r, "\beta"] \arrow[d, "h"'] & T \arrow[dl, dotted, "\phi"] \\
 & T &
\end{tikzcd}$$

**Definition 2.19.[11, 3.1]**A semimodule $T$ is said to be $\pi$-injective if for every two subsemimodules $A$ and $B$ of $T$ with $A\cap B=0$,there exist $h$ and $q$ ϵEnd($T$) such that $h+q=1_T$, $h\subseteq\ker(h)$ and $q\subseteq\ker(q)$, and both of $h$ and $q$ are idempotent.

**Definition 2.20. [9, p.7]** A semimodule $T$ is said to be projective if for every epimorphism $h$:$K\to H$ and for any homomorphism $q$:$T\to H$, then there exists $g$:$T\to K$ such that $hg=q$.

$$\begin{tikzcd} & T \arrow[dl, dashed, "g"'] \arrow[d, "q"] \\ K \arrow[r, "h"'] & H \end{tikzcd}$$

**Definition 2.21.[10]** A semimodule $T$ is said to be quasi-projective if for any semimodule $K$, any epimorphism $f$:$T\to K$ and any homomorphism $q$:$T\to K$, then there exists $h\in$ End($T$) such that $fh=q$.

$$\begin{tikzcd} & T \arrow[dl, dashed, "h"'] \arrow[d, "q"] \\ T \arrow[r, "f"'] & K \end{tikzcd}$$

**Definition 2.22. [12]**Let $S$ be a semiring and let $I$ be a subset of $S$, $I$ will be left (resp. right) ideal of $S$ if for $m$ and $m'\in I$, and $s$ ϵ$S$, then $m+m'\in I$ and $sm\in I$ ($ms\in I$).

**Definition.2.23.[3]** A semiring $S$ is called local semiring, if the set $\{r\in T|\ r$ is (multiplicatively) non-invertible $\}$is an ideal of S.

**Remark 2.24.** A semiring $S$ is local if and only if the set of all noninvertible elements of $S$ is closed under addition.

Proof: By Definition**(2.23)**.

**Definition 2.25.[11, 3.7]** If $E$ is an injective semimodule, and it is essential extension of a semimodule $W$, then $E$ is said to be an injective hull(envelop) of $S$.

## 3. $\pi$-projective semimodule.

In this section the concept of $\pi$-projective semimodule and some of its own results with its proof will be presented.

**Definition 3.1**A semimodule $T$ is $\pi$-projective if for every two subsemimodules $M$ and $L$ with $M+L=T$ , then there exist $f$, $g$ ϵ End ($T$) such that $f+g=1_T$, $f(T)\subseteq M$ and $g$(T)$\subseteq L$.

**Note:**1. If $T=M\oplus L$, then $f=\pi M$ and $g=\pi L$ satisfies the conditions $f+g=1_T$, $f(T)\subseteq M$ and $g(\mathrm{T})\subseteq L$.

2. If $T=M\oplus L$ and $M$, $L$ are the only proper subsemimodules with $T=M+L$, then $T$ is π-projective by (1).

3. $T=\mathbb{Z}_6$ as $\mathbb{N}$-semimodule $T=2\mathbb{Z}_6 \oplus 3\mathbb{Z}_6$ and $2\mathbb{Z}_6$, $3\mathbb{Z}_6$ are the only proper subsemimodules of $T$, then $T$ is π-projective.

4.In fact $T=\mathbb{Z}_{pq}$ (with $p$ and $q$ are prim integers)is π-projective semimodule.

By Definition (3.1), it is clear that the following remark is true.

**Remark 3.2** If $T$ is a π-projective semimodule, with $T=M+L$, then there exist $f$ and $g\in$ End($T$) such that: $i$) $f(t)+g(t)=t$, for all $t \in T$.

$ii$) $t=f(t)+l$ and $t=m+g(t)$, for all $t \in T$, for some $m \in M$ and for some $l \in L$

Recall that a monomorphism $h:A\to B$ is split if there exists a homomorphism $q:B\to A$ such that $qh=1_A$ An epimorphism $q:B\to A$ is split if there exists a homomorphism $h:A\to B$ such that $qh=1_A$. [13, 3.9]]

In [1, p359] a characterization for π-projective modules was given. Analogously, in the following, a characterization for π-projective semimodules will be given.

**Proposition 3.3** Let $T$ be an S-semimodule and $T=M+L$, when $M$ and $L$ are any two subsemimodules of $T$ . Then $T$ is a π-projective if and only if the epimorphism $g$ from $M\oplus L$ onto $T$ which defined by $g(m, l)=m+l$, for all $m \in M$ and for all $l \in L$, splits.

**Proof:** Let $T$ be a π-projective semimodule, with $M+L=T$, then there exist $f$, $h \in$ End($T$) such that $f+h=1_T$, $\mathrm{f}(T)\subseteq M$ and $h(T)\subseteq L$. $g:M\oplus L\to T$ is an epimorphism defined by $g(m, l)=m+l$, for all $m \in M$ and for all $l \in L$. Let $q:T\to M\oplus L$ define by $q(t)=(f(t), h(t))$ , for all $\in T$. Since $gq=1_T$, then one can easy show that the homomorphism $g$ splits.

Conversely, let $M$ and $L$ be any two subsemimodules of $T$ such that $M+L=T$. Assume that $g:M\oplus L\to T$ is an epimorphism, defined by $g(m, l)=m+l$ , for all $m \in M$ and for all $l \in L$ splits. Thus there exists a homomorphism $q:T\to M\oplus L$, such that $gq=1_T$. Let $\pi_1:M\oplus L\to M$ and $\pi_2:M\oplus L\to L$ be the projections map. Now we define

$f'=\pi_1 q$ ,then $f'\epsilon\text{End}(T)$, and for all $t\ \epsilon T$, we have $f'(t)=\pi_1 q(t)=\pi_1(m,\ l)=m\ \epsilon M$ implies $f'(t)\ \epsilon M$, thus $f'(T)\subseteq M$. Similarly we can define $h'=\pi_2 q$, then $h'\ \epsilon\text{End}(T)$ and $h'(T)\subseteq L$. $f'(t)+h'(t)= \pi_1 q(t)+ \pi_2 q(t)=\pi_1 q(m+l)+\pi_2 q(m+l)=\pi_1(m,l)+\pi_2(m,l)= m +l=t$, for all $t\ \epsilon T$, for some $m\epsilon M$ and $l\ \epsilon L$, then $f' +h'=1_T$, hence $T$ is $\pi$-projective semimodule.

In [2, p36] a result for modules was given, in the following an analogous result for semimodules will be given.

**Proposition3.4** Every hollow semimodule is π-projective.

**Proof:** Since in a hollow semimodule, the sum of any proper subsemimodules is not equal to *T*, so *T* is π-projective. ▓

It clear that the converse of Proposition (3.4) in general is not true, see the note after Definition (3.1).

**Remark 3.5** Any local semimodule is hollow.

**Proof:** A local semimodule has a largest proper subsemimodule. So, the sum of any two proper subsemimodules is contained in a largest subsemimodule, hence is proper. this means that , a local semimodule is hollow. ▓

By Remark (3. 5), we have;

**Corollary 3.6** Every local semimodule is π-projective.

Proof: Clear. ▓

A result which appeared for modules in [1, 41.14], will be converted for semimodules in the following, by adding suitable conditions.

**Lemma 3.7** Let *T* be an *S*-semimodule. Then *T* is hollow if and only if every non-zero *T/D* semimodule is indecomposable.

**Proof:** (⟹) Let *T* be hollow semimodule such that it is a non-zero and let *T/H* be a factor semimodule of *T* also it is a non-zero, suppose that $T/D=A/D+B/D$, where *A*, *B* are subsemimodules of *T* containing *D*, since *T* is hollow, then either $A=T$ or $B=T$, hence either $T/D=A/D$ or $T/D=B/D$, therefore *T/D* is indecomposable.

($\Leftarrow$) Assum that every non-zero factor semimodule of $T$ is indecomposable. Let $C$, $D$ be proper subsemimodules of $T$ such that $T=C+D$. Now define $\Psi: T\rightarrow T/C \oplus T/D$ by $\Psi(t)=\Psi(x+ y)=(y+ C , x+ D)$, where $x \in C, y \in D$ and $t=x+ y$. To see that $\Psi$ is well defined, suppose that $t=x+ y=p+ s$, $p \in C$, $s \in D$. Since $T$is semisubtractive, then there exists $a \in T$ such that either $x+ a=p$ or $x=p+ a$, if $x+ a=p$, then $x+ y=x + a+ s$ implies $y=a+ s$ ($T$ is cancellative), since $D$ is subtractive, it follows $a \in D$. If $x=p+ a$, then $p+ a+ y=p+ s$ implies $a+ y=s$ (by $T$ is cancellative), then $a \in D$ ( $D$ is subtractive), then in the two cases $x+ D=p+ D$. Similarly we can write $y+ C=s+ C$ and this implies that$(y+ C, x+ D)=(s+ C, p+ D)$. Hence $\psi(x+ y)=\psi(p+ s)$. We claim that $\psi$ is an epimorphism. To verify this claim,

suppose that $(t_1+C, t_2+D) \in T/C \oplus T/D$ , where $t_1, t_2 \in T$, since $T=C+D$, let $t_1=c_1+d_1$, then $t_1+C=c_1+d_1+C=d_1+C$ and $t_2=c_2+d_2$ implies $t_2+D=c_2+d_2+D=c_2+D$, then $(t_1+C , t_2+D)=(d_1+C , c_1+D)=\Psi(c_1 , d_1)$, hence $\Psi$ is an epimorphism. Now by isomorphism theorem, $T/ \ker \Psi \cong T/C \oplus T/D$. Since $\ker \Psi=\{(x+ y)\in T \mid x, y \in C\cap D\}=C\cap D$ . On the other hand

$\Psi^{-1}(T/C)= \{ t \in T \mid \Psi(t) \in T/C \}=\{t \in T \mid t=x+y, x \in C\cap D, y \in D\}=D$, similarly

$\Psi^{-1}(T/D)=C$ which implies $(C/(C\cap D))\cap(D/(C\cap D))=0$ , hence $(C/(C\cap D))\oplus D/(C\cap D)=T/(C\cap D)$. This contradicts the assumption, so, either $C/(C\cap D)=0$ or $D/(C \cap D)$, that is, either $C\subseteq D$ or $D\subseteq C$ which means, $T=D$ or $T=C$. Hence $T$ is hollow. ▓

By [2, p. 36], there is another characterization of π-projective modules when the ring of endomorphisms of the module is local. Now in the following, this characterization will be converted for semimodules as follows:

**Proposition 3.**8 If $T$ is a semimodule with End($T$) is a local semiring. Then $T$ is a π-projective semimodule if and only if every non-zero factor semimodule $T/D$ of $T$ is indecomposable.

**Proof:** Let $T/D$ be a non-zero factor semimodule of a semimodule $T$, and let End($T$) be a local semiring. Assume that $T/D=(A/D)\oplus(B/D)$, where $A$ and $B$ are proper subsemimodules of $T$ containing $D$, then $T=A+B$, since $T$ is π-projective by assumption, there exist homomorphisms $f$, $g\in$ End($T$) such that $f(T)\subseteq A$ and $g(T)\subseteq B$.

and $f+g=1_T$, then either $f$ or $g$ is invertible (if both are noninvertible then there sum must be noninvertible, too since End($T$) is local semiring), (see Remark**(**2.24**)).** When $f$ is invertible, then $f$ is onto, hence $T=A$, and when $g$ is invertible, then $g$ is onto, hence $T=B$. Both cases contradict with the assumption that $A$ and $B$ are proper. Then $T/D$ is indecomposable.

Conversely, by Lemma (3.7) $T$ is hollow, then $T$ is π-projective
(Proposition**(**3.4**)).** ▓

A similar to the following result, appeared for modules in [2, p.38].

**Proposition 3.9** If $T$ is a quasi-projective semimodule, then it is π-projective.

**Proof:** Let $T$ be a quasi -projective semimodule and let $M$ and $L$ be subsemimodules of $T$ such that $M+L=T$. Consider the following diagram:

$$\begin{tikzcd} & T \arrow[dl, dashed, "g_i"'] \arrow[d, "f_i"] & \\ T \arrow[r, "\pi"'] & \frac{T}{M\cap L} \arrow[r] & 0 \end{tikzcd}$$

Where π is the natural epimorphism and $f_1:T\to\frac{T}{M\cap L}$ defined by $f_1(t)=f_1(m+l)=m+M\cap L$, where $t\in T$, $m\in M$, $l\in L$ and $t=m+l$. First to prove that $f_1$ is well defined. If $m+l=m'+l'$ , since $T$ is CSS, there exists $k\in M$ such that (1) $m=k+m'$, then $k+m'+l=m'+l'$ so $k+l=l'$ hence $k\in L$ and $k\in M\cap L$, or (2) $m+k=m'$, then $m+l=m+k+l'$ so $l=k+l'$ hence $k\in M\cap L$, from (1) and (2) $f_1(m+l)=f_1(m'+l')$. Since $T$ is quasi-projective, there exists a homomorphism $g_1:T\to T$ such that $\pi g_1=f_1$ that is $\pi(g_1(t))=f_1(t)$ which means $g_1(t)+(M\cap L)=m+(M\cap L)$, let $g_1(t)+l=m+l'$. Since $T$ is CSS, there exists $x\in T$ such that: (1) $m=x+g_1(t)$ which implies $l=x+l'$, hence $x\in L$ and so $x\in M\cap L$, or (2) $m+x=g_1(t)$ implies $x+l=l'$, then $x\in L$ hence $x\in M\cap L$. From (1) and (2) $f(g(t)+d)=f(m+l')$ implies $g_1(t)\in M$, hence $g_1(T)\subseteq M$. Similarly, when $f_2(t)=f_2(m+l)=l+(M\cap L)$ and $g_2$ exists with $\pi g_2=f_2$ and $g_2(T)\subseteq L$.

Now, for each $t\in T$ , $t=m+l$, $m\in M$ and $l\in L$, $m+M\cap L=f_1(t)=\pi(g_1(t))=g_1(t)+M\cap L$, this implies $m=g_1(t)+m_1$ for some $m_1\in M\cap L$(note that $m1$ is unique and depends on $t$). Define $h_1(t)=g_1(t)+m_1=m$. Similarly we have $h_2(t)=g_2(t)+l_1=l$, hence $h_1(t)+h_2(t)=$

$m+l=t$, that is $h_1+h_2=1_T$, and it is clear that $h_1(T)\subseteq M$ and $h_2(T)\subseteq L$. Therefore $T$ is $\pi$-projective. ▓

We must know that the converse of the last result is not true in general, for example $\mathbb{Z}_{p^n}$ as $\mathbb{N}$ -semimodule is $\pi$-projective, but not quasi-projective.

Note that: every projective semimodule is quasi-projective, then from Proposition(3.9),we have;

**Corollary 3.10** Every projective semimodule is $\pi$-projective.

**Proof:** By above note ▓

Recall that Hom($A, A'$) is the set of all homomorphisms from $A$ to $A'$ [7].

There are two important notions for a module equipped with $\pi$-projective module: dividing module [14] and uniserial module [15] here it will be converted for a semimodule as follows:

**Definition 3.11** An $S$-semimodule $T$ is dividing if for any two subsemimodules $M$ and $L$ of $T$; Hom($T$, $M+L$)=Hom($T$,$M$)+Hom($T$, $L$).

**Example 3.12** Every simple semimodule is dividing semimodule.

**Definition 3.13** An $S$-semimodule $T$ is called uniserial if for any two subsemimodules $M$ and $L$ of $T$, either $M\subseteq L$ or $L\subseteq M$.

**Example 3.14** $\mathbb{Z}_{p^n}$ as $\mathbb{N}$-semimodule is uniserial. ($\mathbb{Z}_8$, where $4\mathbb{Z}_8$ and $2\mathbb{Z}_8$ are two subsemimodules of $\mathbb{Z}_8$ and $4\mathbb{Z}_8\subseteq 2\mathbb{Z}_8$).

The following result which has been demonstrated by [2, p.40] for modules, in this work it will converted for semimodules.

**Proposition 3.15** Every dividing semimodule is $\pi$-projective.

**Proof:** Let $T$ be dividing semimodule and let $M$ and $L$ be two subsemimodules of $T$ such that $T=M+L$, since $T$ is dividing semimodule, then Hom($T$, $M+L$)= Hom($T$, $M$)+Hom($T$, $L$), but $T=M+L$ and $I\,\epsilon$Hom($T$, $T$), hence $I=f+g$ such that $f\,\epsilon$Hom($T$, $M$) and $g\,\epsilon$Hom($T$, $L$) implies that $f(T)\subseteq M$ and $g(T)\subseteq L$. Hence $f(T)+g(T)=1_T$, so, $T$ is $\pi$-projective. ▓

Note that every uniserial semimodule is dividing semimodule this implies the following corollary:

**Corollary 3.16** Every uniserial semimodule is $\pi$-projective.

**Proof:** Let $T$ be a uniserial semimodule, then $T$ is dividing semimodule and by Proposition (3.1.15) $T$ is $\pi$-projective. $\blacksquare$

The converse of Corollary(3. 16) in general is not true, for example $\mathbb{Z}_6$ as $\mathbb{N}$-semimodule is $\pi$-projective but it is not uniserial (because neither $2\mathbb{Z}_6 \subseteq 3\mathbb{Z}_6$ nor $3\mathbb{Z}_6 \subseteq 2\mathbb{Z}_6$, where all of them are a proper subsemimodules of $\mathbb{Z}_6$). But this Corollary is true under certain condition.

In [2, p.41] the following lemma was appeared for modules. Now it will be converted relative for semimodule.

**Lemma 3.17** If $T$ is $\pi$-injective indecomposable has an injective hull and it is quasi-injective, then $T$ is uniform and End($T$) is a local semiring.

**Proof:** Let $T$ be $\pi$-injective and indecomposable semimodule , then by [11,4.6]$T$ is uniform. To show that End($T$)is local, by Definition(1.24), we must prove that the set of noninvertible elements of End($T$) is closed under addition: assume that ($0\neq f$and $0\neq g$)$\in$ End($T$) such that $f+g$ is invertible, then $\ker(f+g)=0$, and so $\ker f \cap \ker g=0$(since $\ker(f+g)\supseteq \ker f \cap \ker g$), since $T$ is uniform, then either $\ker f=0$ or **ke**r $g$=0, that is either $f$ or $g$ is monomorphism. If $f$ is monomorphism, consider the diagram:

$$\begin{tikzcd}
0 \arrow[r] & f(T) \arrow[r, "i"] \arrow[d, "h"'] & T \arrow[dl, dashed, "\phi"] \\
& T &
\end{tikzcd}$$

where $i$ is the inclusion map and $h$: $f(T)\rightarrow T$ is defined by $h(f(t))=t$, for all $t \in T$, since $T$ is quasi-injective, then $\phi$: $T\rightarrow T$ such that $\phi\ i=h$. Claim that $f\ \phi$ a left inverse of $i$, to verify this claim: since $f\ \phi\ i=f\ (\phi i)=fh=I_{f(T)}$. Hence $T= f(T)\oplus L$, for some subsemimodule $L$ of $T$, since $T$ is indecomposable, then $L$=0$\rightarrow$ $f(T)=T$, that is $f$ is invertable. Similarly when $g$ is monomorphism. Thus End($T$) is a local semiring.

The following result will explain that the converse of Corollary (3.16) is true under certain conditions for a semimodule (for the module version see [2, p.42]).

**Proposition (3.18)** Let $T$ be π-projective and every factor semimodule of $T$ is π-injective has an injective hull, then the following cases hold:
*a*) When End($T$) is local, then $T$ is uniserial semimodule.

*b*) If $T$ is indecomposable, then $T$ is uniserial.

**Proof:** *a*) Assume that End($T$) is a local semiring, since $T$ is π-projective, then by proposition(3.7) all non-zero factor semimodule of $T$ are indecomposable and since all factor semimodule of $T$ is π-injective , then by [11, 2.6] every non-zero factor semimodule of $T$ is uniform. Let $K$ and $H$ be non-zero proper subsemimodules of $T$, then $T/(K\cap H)$ is a non-zero factor semimodule of $T$ which is uniform. Since $\frac{K}{K\cap H}\cap\frac{H}{K\cap H}=0$, then either $(K/(K\cap H))=0$ or $(H/(K\cap H))=0$, if $K/(K\cap H)=0\rightarrow K\cap H=K\rightarrow K\subseteq H$ and if $H/K\cap H=0\rightarrow K\cap H=H\rightarrow H\subseteq K$. Thus $T$ is uniserial.

b) Since all factor semimodule of $T$ is π-injective and $T$ is indecomposable, then by Lemma(3.17) End($T$) is local semiring, then by (a)$T$ is uniserial semimodule.

## 4. Some properties of π-projective semimodules.

This section will gives some properties of π-projective semimodule with the detail of proofs. It will start with the following proposition, which was appeared for modules in [1, 41.14].

**Proposition 4. 1** Let $T=M+L$ be π-projective semimodule and if $M$ is a direct summand of $T$, then there exists a subsemimodule $L'$ of $L$ such that $T=M\oplus L'$.

**Proof:** Since $M$ is a direct summand of $T$, then $T=M\oplus K$ for a suitable subsemimodule $K$ of $T$. Since $T$ is π-projective semimodule with $T=M+L$, there exist $h$ and $q\in$End($T$) such that $h+q=1_T$, $h(T)\subseteq M$ and q($T$)$\subseteq L$. Claim that $q(M)\subseteq M$ and $T=M\oplus q(K)$. To verify this claim: let $k\in q(M)$, then $q(m)=k$ for some $m\in M$, by Remark**(**3. 2**)**, $h(m)+q(m)=m$, then $h(m)+k=m$, since $h(T)\subseteq M$ implies $h(m)\in M$, and $T$ is subtractive semimodule, then $k\in M$. It is clear that $q(K)\subseteq L$. Now to prove $T=M+q(K)$, since $T=M\oplus K$, then $q(T)=q(M)+q(K)\subseteq M+q(K)$. Hence $T=h(T)+q(T)\subseteq M+M+q(K)=M+q(K)$, which implies $T=M+q(K)$. Let $t\in(M\cap q(K))$, then $t\in M$ and $t\in q(K)$, then $t=q(k)$ for some $k\in K$, since $h(k)+q(k)=k$, so $h(k)+t=k\in M$ ($t\in M$ and $h(k)\in M$), hence $k\in$

$M\cap K=0$, then $k=0=h(k)+t$ .Thus $t=0$ and $M\cap q(k)=0$. Hence $T=M\oplus q(K)$. Let $L'=q(K)$, then $T=M\oplus L'$.

The two following results which are needed later in this work, have module versions in [9, p.17].

**Proposition 4.2**Let $T$ be an $S$- semimodule and let$\{X_i\}_{i\in I}$ be a set of $S$-semimodules, then:

1) $\oplus_{i\in I} Xi$ is T-projective if and only if $X_i$ is $T$-projective for all $i\in I$

2) If the semimodule $T$ is $X_i$-projective for finitely many semimodules $X_1$, $X_2$, …, $X_n$, then $T$ is $\oplus_{i=1}^{n} X_i$ -projective.

**Proof:**1) $\Longrightarrow$ Suppose that $\oplus_{k\epsilon I} X_k$ is $T$-projective and consider the following diagram:

$$\begin{tikzcd}
\oplus_{k\in I} X_k \arrow[r, shift left, "\pi_i"] \arrow[d, "\beta"'] & X_i \arrow[l, shift left, "j_i"] \arrow[d, "q"] \arrow[dl, dashed, "\beta_i"'] & \\
T \arrow[r, "g"'] & K \arrow[r] & 0
\end{tikzcd}$$

where $q:X_i \to K$ is any homomorphism($K$ is any semimodule), $g:T\to K$ is an epimorphism, $\pi_i$ are the projection map from $\oplus_{k\epsilon I} X_k$ onto $X_i$ and $j_i$ are the injection map of $X_i$ into $\oplus_{k\epsilon I} X_k$, $k\epsilon I$ . Since $\oplus_{k\epsilon I} X_k$ is $T$-projective, then there exists a homomorphism $\beta:\oplus_{k\epsilon I} X_k\to T$ such that $g\beta = q\,\pi_i$. Define $\beta_i:X_i\to T$ by $\beta_i=\beta\, j_i$, hence $g\,\beta_i=g\,\beta\, j_i= q\,\pi_i\, j_i= q$ ($\pi_i\, j_i= 1_{X_i}$). Thus $X_i$ is $T$-projective for every $i\epsilon I$

suppose that $X_i$ is $T$-projective for every $i\epsilon I$ and consider the following diagram: $\Longleftarrow$

$$\begin{tikzcd}
X_i \arrow[r, "j_i"] \arrow[d, "\delta_i"'] & \oplus_{k\in I} X_k \arrow[d, "q"] \arrow[dl, dashed, "\delta"'] & \\
T \arrow[r, "g"'] & K \arrow[r] & o
\end{tikzcd}$$

where $K$ is a semimodule and $g:T\to K$ is an epimorphism, $q:\oplus_{k\epsilon I} X_k\to K$ is any homomorphism and $j_i:X_i\to\oplus_{k\epsilon I} X_k$ is the injection map. Since $X_i$ is $T$-projective for all $i\,\epsilon I$, there exists a homomorphism $\delta_i :X_i \to T$ for each $i\epsilon I$ such that $g\,\delta_i =qj_i$ for all $i\epsilon I$.

Define $\delta:\oplus_{k\epsilon I} X_k \to T$ by $\delta\ ((x_i)) = \sum_{k\epsilon I} \delta_k(x_k)$, where $(x_i)\ \epsilon \oplus_{k\epsilon I} X_k$. Since the sum is finite, then $\delta$ is well defined and it is clear that $\delta$ is a homomorphism. Let $(x_k)\epsilon \oplus_{k\epsilon I} X_k$ then $g(\delta\ ((x_k))) = g(\sum_{k\epsilon I} \delta_k(x_k) = \sum_{k\epsilon I} g\delta_k(x_k) = \sum_{k\epsilon I} q\ jk(x_k)) = q((x_k))$, where $\sum_{k\epsilon I} jk\ (x_k)) = (x_k)$.Hence $g\ \delta\ ((x_i)) = q((x_i))$. Thus $g\ \delta = q$. That is, $\oplus_{k\epsilon I} X_k$ is a $T$-projective semimodule.

2) The proof can be found in [9,p.17].

By[1, 41.14] for modules the following results were appeared. Here it will be proved for semimodules.

**Proposition 4.3** Let $T=K\oplus D$ be a $\pi$-projective semimodule, then $D$ is $K$-projective(and $K$ is $D$-projective).

Proof: let $q:K\to L$ be an epimorphism where $L$ is an $S$-semimodule, and let $h:D\to L$ be any homomorphism. Consider the following diagram:

$$\begin{tikzcd} & D \arrow[dl, dashed, "g"'] \arrow[d, "h"] & \\ K \arrow[r, "q"'] & L \arrow[r] & 0 \end{tikzcd}$$

Now to show that there e $g$ $\to K$ such that $qg=h$. Since $q$ is epimorphism, then for each $d\epsilon D$, there exits $k\epsilon K$ such that $q(k)=h(d)$. Let $X=\{b\epsilon T|\ b+k=d$, for $d\epsilon D$, $k\epsilon K$ and $q(k)=h(d)\}$. surly that $X\neq \phi$ and is a subsemimodule of $T$, so $T=K+X$ to see this , let $t\epsilon T$, then $t=k+d$ for some $k\epsilon K$ and for some $d\epsilon D$, $h(d)\epsilon L$, since $q$ is epimorphism and there exists $k'\epsilon K$ such that $q(k')=h(d)$ there exists $b\epsilon X$ such that $b+k'=d$ , but $t=k+d=k+b+k'=((k+k')+b)\epsilon K+X$, then $T=K+X$. By Proposition(3.2. 1) there exists $X'\subseteq X$ with $T=K\oplus X'$. Let $i: D\to T$ be the inclusion homomorphism and let $\pi:K\oplus X'\to K$ be the natural projection map. Let $g=\pi\ I$, then $(q\ g)(y)=(q\ \pi\ i)(y)=q\ \pi(k+a)$ for some $k\epsilon K$ and for some $a\ \epsilon\ X'$ with $y=k+a$, $(q\ g)(y)=(q\ \pi)(k+a)=q(k)$ , since $y=k+a$ and $a\epsilon X'\subseteq X$ implies that $q(k)=h(y)$ , thus $qg=h$. ▓

**Proposition 4.4** Let $T=K\oplus H$ be a $\pi$-projective semimodule with $K\simeq H$, then $T$ is quasi-projective.

**Proof:** By Proposition(4.3) $K$ is $H$-projective, since $K\simeq H$, then $K$ is $K$-projective. Similarly $H$ is $H$- projective. By Proposition(4.2) $K$ is $K\oplus H$-projective and $H$ is

$K \oplus H$-projective. Also by Proposition(4.1) $K \oplus H$ is $K \oplus H$-projective, hence $T$ is quasi-projective. ▓

The next definition which is needed to prove the following proposition analogues to that in modules [16].

**Definition 4.5** An $S$-semimodule $T$ is said to be completely $\pi$-projective if every subsemimodules of $T$ are $\pi$-projective.

**Example 4.6** $\mathbb{Z}_6$ as $\mathbb{N}$-semimodule is $\pi$-projective, and [$\{0\}$, $2\mathbb{Z}_6$ and $3\mathbb{Z}_6$] which are only proper subsemimodules of $\mathbb{Z}_6$ are $\pi$-projective, then $\mathbb{Z}_6$ is a completely $\pi$-projective.

The end of this section will be with the following proposition for semimodules. The module version appeared in [2, p52] .

**proposition 4.6** Let $T$ be a completely $\pi$-projective semimodule and $T=T_1 \oplus T_2 \oplus \ldots \oplus T_n$ with hollow semimodules $T_i$, for all $i$, $i$=1, 2, …,$n$. Then:

1) Every non-zero $h \in$ Hom($T_i$, $T_j$), $i \neq j$ is a monomorphism. If $T_i$ is $T_j$-injective, then $h$ is an isomorphism.

2) If some of the non-zero $h \in$ End($T_j$) is monomorphism, then Hom($T_i$, $T_j$)=0, for all $i \neq j$.

**Proof:**1) Let $h{:}T_i \to$h($T_j$) be a non-zero homomorphism where , $i \neq j$ then $T_i \oplus h(T_i)$ is a subsemimodule of $T$, since $T$ is completely $\pi$-projective , then $T_i \oplus h(T_i)$ is $\pi$-projective and by Proposition (4.3) $h(T_i)$ is $T_i$-projective, hence there exists a homomorphism $g{:}h(T_i) \to T_i$ such that the following diagram is commutative:

$$\begin{tikzcd} & h\,(T_i) \arrow[dl, dashed, "g"'] \arrow[d, "I"] & \\ T_i \arrow[r, "h"'] & h(T_i) \arrow[r] & 0 \end{tikzcd}$$

Then $h\,g=I$, where $I$ is the identity map. Thus $T_i=g(h(T_i)) \oplus \ker h$, but by Lemma(3.7) $T_i$ is indecomposable , since $g(h(T_i)) \neq 0$, then ker $h$=0, thus $h$ is (one to one)..

Let $T_i$ is $T_j$-injective and consider the next diagram:

$$\begin{tikzcd} 0 \arrow[r] & T_i \arrow[r, "h"] \arrow[d] & T_j \arrow[dl, dashed, "q"] \\ & T_i & \end{tikzcd}$$

There exists a homomorphism $q$: $T_j \to T_i$ such that $qh=I$, then $h(T_i)$ is a direct summand of $T_j$, but $T_j$ is indecomposable , then $h$ is onto . Hence $h$ is isomorphism.

2) Let $p:T_j \to P(T_j)$ be a homomorphism and is not one- to- one, assume that there is a non-zero homomorphism $h:T_j \to T_i$, where $i \neq j$. By 1) $h$ is monomorphism. , since $T$ is completely π-projective , then $T_i \oplus p(T_j)$ is π-projective and by Proposition (4.3) $p(T_j)$ is $T_i$-projective, since $h:T_j \to T_i$ is monomorphism, then by Proposition (4.4) $p(T_j)$ is $T_j$-projective, consider the following diagram:

$$\begin{tikzcd} & p(T_j) \arrow[dl, "g"'] \arrow[d, "I"] \\ T_j \arrow[r, "P"'] & p(T_j) \end{tikzcd}$$

But $p(T_j)$ is $Ti$ -projective, there exists $g$: $p(T_j) \to T_j$ such that $p\, g=I$ and hence ker$p$ is direct summand of $T_j$, ker $p \neq 0$ and ker $p \neq T_j$, and this a contradiction(since $T_j$ is indecomposable). ▓

**Conflict of Interests.**
**There are non-conflicts of interest .**

## References


(1) R. Wisbauer, Foundations of module and ring theory, Gordon and Breach science publishers, Raiding 1991.

(2) A. Alaa . Elewi "Some result of π-projective modules", Ph. D.2006; dissertation University of Baghdad.

(3) NX. H. Tuyen, Thang HX. On superfluous subsemimodules. Georgian Math J. 2003; 10(4)763-77.

(4) A. M. Alhossaini and Z. A. Aljebory, "Fully Dual Stable Semimodule", rnaljou of Iraqi Al-khwarizmi, vol. 1, no. 1, 92-100, 2017.

(5) A. M. Alhossaini and Z. A. Aljebory "On P-duo semimodule" 2018; Journal of University of Babylon, pure and applied science Vol.26 , no. 4, 27-35.

(6) A. M. Alhossaini, K. S. Aljebory, "The Jacobson Radical of The Endomorphism Semiring of a P.Q.-Injective Semimodules", Baghdad Science Journal, to appear, 2019.

(7) H . Abdul Ameer, Husain A. M. Fully stable semimodule. Al-Bahir Quarterly Adj J for Natural an Engineering Research and studies. 2017; 5(9 and10)

(8) E. Diop. Sow, On Essential Subsemimodules and Weakly Co-Hopfion Semimodules. European Journal of pur and applied Mathematics. 2016, 9(3):250-265.

(9) H. M. J. Al-thani, "Projective and Injective Semimodules over Semirings", .Ph. D dissertation, East London Univ., 1998.

(10) Ahsan, shabeir and Weinert. characterizations semiring by semimodule P-injective and projective semimodules communication algebra. 26(7)2199-2209(1998).

(11) A. M. Alhossaini , M. T. Altaee."π-injective semimodule over semiring" Journal of Engineering and Applied sciences vol :16, no. 11, 2019.

(12) A. M. Alhossaini, K. S. Aljebory "Principally Q-injective semimodule", sci J, 2019.

(13) A. M. Alhossaini, K. S. Aljebory" Principally Pseudo-Injective Semimodule" Journal of University of Babylon for Pure and Applied Sciences,Vol.(27), No.(4): 2019.

(14) A.A. Tuganbaev,Modules over bounded Dedekind Prim rings, Mat.sb. 192:5(2001) 65-86

(15) F.G. Ivannov, Decomposition of modules over serial rings, comm. Algebra. Algebra. 3(11)(1975), 1031-036.

(16) A.A. Tuganbaev, Modules over hereditary rings, Mat. Sb.189:4(1998), 143-160.



## الخلاصة

سابقا تم دراسة مفهوم المقاس الاسقاطي من النوع π على الحلقة من قبل عدة مؤلفين. في هذا البحث هذا المفهوم سيقدم ويعمم لشبه المقاس على شبه الحلقة. ليكن*T* شبه مقاس يساري وحدوي, فنقول انه اسقاطي من النوع π اذا كان لكل شبه مقاسين جزئيين منه بشرط ان شبه المقاس يساوي مجموع هذين الشبه المقاسين الجزئيين, فيوجد تشاكلين بحيث ان التشاكل الاول مجموعة جزئية من احد شبه المقاسين الجزئيين, والتشاكل الثاني مجموعة جزئية من الاخر و مجموع التشاكلين يساوي الدالة الاحادية بالنسبة لشبه المقاس المعطى.


**الكلمات الدالة:** شبه مقاس جزئي شبه طرح, شبه مقاس طرح, شبه مقاس اسقاطي نمط π , شبه مقاس شبه اسقاطي, شبه مقاس تقسيمي.