Solution Techniques Based on Adomian and Modified Adomian Decomposition for Nonlinear Integro-Fractional Differential Equations of the Volterra-Hammerstein Type

Shokhan A. Hamasalih
Department of Mathematics, College of Science, University of Sulaimani, Sulaimanyah,
Iraq.shokhan.hamasalih@univsul.edu.iq
Mariwan R. Ahmed
Department of Science, College of Basic Education, Charmo University, Chamchamal,Kurdistan Region
Iraq.mariwanmath96@yahoo.com
Shazad Shawki Ahmed
Department of Mathematics, College of Science, University of Sulaimani, Sulaimanyah, Iraq.
shazad.ahmed@univsul.edu.iq

ARTICLE INFO
Submission date: 20 / 11/ 2019
Acceptance date: 31 /12/ 2019
Publication date: 31/ 3 / 2020

Abstract

This paper efficiently applies the Adomian Decomposition Method and Modified Adomian Decomposition Method as computational techniques to locate the semi-analytical solution or semi-approximate solution for the considered nonlinear Integro Differential Equations for the fractional-order (IFDE) of the Volterra-Hammerstein (V-H) type, in which the higher-multi fractional derivative is described in the Caputo sense.In this procedure, we radically change the IFDE's of V-H type into some iterative algebraic equations and the solution of this equations is considered as the sum of the countless sequence of components typically converging to the solution based on the noise terms where a closed-form solution is not obtainable, a truncated number of terms is usually used for numerical purposes.Finally, examples are prepared to illustrate these considerations.

Keywords : Integro-fractional differential equation, Caputo-fractional derivative, Adomian decomposition method, Modified Adomian decomposition method, Noise term phenomenon.

1. Introduction

The nonlinear integro-fractional differential equation is an important branch of modern mathematics. Such equations occur in various areas of applied mathematics, physical phenomena and bioengineering [1,2,3]. Volterra-Hammerstein is one of important nonlinear types, which arises in various branches of applications such as heat conduction in materials with memory. Moreover, these equations are encountered in combined conduction, convection and radiation problems [4].

Adomian decomposition method (ADM) and Modified Adomian Decomposition method (MADM) have gained great interest in the user through many authors and researches to solve the problems in applied sciences such as the differential, fractional-order derivatives, integral and integro-differential equations [4,5,6,7,8]. In this paper, we extend this technique to further deal with considering problems.

The idea of this work is to find a semi-analytical solution or semi-approximate solution for multi-higher nonlinear Integro-Fractional Differential Equations (IFDE) of Volterra-Hammerstein (V-H) type with the variable coefficients in the general form as:

$$ {}_a^CD_t^{\alpha_n}u(t) + \sum_{i=1}^{n-1} P_i(t)\,{}_a^CD_t^{\alpha_i}u(t) + P_n(t)u(t) = f(t) + \sum_{\ell=0}^{m} \lambda_\ell \int_a^t \mathcal{K}_\ell(t,s)\mathcal{H}_\ell\left(s\,,{}_a^CD_s^{\beta_\ell}u(s)\right)ds \quad (1) $$

For all $a \le t \le \mathcal{T}$, with the initial conditions:

$$ u^k(a) = u_k \in \mathbb{R}\ , k = 0,1,\cdots,\mu-1\ ;\ \mu = \max\{\lceil\alpha_n\rceil, \lceil\beta_m\rceil\} \quad (2) $$

Where $u(t)$is the unknown function which is the solution of equation (1) under initial condition (2), as well as, the functions $\mathcal{K}_\ell: S \to \mathbb{R}$ with $(S = \{(t,s): a \le s \le t \le \mathcal{T}\}\ ;\ \mathcal{H}_\ell: S_* \times \mathbb{R} \to \mathbb{R}\ (S_* = \{s: a \le s \le t\ ;\ t \le \mathcal{T}\});\ \ell = 0\,,1\,,\cdots,m\,,$ and$f, P_i \in C([a,\mathcal{T}],\mathbb{R})$ for all $i = 1,2,\cdots,n$. In addition, $\alpha_i, \beta_\ell \in \mathbb{R}^+, \ell = 1,2,\cdots,m$and $i = 1,2,\cdots,n$with property that $\alpha_n > \alpha_{n-1} > \cdots > \alpha_1 > \alpha_0 = 0$, $\beta_m > \beta_{m-1} > \cdots > \beta_1 > \beta_0 = 0$ and λ_ℓ(for allℓ) are the scalar parameters. Where ${}_a^CD_t^{\rho}$denotes the Caputo fractional derivative of order $\rho \in \mathbb{R}^+$.

The structure of this paper is organized as follows: Section 2 presents the necessary definitions and basic preliminaries of the fractional derivatives and fractional integration, section 3 the basic concept of the Adomian decomposition method, section 4 devoted to formulation of ADM and MADM for solving nonlinear IFDE of V-H type and our results illustrated throughout examples in section 5. Finally, section 6 includes a discussion for these methods.

2. Fractional Order Derivative and Integral

For the concept of fractional derivative and fractional integration, we present some basic definitions and properties about these operators which are used throughout this paper [1,2,3,9]. We will adopt one of the most useable fractional derivatives which are a modification of the Riemann-Liouville operator namely Caputo derivative and has the advantage of dealing properly with initial value problems in which the initial conditions are given in their integer-order which is the case in most applied processes [10].For more details, see [1,2,3,9,10,11,12]:

Definition 2.1:A real-valued function u defined on a closed bounded interval$[a,b]$ be in the spaceC_μ, $\mu \in \mathbb{R}$, if there exists a real number $k > \mu$, such that $u(t) = (t-a)^k u_c(t)$, where $u_c \in C[a,b]$ and it is said to be in the space C_μ^n on interval $[a,b]$ where n-positive integer number with zero, if and only if $u^{(n)} \in C_\mu$.

Definition 2.2: Let $u \in C_\mu, \mu \geq -1$on a closed bounded interval$[a,b]$and $\alpha \in \mathbb{R}^+$. Then the operator ${}_aJ_t^\alpha u$ is the Riemann-Liouville (R-L) fractional integral operator of order α of a function u, is defined as:

$${}_aJ_t^\alpha u(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_a^t (t-\vartheta)^{\alpha-1} u(\vartheta) d\vartheta, & \alpha > 0 \\ u(t), \quad whenver \ \alpha = 0 \end{cases}$$

Hence, For all $\alpha, \beta \geq 0,\ \gamma > -1$ and $u \in C_\mu, \mu \geq -1$ on the closed interval $[a,b]$, we have:

$${}_aJ_t^\alpha\ {}_aJ_t^\beta u(t) = {}_aJ_t^\beta\ {}_aJ_t^\alpha u(t) = {}_aJ_t^{\alpha+\beta} u(t)$$

$${}_aJ_t^\alpha (t-a)^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} (t-a)^{\gamma+\alpha}, t > a$$

Definition 2.3: The Caputo fractional derivative operator ${}_a^CD_t^\alpha$ of order $\alpha \in \mathbb{R}^+$ of a function $u \in C_{-1}^m$ on the closed bounded interval $[a,b]$ and $m-1 < \alpha \leq m, m \in \mathbb{Z}^+$ is defined as:

$${}_a^CD_t^\alpha u(t) = \begin{cases} {}_aJ_t^{m-\alpha}[D_t^m u(t)] = \dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_a^t (t-\vartheta)^{m-\alpha-1} u^{(m)}(\vartheta) d\vartheta, \alpha > 0, t > a \\ u(t)\text{whenever} \quad \alpha = 0 \\ \qquad u^{(m)}(t), \text{If } \alpha = m (\in \mathbb{Z}^+) \ \text{ and } \ u \in \ C^m[a,b] \end{cases}$$

Hence, we mention only the following properties about the derivative operator ${}_a^CD_t^\alpha$:

- For any $\alpha \geq 0, \alpha \notin \mathbb{N}$ and $\mathcal{C}$ any real constant then${}_a^CD_t^\alpha \mathcal{C} = 0$.
- Assume that on any closed bounded interval $[a,b]$,$u \in C_{-1}^m$; $\alpha \geq 0, \alpha \notin \mathbb{N}$ and $m = \lceil \alpha \rceil$ then ${}_a^CD_t^\alpha u(t)$ is continuous on $[a,b]$, and$[{}_a^CD_t^\alpha u(t)]_{t=a} = 0$.
- Let $\alpha \geq 0$; $m = \lceil \alpha \rceil$ and for $u(t) = (t-a)^\gamma$for some $\gamma \geq 0$. Then:

$${}_a^CD_t^\alpha u(t) = \begin{cases} 0 & if\ \gamma \in \{0,1,2,\cdots,m-1\} \\ \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} (t-a)^{\gamma-\alpha} & if\ \gamma \in \mathbb{N} \text{ and } \gamma \geq m \\ & or\ \gamma \notin \mathbb{N} \text{ and } \gamma > m-1 \end{cases}$$

Also, here we need some basic lemmas [1,2,10]:

Lemma (1): Let F be continuous function on$[a,b] \times [a,b]$. Then for $\alpha \geq 0$

$$ {}_aJ_t^\alpha \int_a^t F(t,s)ds = \int_a^t {}_sJ_t^\alpha F(t,s)ds \ , t \in [a,b] $$

Lemma (2): (The Caputo derivative is left inverse of the RL-integral but not right inverse):

i. If u is continuous on closed bounded interval $[a,b]$ and $\alpha \geq 0$ with $m-1 < \alpha \leq m \ (\in \mathbb{N})$, then ${}_a^CD_t^\alpha \ {}_aJ_t^\alpha u(t) = u(t)$.

ii. Assume that $\alpha \geq 0$, $m = \lceil \alpha \rceil$, and $u \in C^m[a,b]$. Then

$$ {}_aJ_t^\alpha \, {}_a^CD_t^\alpha u(t) = u(t) - \sum_{k=0}^{m-1} \frac{u^{(k)}(a)}{k!}(t-a)^k $$

Lemma (3):Let $\alpha > \beta \geq 0$, $m_\alpha - 1 < \alpha \leq m_\alpha$ and $m_\beta - 1 < \beta \leq m_\beta$ $(m_\alpha, m_\beta \in \mathbb{N})$ be such that $u(t) \in C^{m_\beta}[a,b]$. Then

$$ {}_aJ_t^\alpha \, {}_a^CD_t^\beta u(t) = \ {}_aJ_t^{\alpha-\beta} u(t) - \sum_{k=0}^{m_\beta - 1} \frac{u^{(k)}(a)}{\Gamma(k+\alpha-\beta+1)}(t-a)^{k+\alpha-\beta} $$

3. Basic Idea of the Adomian Decomposition Method: [8,13,14,15,16]

In the 1980t's, George Adomian introduced a very simple and effective technique for solving nonlinear functional equations. His technique is known as the Adomian decomposition method (ADM) [13]. This method is based totally on the representation of the unknown function $u(t)$ in the functional equation as an infinite number of functions $u_r(t), r \geq 0$ described through the decomposition series:

$$ u(t) = \sum_{r=0}^{\infty} u_r(t) \quad (3) $$

Each term of the series $u_0(t), u_1(t), u_2(t), \cdots$ are to be determined in a recursive manner. However the nonlinear term $N(u(t))$ in the functional equationscan be decomposed into an infinite series of Adomian polynomials $A_n's$ which depending on $u_0, u_1, \cdots, u_n$:

$$ N\big(u(t)\big) = \sum_{n=0}^{\infty} A_n[u_0(t), u_1(t), \cdots, u_n(t)] $$

The uniqueness of the Adomian polynomial isn't always required at all which we are going to apply the Taylor expansion of $N(u(t))$ regarding the first component-part $u_0(t)$ to induce the forms as follows:

$$ N\big(u(t)\big) = \sum_{n=0}^{\infty} \frac{\big(u(t) - u_0(t)\big)^n}{n!} N^{(n)}\big(u_0(t)\big) $$

Since from ADM $u(t) = \sum_{r=0}^{\infty} u_r(t) = u_0(t) + u_1(t) + u_2(t) + \cdots$ substituting this in above expansion we get

$$N\big(u(t)\big) = N\big(u_0(t)\big) + N'\big(u_0(t)\big)\big(u_1(t) + u_2(t) + u_3(t) + \cdots\big) + \frac{1}{2!}N''\big(u_0(t)\big)\big(u_1(t) + u_2(t) + u_3(t) + \cdots\big)^2 + \frac{1}{3!}N'''\big(u_0(t)\big)\big(u_1(t) + u_2(t) + u_3(t) + \cdots\big)^3 + \cdots$$

Then by expanding all terms we get:

$$\begin{aligned} N\big(u(t)\big) = {} & N\big(u_0(t)\big) + N'\big(u_0(t)\big)u_1(t) + N'\big(u_0(t)\big)u_2(t) + N'\big(u_0(t)\big)u_3(t) + \cdots \\ & + \frac{1}{2!}N''\big(u_0(t)\big)u_1^2(t) + \frac{2}{2!}N''\big(u_0(t)\big)u_1(t)u_2(t) + \frac{1}{2!}N''\big(u_0(t)\big)u_2^2(t) \\ & + \frac{2}{2!}N''\big(u_0(t)\big)u_1(t)u_3(t) + \cdots + \frac{1}{3!}N'''\big(u_0(t)\big)u_1^3(t) \\ & + \frac{3}{3!}N'''\big(u_0(t)\big)u_1^2(t)u_2(t) + \frac{1}{3!}N'''\big(u_0(t)\big)u_1^2(t)u_3(t) \\ & + \frac{1}{3!}N'''\big(u_0(t)\big)u_1(t)u_2(t)u_3(t) + \cdots \end{aligned}$$

And by reordering the terms and determining the order of each term which depends on both the subscripts and the exponent of the u_n's. The order of $u_n^m u_k^l$ is $mn + kl$, for example $u_2^3 u_1^2$ is of order $(2 * 3) + (1 * 2) = 6 + 2 = 8$ and so on. Therefore, we obtain

$$\begin{aligned} N\big(u(t)\big) = {} & N\big(u_0(t)\big) + N'\big(u_0(t)\big)u_1(t) + N'\big(u_0(t)\big)u_2(t) + \frac{1}{2!}N''\big(u_0(t)\big)u_1^2(t) \\ & + N'\big(u_0(t)\big)u_3(t) + \frac{2}{2!}N''\big(u_0(t)\big)u_1(t)u_2(t) + \frac{1}{3!}N'''\big(u_0(t)\big)u_1^3(t) \\ & + N'\big(u_0(t)\big)u_4(t) + \frac{1}{2!}N''\big(u_0(t)\big)u_2^2(t) + \frac{2}{2!}N''\big(u_0(t)\big)u_1(t)u_3(t) \\ & + \frac{3}{3!}N'''\big(u_0(t)\big)u_1^2(t)u_2(t) + \cdots \end{aligned}$$

By comparing the terms from the previous formula with the terms of the assumption $N(u) = \sum_{n=0}^{\infty} A_n[u_0, u_1, \cdots, u_n]$ the values of $A_n's$ can be constructed as follow:

$A_0 = N(u_0)$

$A_1 = u_1 N'(u_0)$

$A_2 = u_2 N'(u_0) + \frac{1}{2!}u_1^2 N''(u_0)$

$A_3 = u_3 N'(u_0) + u_1 u_2 N''(u_0) + \frac{1}{3!}u_1^3 N'''(u_0)$

$A_4 = u_4 N'(u_0) + \left(\frac{1}{2!}u_2^2 + u_1 u_3\right) N''(u_0) + \frac{1}{2!}u_1^2 u_2 N'''(u_0) + \frac{1}{4!}u_1^4 N^{(4)}(u_0)$

and so on. For this case the Adomian polynomial $A_n(t) = A_n[u_0(t), u_1(t), \cdots, u_n(t)], n \geq 1$, can be listed in general formula, [14]:

$$A_n(t) = \sum_{k=1}^{n} C_n^k N^{(k)}(u_0)$$

Where

$$C_n^k = \begin{cases} u_n, & k = 1 \\ \dfrac{1}{n}\sum_{j=0}^{n-k}(j+1)u_{j+1}C_{n-1-j}^{k-1}, & k = 2,3,\dots,n \end{cases}$$

Through the use of above expansions, from the simple analytic nonlinearity $N(u(t))$, the Adomian polynomials $A_n's$ are arranged to have the form, [8,15]. To find the $A_n's$ by Adomian general formula, these polynomial will be computed as follows:

$$A_0 = N(u_0) = \frac{1}{0!}\frac{d^0}{d\lambda^0}\left[N\left(\sum_{i=0}^{0}\lambda^i u_i\right)\right]_{\lambda=0}$$

Since

$$\frac{1}{1!}\frac{d^1}{d\lambda^1}\left[N\left(\sum_{i=0}^{1}\lambda^i u_i\right)\right]_{\lambda=0} = \frac{1}{1!}\frac{d^1}{d\lambda^1}[N(\lambda^0 u_0 + \lambda^1 u_1)]_{\lambda=0}$$
$$= [N'(\lambda^0 u_0 + \lambda^1 u_1)]_{\lambda=0}(u_1)$$

So

$$A_1 = u_1 N'(u_0) = \frac{1}{1!}\frac{d^1}{d\lambda^1}\left[N\left(\sum_{i=0}^{1}\lambda^i u_i\right)\right]_{\lambda=0}$$

Also, Since

$$\frac{1}{2!}\frac{d^2}{d\lambda^2}\left[N\left(\sum_{i=0}^{2}\lambda^i u_i\right)\right]_{\lambda=0} = \frac{1}{2!}\frac{d^2}{d\lambda^2}[N(\lambda^0 u_0 + \lambda^1 u_1 + \lambda^2 u_2)]_{\lambda=0}$$
$$= \frac{1}{2!}\frac{d}{d\lambda}[N'(\lambda^0 u_0 + \lambda^1 u_1 + \lambda^2 u_2)(u_1 + 2\lambda u_2)]_{\lambda=0}$$
$$= \frac{1}{2!}[N'(\lambda^0 u_0 + \lambda^1 u_1 + \lambda^2 u_2)(2u_2)$$
$$+ N''(\lambda^0 u_0 + \lambda^1 u_1 + \lambda^2 u_2)(u_1 + 2\lambda u_2)^2]_{\lambda=0}$$
$$= u_2 N'(u_0) + \frac{1}{2!}u_1^2 N''(u_0)$$

So

$$A_2 = u_2 N'(u_0) + \frac{1}{2!}u_1^2 N''(u_0) = \frac{1}{2!}\frac{d^2}{d\lambda^2}\left[N\left(\sum_{i=0}^{2}\lambda^i u_i\right)\right]_{\lambda=0}$$

Hence, by same techniques we can obtained $A_n's$ for the nonlinearity $N(u(t))$ by formula:

$$A_n(t) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i u_i(t)\right)\right]_{\lambda=0}, \qquad n = 0,1,2,\cdots$$

Where λ is a parameter introduced for convenience.

A useful device in an effort to speed up the convergence of the ADM is developed; the new technique relies upon in most instances the so-called “Noise Term Phenomenon” [13] that display a rapid convergence of the solution. The noise terms phenomenon may be used for all functional equations like the differential or integral equations. The noise terms are described as the identical terms with contrary signs that can also appear within the components $u_0(t)$ and $u_1(t)$ and in the other components as well, Noise terms may also seem if the exact solution of the equation is part of the zeroth component$u_0(t)$. Verification that the remaining non-canceled terms satisfy the integral equation is necessary and essential.

4. Analysis Technique of Method

In this section we try to find general solution form of multi-higher nonlinear IFDE's of V-H type of the form (1) under the initial condition (2) by applying the standard Adomian decomposition method (SADM) and Modified Adomian decomposition method (MADM) as follows:

4.1Applying the SADM for Solving nonlinear IFDE of V-H Type

Our approach begins by taking α_n-order of R-L fractional integral ${}_aJ_t^{\alpha}$to both sides of equation (1) and using lemma (1) and lemma (2, part-ii) we obtain:

$$u(t) = {}_aJ_t^{\alpha} f(t) + \sum_{k=0}^{m_{\alpha_n}-1} \frac{u^{(k)}(a)}{k!}(t-a)^k + \sum_{i=1}^{n-1} {}_aJ_t^{\alpha_n}\left[\bar{P}_i(t)\, {}_a^CD_t^{\alpha_i} u(t)\right] + {}_aJ_t^{\alpha_n}[\bar{P}_n(t)u(t)] + \sum_{\ell=0}^{m} \lambda_\ell \int_a^t \bar{\mathcal{K}}_\ell(t,s)\mathcal{H}_\ell\left(s\,, {}_a^CD_s^{\beta_\ell}u(s)\right)ds \quad (4)$$

Where$m_{\alpha_n} = \lceil \alpha_n \rceil$, $\bar{P}_i(t) = -P_i(t)$for all $i = 1,2,\cdots,n$;$\bar{\mathcal{K}}_\ell(t,s) = {}_sJ_t^{\alpha_n}\mathcal{K}_\ell(t,s)$ for all $s > a$ and let $N_\ell(u(s)) = \mathcal{H}_\ell\left(s\,, {}_a^CD_s^{\beta_\ell}u(s)\right)$, for all$\ell = 0,1,\cdots,m$.

Second, according to the decomposition method, we assume series solution for the unknown function $u(t)$ in the form (3) and the nonlinear terms $N_\ell(u(s))$ can be decomposed into the infinite series of Adomian polynomials, for all $\ell = 0,1,\cdots,m$given by:

$$N_\ell(u(s)) = \sum_{r=0}^{\infty} A_r^\ell[u_0(s), u_1(s), \cdots, u_r(s)] \quad (5)$$

Then substituting the decomposition series (3) and (5) into equation (4), yields we advised the subsequent recursive formula (6):

$$\left.\begin{aligned} u_0(t) &= {}_aJ_t^{\alpha_n} f(t) + \sum_{k=0}^{m_{\alpha_n}-1} \frac{u^{(k)}(a)}{k!}(t-a)^k \\ u_{r+1}(t) &= \sum_{i=1}^{n-1} {}_aJ_t^{\alpha_n}\left[\bar{P}_i(t)\,{}_a^CD_t^{\alpha_i}u_r(t)\right] + {}_aJ_t^{\alpha_n}[\bar{P}_n(t)u_r(t)] \\ &+ \sum_{\ell=0}^{m} \lambda_\ell \int_a^t \bar{\mathcal{K}}_\ell(t,s) A_r^\ell[u_0(s), u_1(s), \cdots, u_r(s)]ds, \quad \text{for all } r \geq 0 \end{aligned}\right\} \quad (6)$$

Where

$$\left.\begin{array}{l} A_0^{\ell}[u_0] = N_{\ell}(u_0) \quad ; \quad A_1^{\ell}[u_0, u_1] = u_1 N_{\ell}'(u_0) \; ; \\ A_2^{\ell}[u_0, u_1, u_2] = u_2 N_{\ell}'(u_0) + \frac{u_1^2}{2!}\left(N_{\ell}''(u_0)\right) ; \; \cdots \end{array}\right\} \tag{7}$$

Now, as a special case if $\bar{P}_i(t) = \bar{C}_i = -C_i (i = 1,2, \cdots, n-1)$ in which $\bar{C}_i$ are any real constants then we are able to write equation (6), after making use of lemma (3) as follows:

$$\left.\begin{array}{l} u_0(t) = {}_aJ_t^{\alpha_n} f(t) + \sum_{k=0}^{m_{\alpha_n}-1} \frac{u^{(k)}(a)}{k!}(t-a)^k \\ + \sum_{i=1}^{n-1} \bar{C}_i \left[\sum_{k=0}^{m_{\alpha_i}-1} \frac{u^{(k)}(a)}{\Gamma(k+\alpha_n-\alpha_i+1)}(t-a)^{k+\alpha_n-\alpha_i} \right] \\ u_{r+1}(t) = \sum_{i=1}^{n-1} \bar{C}_i \left[{}_aJ_t^{\alpha_n-\alpha_i} u_r(t) \right] + {}_aJ_t^{\alpha_n}[\bar{P}_n(t) u_r(t)] \\ + \sum_{\ell=0}^{m} \lambda_{\ell} \int_a^t \bar{\mathcal{K}}_{\ell}(t,s) A_r^{\ell}[u_0(s), u_1(s), \cdots, u_r(s)] ds, \quad r \geq 0 \end{array}\right\} \tag{8}$$

In computational practice for the Adomian polynomials, A_r^{ℓ}'s we truncate the series after $r = M$ for positivefinite quantity M. Thus:

$$A_r^{\ell}(t) = \frac{1}{r!}\frac{d^r}{d\lambda^r}\left[N_{\ell}\left(\sum_{i=0}^{M} \lambda^i u_i\right)\right]_{\lambda=0}, 0 \leq r \leq M; \text{ for all} \ell = 0,1 \cdots, m$$

So, all terms of the series in equation (3) want not to be determined and so we use an approximation of the solution with the aid of the use of the following truncated series:

$$u(t) \cong \hat{u}_M(t) = \sum_{r=0}^{M} u_r(t), \qquad M \in \mathbb{Z}^+ \tag{9}$$

The components $u_0, u_1, \cdots, u_M$ are determined recursively by waythe above formula (6 or 8) or the usage of the noise terms approach concept.It is important to phrase that the decomposition method indicates that the zeroth factor $u_0(t)$usually be defined by the initial conditions and the α_n-order of R-L fractional integral operator of the function$f(t)$ as described above. The other components namely $u_1, u_2, \cdots, u_M$are derived recurrently.

4.2Applying the MADM for Solving nonlinear IFDE of V-H Type:

As stated before, the standard Adomian decomposition method affords the solutions in an infinite series of components. The components $u_j, j \geq 0$ are easily computed if the inhomogeneous term in equation (4) contains a few terms. However, if the inhomogeneous term contains two or more terms, the evaluation of the components $u_j, j \geq 0$ requires more work.The modified Adomian decomposition method will facilitate the computational process and further accelerate the convergence of the series solution,the assumptions made by Adomian [13] were modified recently by Wazwaz [17,18].

The Modified Decomposition Method 1 (MADM1). It is interesting to note that the MADM1 depends mainly on splitting the inhomogeneous term into two parts, consequently,it cannot be used if the inhomogeneous term consists of the simplest one term. The achievement of this modification relies upon most effective at the right choice of assumption the two functions and this will be made via trials only. One of the disadvantages of this method is that a rule which can help for the proper choice of sub-functions could not be observed yet, [8,17]. By same stages as in SADM we obtain the following equation:

$$u(t) = g(t) + \sum_{i=1}^{n-1} {}_aJ_t^{\alpha_n}\left[\bar{P}_i(t)\,{}_a^CD_t^{\alpha_i}u(t)\right] + {}_aJ_t^{\alpha_n}[\bar{P}_n(t)u(t)] + \sum_{\ell=0}^{m} \lambda_\ell \int_a^t \bar{\mathcal{K}}_\ell(t,s)\mathcal{H}_\ell\left(s\,,{}_a^CD_s^{\beta_\ell}u(s)\right)ds \quad (10)$$

Where$m_{\alpha_n} = \lceil \alpha_n \rceil, \bar{P}_i(t) = -P_i(t)$for all $i = 1,2,\cdots,n$;$\bar{\mathcal{K}}_\ell(t,s) = {}_sJ_t^{\alpha_n}\mathcal{K}_\ell(t,s)$ for all $s > a$ and let $N_\ell(u(s)) = \mathcal{H}_\ell\left(s\,,{}_a^CD_s^{\beta_\ell}u(s)\right)$, for all$\ell = 0,1,\cdots,m$. and

$$g(t) = {}_aJ_t^{\alpha_n}f(t) + \sum_{k=0}^{m_{\alpha_n}-1} \frac{u^{(k)}(a)}{k!}(t-a)^k \quad (11)$$

According to the function $g(t)$ in equation (11) can be set as the sum of two partial functions, namely $f_1(t)$ and $f_2(t)$. In other words, we can set:

$$g(t) = f_1(t) + f_2(t) \quad (12)$$

To reduce the size of calculations, a slight variation becomes proposed handiest for the components $u_0(t)$ and $u_1(t)$. We identify that only the part$f_1(t)$ might be assigned to the zeroth component $u_0(t)$, by means of one part of $g(t)$, where the other remaining part of $g(t)$ can be introduced to the component $u_1(t)$ among different terms say $f_2(t)$. Consequently, the MADM1 introduces the modified recurrence relation:

$$\left.\begin{aligned} u_0(t) &= f_1(t) \\ u_1(t) &= \begin{bmatrix} f_2(t) + \sum_{i=1}^{n-1} {}_aJ_t^{\alpha_n}\left[\bar{P}_i(t)\,{}_a^CD_t^{\alpha_i}u_0(t)\right] + {}_aJ_t^{\alpha_n}[\bar{P}_n(t)u_0(t)] \\ + \sum_{\ell=0}^{m} \lambda_\ell \int_a^t \bar{\mathcal{K}}_\ell(t,s)A_0^\ell[u_0(s)]ds \end{bmatrix} \\ u_{r+1}(t) &= \begin{bmatrix} \sum_{i=1}^{n-1} {}_aJ_t^{\alpha_n}\left[\bar{P}_i(t)\,{}_a^CD_t^{\alpha_i}u_r(t)\right] + {}_aJ_t^{\alpha_n}[\bar{P}_n(t)u_r(t)] \\ + \sum_{\ell=0}^{m} \lambda_\ell \int_a^t \bar{\mathcal{K}}_\ell(t,s)A_r^\ell[u_0(s),u_1(s),\cdots,u_r(s)]ds, \quad r \geq 1 \end{bmatrix} \end{aligned}\right\} \quad (13)$$

For the special case, IFDE's of V-H type with constant coefficients $\bar{P}_i(t) = \bar{C}_i = -C_i (i = 1,2,\cdots,n-1)$ in which $\bar{C}_i$ are any real constants, after applying the lemma (3) then we defined the inhomogeneous part $g(t)$ as follows:

$$g(t) = {}_aJ_t^{\alpha_n} f(t) + \sum_{k=0}^{m_{\alpha_n}-1} \frac{u^{(k)}(a)}{k!}(t-a)^k + \sum_{i=1}^{n-1} \bar{C}_i \left[\sum_{k=0}^{m_{\alpha_i}-1} \frac{u^{(k)}(a)}{\Gamma(k+\alpha_n-\alpha_i+1)} (t-a)^{k+\alpha_n-\alpha_i} \right] \quad (14)$$

After setting equation (14) as on the structure in equation (11), Also the suggestion was that only the element $f_1(t)$ may be assigned to the zeroth element $u_0(t)$, wherein the final element $f_2(t)$ might be combined with the other terms given $u_1(t)$:

$$\left.\begin{array}{l} u_0(t) = f_1(t) \\ u_1(t) = \left[f_2(t) + \sum_{i=1}^{n-1} \bar{C}_i \left[{}_aJ_t^{\alpha_n-\alpha_i} u_0(t) \right] + {}_aJ_t^{\alpha_n} [\bar{P}_n(t) u_0(t)] + \sum_{\ell=0}^{m} \lambda_\ell \int_a^t \bar{\mathcal{K}}_\ell(t,s) A_0^\ell [u_0(s)] ds \right] \\ u_{r+1}(t) = \left[\sum_{i=1}^{n-1} \bar{C}_i \left[{}_aJ_t^{\alpha_n-\alpha_i} u_r(t) \right] + {}_aJ_t^{\alpha_n} [\bar{P}_n(t) u_r(t)] + \sum_{\ell=0}^{m} \lambda_\ell \int_a^t \bar{\mathcal{K}}_\ell(t,s) A_r^\ell [u_0(s), u_1(s), \cdots, u_r(s)] ds, \quad r \geq 1 \right] \end{array}\right\} \quad (15)$$

So, all terms of the series in equation (14) need now not to be determined and so we use an approximation of the solution by the subsequent truncated collection:

$$u(t) \cong \hat{u}_M(t) = \sum_{r=0}^{M} u_r(t), \qquad M \in \mathbb{Z}^+ \quad (16)$$

The components $u_0, u_1, \cdots, u_M$ are usually decided recursively by equations (13 or 15). However, the success of the MADM1 relies upon totally on the right choice of the functions $f_0(t)$ and $f_1(t)$. It appears that trials are the handiest standards that may be applied thus far.

More generalization for some time: *The Modified Decomposition Method 2* (MADM2). In the second kind of modification, we update the process of dividing the inhomogeneous part $g(t)$, equation (11 or 14) identical, into two components with the aid of a sequence of infinite components. We, therefore, pointed out that sometimes it is able to be beneficial to specific $g(t)$ in Taylor series. In this method, MADM2, the function $g(t)$ can be useful to express in Taylor series for components functions $f_i(t), i = 0,1,2$ .... In other words, we can set:

$$g(t) = f_0(t) + f_1(t) + \cdots + f_N(t) + \cdots \qquad (17)$$

In view of (17), we introduce a qualitative change in the formation of the recurrence relation (6) to suggest a new recursive relationship expressed in the form:

$$\left.\begin{array}{l} u_0(t) = f_0(t) \\ u_1(t) = \begin{bmatrix} f_1(t) + \sum_{i=1}^{n-1} {}_aJ_t^{\alpha_n}\left[\bar{P}_i(t)\,{}_a^CD_t^{\alpha_i}u_0(t)\right] + {}_aJ_t^{\alpha_n}[\bar{P}_n(t)u_0(t)] \\ + \sum_{\ell=0}^{m} \lambda_\ell \int_a^t \bar{\mathcal{K}}_\ell(t,s)A_0^\ell[u_0(s)]ds \end{bmatrix} \\ u_{r+1}(t) = \begin{bmatrix} f_{r+1}(t) + \sum_{i=1}^{n-1} {}_aJ_t^{\alpha_n}\left[\bar{P}_i(t)\,{}_a^CD_t^{\alpha_i}u_r(t)\right] + {}_aJ_t^{\alpha_n}[\bar{P}_n(t)u_r(t)] \\ + \sum_{\ell=0}^{m} \lambda_\ell \int_a^t \bar{\mathcal{K}}_\ell(t,s)A_r^\ell[u_0(s),u_1(s),\cdots,u_r(s)]ds \end{bmatrix}, \quad r \geq 1 \end{array}\right\} (18)$$

For the particular case, wherein we have the type of the constant coefficients, then we need to describe the inhomogeneous term g(t) as in equation (14) then we've got the to suggest a new recursive courting expressed in the form:

$$\left.\begin{array}{l} u_0(t) = f_0(t) \\ u_1(t) = \begin{bmatrix} f_1(t) + \sum_{i=1}^{n-1} \bar{C}_i\left[{}_aJ_t^{\alpha_n-\alpha_i}u_0(t)\right] + {}_aJ_t^{\alpha_n}[\bar{P}_n(t)u_0(t)] \\ + \sum_{\ell=0}^{m} \lambda_\ell \int_a^t \bar{\mathcal{K}}_\ell(t,s)A_0^\ell[u_0(s)]ds \end{bmatrix} \\ u_{r+1}(t) = \begin{bmatrix} f_{r+1}(t) + \sum_{i=1}^{n-1} \bar{C}_i\left[{}_aJ_t^{\alpha_n-\alpha_i}u_r(t)\right] + {}_aJ_t^{\alpha_n}[\bar{P}_n(t)u_r(t)] \\ + \sum_{\ell=0}^{m} \lambda_\ell \int_a^t \bar{\mathcal{K}}_\ell(t,s)A_r^\ell[u_0(s),u_1(s),\cdots,u_r(s)]ds \end{bmatrix}, \quad r \geq 0 \end{array}\right\} \qquad (19)$$

According to equation (18 or 19), the terms $u_0(t), u_1(t), u_2(t), \ldots$ of the solution $u(t)$ follow immediately, and the solution can be obtained using (16).

It is essential to note that if inhomogeneous time period consists of a one-time period only, then scheme (18 or 19) reduces to relation (6 or 8) respectively. Moreover, if the inhomogeneous part includes two terms, then relation (18 or 19) reduces to the modification relation (13 or 15) respectively.

5. Numerical Experiments

In this section, we shall provide some illustrative examples so that it will clarify our approach ADM, MADM1 and MADM2 to solve Volterra-Hammerstein nonlinear integro-fractional differential equations. We do not forget about the following check problems:

Example (1): We first consider the linear IFDE of V-H type for fractional orders, where allα,β are lie in (0,1)and$\lambda \in \mathbb{R}$:

$$ {}_0^C D_t^{\alpha} u(t) = f(t) + \lambda \int_0^t s t^2 \, {}_0^C D_s^{\beta} u(s) ds , \qquad 0 \le t \le 1 $$

Together with initial condition: $u(0) = 0$; and the inhomogeneous term formed as:

$$ f(t) = \frac{6}{\Gamma(3-\alpha)} t^{2-\alpha} - \frac{6\lambda}{(4-\beta)\Gamma(3-\beta)} t^{6-\beta} $$

By comparison with the fundamental equation (1), we will see that $n = 1, m = 0$ and $P_i(t) = 0$, for all i, so first case, if we take $\alpha_1 = \alpha = 0.5$, $\beta_0 = \beta = 0$; $\lambda_0 = \lambda = \frac{1}{3}$ then from the equation above we've $m_{\alpha_1} = 1$; and we have one kernel $\mathcal{K}_0(t,s) = st^2$; with $N_0\big(u(s)\big) = u(s)$;and the inhomogeneous time become $f(t) = \frac{6}{\Gamma(2.5)} t^{1.5} - \frac{1}{4} t^6$.

Applying the SADM for solving our problem, the recursive formula (6) with initial condition $u(0) = 0$leads to the following scheme:

$$ \left. \begin{aligned} u_0(t) &= {}_0J_t^{0.5} f(t) \\ u_{r+1}(t) &= \frac{1}{3} \int_0^t \bar{\mathcal{K}}_0(t,s) A_r^0[u_0(s), u_1(s), \cdots, u_r(s)] ds , \qquad \text{for all } r \ge 0 \end{aligned} \right\} \quad (20) $$

So that:

$$ u_0(t) = {}_0J_t^{0.5} \left[\frac{6}{\Gamma(2.5)} t^{1.5} - \frac{1}{4} t^6 \right] = 3t^2 - \frac{\Gamma(7)}{4\Gamma(7.5)} t^{6.5} $$

and, using equation (7) where $\ell = 0$, we get:

$$ A_0^0[u_0] = N_0\big(u_0(s)\big) = u_0(s) = 3s^2 - \frac{\Gamma(7)}{4\Gamma(7.5)} s^{6.5} $$

Also,

$$ \begin{aligned} \bar{\mathcal{K}}_0(t,s) &= {}_sJ_t^{\alpha_1} \mathcal{K}_0(t,s) = {}_sJ_t^{0.5} s t^2 \\ &= \frac{2}{\Gamma(3.5)} s (t-s)^{2.5} + \frac{2}{\Gamma(2.5)} s^2 (t-s)^{1.5} + \frac{1}{\Gamma(1.5)} s^3 (t-s)^{0.5} \end{aligned} $$

Thus, applying the second part of recursive relation (20) with $r = 0$, we obtain:

$$ u_1(t) = \frac{1}{3} \int_0^t \bar{\mathcal{K}}_0(t,s) A_0^0[u_0(s)] ds = \frac{\Gamma(7)}{4\Gamma(7.5)} t^{6.5} - \frac{\Gamma(7)\Gamma(11.5)}{102\Gamma(7.5)\Gamma(12)} t^{11} $$

From equation (7) to find A_1^0 put $\ell = 0$, thus:

$$A_1^0[u_0,u_1]=u_1(s)N_0'\big(u_0(s)\big)=u_1(s)*1=\frac{\Gamma(7)}{4\Gamma(7.5)}s^{6.5}-\frac{\Gamma(7)\Gamma(11.5)}{102\Gamma(7.5)\Gamma(12)}s^{11}$$

Applying the second part of recursive relation (20) with $r=1$, we get:

$$u_2(t)=\frac{1}{3}\int_0^t \bar{\mathcal{K}}_0(t,s)A_1^0[u_0(s),u_1(s)]ds$$
$$=\frac{\Gamma(7)\Gamma(11.5)}{102\Gamma(7.5)\Gamma(12)}t^{11}-\frac{\Gamma(7)\Gamma(11.5)\Gamma(16)}{3978\Gamma(7.5)\Gamma(12)\Gamma(16.5)}t^{15.5}$$

Using equation (7) for finding A_2^0, putting$\ell=0$ we obtain:

$$A_2^0[u_0,u_1,u_2]=u_2N_0'(u_0)+\frac{1}{2!}u_1^2N_0''(u_0)=\ u_2(s)*1+0$$
$$=\frac{\Gamma(7)\Gamma(11.5)}{102\Gamma(7.5)\Gamma(12)}s^{11}-\frac{\Gamma(7)\Gamma(11.5)\Gamma(16)}{3978\Gamma(7.5)\Gamma(12)\Gamma(16.5)}s^{15.5}$$

Applying the second part of recursive relation (20) with $r=2$, we gain:

$$u_3(t)=\frac{1}{3}\int_0^t \bar{\mathcal{K}}_0(t,s)A_2^0[u_0,u_1,u_2]ds$$
$$=\frac{\Gamma(7)\Gamma(11.5)\Gamma(16)}{3978\Gamma(7.5)\Gamma(12)\Gamma(16.5)}t^{15.5}$$
$$-\frac{\Gamma(7)\Gamma(11.5)\Gamma(16)\Gamma(20.5)}{208845\Gamma(7.5)\Gamma(12)\Gamma(16.5)\Gamma(21)}t^{20}$$

and so on. Considering (16), the approximated solution with two, three and four terms is:

$$u(t)\cong\hat{u}_1(t)=\sum_{r=0}^{1}u_r(t)=3t^2-\frac{\Gamma(7)\Gamma(11.5)}{102\Gamma(7.5)\Gamma(12)}t^{11}$$
$$u(t)\cong\hat{u}_2(t)=\sum_{r=0}^{2}u_r(t)=3t^2-\frac{\Gamma(7)\Gamma(11.5)\Gamma(16)}{3978\Gamma(7.5)\Gamma(12)\Gamma(16.5)}t^{15.5}$$
$$u(t)\cong\hat{u}_3(t)=\sum_{r=0}^{3}u_r(t)=3t^2-\frac{\Gamma(7)\Gamma(11.5)\Gamma(16)\Gamma(20.5)}{208845\Gamma(7.5)\Gamma(12)\Gamma(16.5)\Gamma(21)}t^{20}$$

The following table presents a comparison between the exact solution and the approximate analytical solution $\hat{u}_1(t),\hat{u}_2(t)$ and $\hat{u}_3(t)$ respectively, depending on the least square error.

t	Exact solution $u(t)=3t^2$	Approximate analytical solution		
		$\hat{u}_1(t)$	$\hat{u}_2(t)$	$\hat{u}_3(t)$
0.0	0.0	0.0	0.0	0.0
0.1	0.03	0.029999999999988	0.03	0.03
0.2	0.12	0.119999999976969	0.12	0.12
0.3	0.27	0.269999998007935	0.269999999999942	0.27
0.4	0.48	0.479999952833939	0.479999999995066	0.48
0.5	0.75	0.749999450914787	0.749999999843226	0.749999999999970
0.6	1.08	1.079995920250909	1.079999997354048	1.079999999998875
0.7	1.47	1.469977764436483	1.469999971142606	1.469999999975463
0.8	1.92	1.919903403907878	1.919999771370495	1.919999999645473
0.9	2.43	2.429647111663961	2.429998580954332	2.429999996261483
1.0	3.0	2.998875473484848	2.999992734968932	2.999999969249717
L.S.E		$1.398932\ e-006$	$5.484747\ e-011$	$9.596827\ e-016$

Applying the MADM 1 for solving our problem, first from equation (11) with initial situation $u(0)=0$, given $f(t)$ and using the definition of R-L integral for order $\alpha_1 = 0.5$, we obtain:

$$g(t) = {}_aJ_t^{\alpha_1} f(t) + \sum_{k=0}^{m_{\alpha_1}-1} \frac{u^{(k)}(0)}{k!} t^k = 3t^2 - \frac{\Gamma(7)}{4\Gamma(7.5)} t^{6.5}$$

In other words, we can set: $f_1(t) = 3t^2$ and $f_2(t) = -\frac{\Gamma(7)}{4\Gamma(7.5)} t^{6.5}$. The recursive components (13)leads to the following scheme:

$$\left.\begin{aligned} u_0(t) &= f_1(t) \\ u_1(t) &= f_2(t) + \lambda_0 \int_0^t \overline{\mathcal{K}}_0(t,s) A_0^0[u_0(s)]ds \\ u_{r+1}(t) &= \lambda_0 \int_0^t \overline{\mathcal{K}}_0(t,s) A_r^0[u_0(s), u_1(s), \cdots, u_r(s)]ds, \qquad r \geq 1 \end{aligned}\right\} \quad (21)$$

So that:

$$u_0(t) = f_1(t) = 3t^2$$

and, using equation (7) where $\ell = 0$, we find the different A_0^0 as:

$$A_0^0[u_0] = N_0\big(u_0(s)\big) = u_0(s) = 3s^2$$

$$\begin{aligned} u_1(t) &= f_2(t) + \lambda_0 \int_0^t \overline{\mathcal{K}}_0(t,s) A_0^0[u_0(s)]ds \\ &= -\frac{\Gamma(7)}{4\Gamma(7.5)} t^{6.5} \\ &+ \frac{1}{3} \int_0^t \left[\frac{2}{\Gamma(3.5)} s(t-s)^{2.5} + \frac{2}{\Gamma(2.5)} s^2 (t-s)^{1.5} \right. \\ &\left. + \frac{1}{\Gamma(1.5)} s^3 (t-s)^{0.5} \right] [3s^2] ds = -\frac{\Gamma(7)}{4\Gamma(7.5)} t^{6.5} + \frac{\Gamma(7)}{4\Gamma(7.5)} t^{6.5} = 0 \end{aligned}$$

Thus, all second part of recursive relation (21) with each $r \geq 1$, we gain:

$$u_{r+1}(t) = \lambda_0 \int_0^t \bar{\mathcal{K}}_0(t,s) A_r^0[u_0(s), u_1(s), \cdots, u_r(s)] ds = 0$$

So, it is obvious that each component of $u_r, r \geq 1$ is zero. The solution is: $u(t) = 3t^2$. which is the exact solution for our problem.

second case, if we take $\alpha_1 = \alpha = 0.5, \beta_0 = 0$ and $\beta_1 = \beta = 0.8$; $\lambda_1 = \lambda = \frac{1}{3}$ then from the equation above we've $m_{\alpha_1} = 1$; and we have two kernels $\mathcal{K}_1(t,s) = st^2$ and $\mathcal{K}_0(t,s) = 0$; with $N_1(u(s)) = {}_0^C D_s^{0.8} u(s)$ and $N_0(u(s)) = 0$. Respectively, the inhomogeneous time become $f(t) = \frac{6}{\Gamma(2.5)} t^{1.5} - \frac{5}{8\,\Gamma(2.2)} t^{5.2}$.

Applying the SADM, the recursive formula (6) with $u(0) = 0$ leads to the following scheme:

$$\left.\begin{aligned} u_0(t) &= {}_0J_t^{0.5} f(t) \\ u_{r+1}(t) &= \frac{1}{3}\int_0^t \bar{\mathcal{K}}_1(t,s) A_r^1[u_0(s), u_1(s), \cdots, u_r(s)] ds, \qquad \text{for all } r \geq 0 \end{aligned}\right\} \quad (22)$$

So that:

$$u_0(t) = {}_0J_t^{0.5}\left[\frac{6}{\Gamma(2.5)} t^{1.5} - \frac{5}{8\,\Gamma(2.2)} t^{5.2}\right] = 3t^2 - \frac{5\Gamma(6.2)}{8\,\Gamma(2.2)\Gamma(6.7)} t^{5.7}$$

And for $r = 0$, using equation (7) where $\ell = 1$, we get:

$$A_0^1[u_0] = N_1(u_0(s)) = {}_0^C D_s^{0.8} u_0(s) = \frac{6}{\Gamma(2.2)} s^{1.2} - \frac{5\Gamma(6.2)}{8\Gamma(2.2)\Gamma(5.9)} s^{4.9}$$

Also,

$$\begin{aligned} \bar{\mathcal{K}}_1(t,s) &= {}_sJ_t^{\alpha_1} \mathcal{K}_1(t,s) = {}_sJ_t^{0.5} st^2 \\ &= \frac{2}{\Gamma(3.5)} s(t-s)^{2.5} + \frac{2}{\Gamma(2.5)} s^2(t-s)^{1.5} + \frac{1}{\Gamma(1.5)} s^3(t-s)^{0.5} \end{aligned}$$

Thus, we obtain:

$$\begin{aligned} u_1(t) &= \frac{1}{3}\int_0^t \bar{\mathcal{K}}_1(t,s) A_0^1[u_0(s)] ds \\ &= \frac{1092\Gamma(3.2)}{25\Gamma(2.2)\Gamma(6.7)} t^{5.7} - \frac{7031\Gamma(6.2)\Gamma(6.9)}{480\Gamma(2.2)\Gamma(5.9)\Gamma(10.4)} t^{9.4} \end{aligned}$$

Put $\ell = 1$ in equation (7) to computing A_1^1. Thus

$$\begin{aligned} A_1^1[u_0, u_1] &= \frac{d}{d\lambda}\left[N_1(u_0(s) + \lambda u_1(s))\right]_{\lambda=0} = {}_0^C D_s^{0.8} u_1(s) \\ &= \frac{1092\Gamma(3.2)}{25\Gamma(2.2)\Gamma(5.9)} s^{4.9} - \frac{7031\Gamma(6.2)\Gamma(6.9)}{480\Gamma(2.2)\Gamma(5.9)\Gamma(9.6)} s^{8.6} \end{aligned}$$

So, from the recursive relation (22) with $r = 1$, we get:

$$u_2(t)=\frac{1}{3}\int_0^t \bar{\mathcal{K}}_1(t,s)A_1^1[u_0(s),u_1(s)]ds$$
$$=\frac{639821\Gamma(3.2)\Gamma(6.9)}{625\Gamma(2.2)\Gamma(5.9)\Gamma(10.4)}t^{9.4}-\frac{1427293\Gamma(6.2)\Gamma(6.9)\Gamma(10.6)}{2000\Gamma(2.2)\Gamma(5.9)\Gamma(9.6)\Gamma(14.1)}t^{13.1}$$

Using equation (7) for finding A_2^1, we obtain:

$$A_2^1[u_0,u_1,u_2]=\frac{1}{2}\frac{d^2}{d\lambda^2}\left[N_1\left(u_0(s)+\lambda u_1(s)+\lambda^2 u_2(s)\right)\right]_{\lambda=0}={}_0^C D_s^{0.8}u_2(s)$$
$$=\frac{639821\Gamma(3.2)\Gamma(6.9)}{625\Gamma(2.2)\Gamma(5.9)\Gamma(9.6)}s^{8.6}-\frac{1427293\Gamma(6.2)\Gamma(6.9)\Gamma(10.6)}{2000\Gamma(2.2)\Gamma(5.9)\Gamma(9.6)\Gamma(13.3)}s^{12.3}$$

From recursive relation (20) with $r=2$, we gain:

$$u_3(t)=\frac{1}{3}\int_0^t \bar{\mathcal{K}}_1(t,s)A_2^1[u_0(s),u_1(s),u_2(s)]ds$$
$$=\frac{779301978\Gamma(3.2)\Gamma(6.9)\Gamma(10.6)}{15625\Gamma(2.2)\Gamma(5.9)\Gamma(9.6)\Gamma(14.1)}t^{13.1}$$
$$-\frac{(1427293)(8313)\Gamma(6.2)\Gamma(6.9)\Gamma(10.6)\Gamma(14.3)}{(200000)\Gamma(2.2)\Gamma(5.9)\Gamma(9.6)\Gamma(13.3)\Gamma(17.8)}t^{16.8}$$

Also, by same procedure, we get:

$$A_3^1[u_0,u_1,u_2,u_3]=\frac{1}{3!}\frac{d^3}{d\lambda^3}\left[N_1\left(u_0(s)+\lambda u_1(s)+\lambda^2 u_2(s)+\lambda^3 u_3(s)\right)\right]_{\lambda=0}$$
$$={}_0^C D_s^{0.8}u_3(s)$$
$$=\frac{779301978\Gamma(3.2)\Gamma(6.9)\Gamma(10.6)}{15625\Gamma(2.2)\Gamma(5.9)\Gamma(9.6)\Gamma(13.3)}s^{12.3}$$
$$-\frac{(1427293)(8313)\Gamma(6.2)\Gamma(6.9)\Gamma(10.6)\Gamma(14.3)}{(200000)\Gamma(2.2)\Gamma(5.9)\Gamma(9.6)\Gamma(13.3)\Gamma(17)}s^{16}$$

From recursive relation (20) with $r=2$, we gain:

$$u_4(t)=\frac{1}{3}\int_0^t \bar{\mathcal{K}}_1(t,s)A_3^1[u_0(s),u_1(s),u_2(s),u_3(s)]ds$$
$$=\frac{(259767326)(24939)\Gamma(3.2)\Gamma(6.9)\Gamma(10.6)\Gamma(14.3)}{1562500\Gamma(2.2)\Gamma(5.9)\Gamma(9.6)\Gamma(13.3)\Gamma(17.8)}t^{16.8}$$
$$-\frac{(1427293)(52649)\Gamma(6.2)\Gamma(6.9)\Gamma(10.6)\Gamma(14.3)\Gamma(18)}{(10000)\Gamma(2.2)\Gamma(5.9)\Gamma(9.6)\Gamma(13.3)\Gamma(17)\Gamma(21.5)}t^{20.5}$$

and so on. The following table presents a comparison between the exact solution and the approximate analytical solution $\hat{u}_k(t), k \geq 0$ respectively, depending on the least square error.

t	Exact solution $u(t) = 3t^2$	Approximate analytical solution $\hat{u}_M(t) = \sum_{k=0}^{M} u_k(t)$, $M \in \mathbb{Z}^+$				
		$\hat{u}_0(t)$	$\hat{u}_1(t)$	$\hat{u}_2(t)$	$\hat{u}_3(t)$	$\hat{u}_4(t)$
0.0	0.00	0.0	0.0	0.0	0.0	0.0
0.1	0.03	0.0299995362	0.029999999999	0.03	0.03	0.03
0.2	0.12	0.11997589	0.119999996	0.12	0.12	0.12
0.3	0.27	0.2697568255	0.2699998206	0.26999999999	0.27	0.27
0.4	0.48	0.47874666	0.4799973191	0.4799999953	0.48	0.48
0.5	0.75	0.7455284719	0.7499781609	0.7499999124	0.7499999997	0.75
0.6	1.08	10.067358778	1.07987879	1.079999046	1.079999994	1.08
0.7	1.47	1.439564127	1.469483776	1.46999281	1.469999915	1.4699999999
0.8	1.92	1.854846182	1.918188815	1.919958655	1.919999195	1.9199999986
0.9	2.43	2.302500276	2.424519799	2.42980657	2.429994174	2.429999846
1.0	3.00	2.767551396	2.985245757	2.999230962	2.99965795	2.999998665
$L.S.E$		7.564132 $e-02$	2.512823 $e-04$	6.305965 $e-07$	1.204603 $e-09$	1.806304 $e-12$

Applying the MADM 1 for solving our problem as in second case, The equation (11) with initial situation $u(0) = 0$, given $f(t) = \frac{6}{\Gamma(2.5)} t^{1.5} - \frac{5}{8\,\Gamma(2.2)} t^{5.2}$ and using the definition of R-L integral for order $\alpha_1 = 0.5$ and $m_{\alpha_1} = 1$, we obtain:

$$g(t) = {}_aJ_t^{\alpha_1} f(t) + \sum_{k=0}^{m_{\alpha_1}-1} \frac{u^{(k)}(0)}{k!} t^k = 3t^2 - \frac{5\Gamma(6.2)}{8\,\Gamma(2.2)\Gamma(6.7)} t^{5.7}$$

In other words, we can set: $f_1(t) = 3t^2$ and $f_2(t) = -\frac{5\Gamma(6.2)}{8\,\Gamma(2.2)\Gamma(6.7)} t^{5.7}$. The recursive components (13) leads to the following scheme:

$$\left.\begin{aligned} u_0(t) &= f_1(t) \\ u_1(t) &= f_2(t) + \lambda_1 \int_0^t \bar{\mathcal{K}}_1(t,s) A_0^1[u_0(s)]ds \\ u_{r+1}(t) &= \lambda_1 \int_0^t \bar{\mathcal{K}}_1(t,s) A_r^1[u_0(s), u_1(s), \cdots, u_r(s)]ds, \qquad r \geq 1 \end{aligned}\right\} \quad (21)$$

So that:

$$u_0(t) = f_1(t) = 3t^2$$

Using equation (7) where $r = 0$ and $\ell = 1$, we get:

$$A_0^1[u_0] = N_1\big(u_0(s)\big) = {}_0^CD_s^{0.8} u_0(s) = \frac{6}{\Gamma(2.2)} s^{1.2}$$

$$u_1(t) = f_2(t) + \lambda_1 \int_0^t \bar{\mathcal{K}}_1(t,s) A_0^1[u_0(s)]ds$$
$$= -\frac{5\Gamma(6.2)}{8\,\Gamma(2.2)\Gamma(6.7)} t^{5.7}$$
$$+ \frac{1}{3}\int_0^t \left[\frac{2}{\Gamma(3.5)} s(t-s)^{2.5} + \frac{2}{\Gamma(2.5)} s^2(t-s)^{1.5} + \frac{1}{\Gamma(1.5)} s^3(t-s)^{0.5}\right]\left[\frac{6}{\Gamma(2.2)} s^{1.2}\right] ds$$
$$= -\frac{5\Gamma(6.2)}{8\,\Gamma(2.2)\Gamma(6.7)} t^{5.7} + \frac{5\Gamma(6.2)}{8\,\Gamma(2.2)\Gamma(6.7)} t^{5.7} = 0$$

Thus, all second part of recursive relation (21) with each $r \geq 1$, we gain:

$$u_{r+1}(t) = \lambda_1 \int_0^t \bar{\mathcal{K}}_1(t,s) A_r^1[u_0(s), u_1(s), \cdots, u_r(s)]ds = 0$$

So, it is obvious that each component of $u_r, r \geq 1$ is zero. The solution is: $u(t) = 3t^2$. which is the exact solution for our problem.

Example (2): Let us take the nonlinear IFDE of V-H typewith variable coefficients for multi-higher fractional orders:

$${}_0^C D_t^{1.4} u(t) + t u(t) = f(t) + \int_0^t ts[{}_0^C D_s^{0.5} u(s)]^2 ds$$

Where theinhomogeneous term is

$$f(t) = \frac{6}{\Gamma(2.6)} t^{1.6} - \frac{36}{7\Gamma^2(3.5)} t^8 + t^4 - t$$

with the initial conditions $u(0) = -1$; $u'(0) = 0$ and has the exact solution $u(t) = t^3 - 1$.

By comparison with the fundamental equation (1), we will see that $n = m = 1$ and $P_1(t) = t$; $\alpha_1 = 1.4$, $\beta_0 = 0, \beta_1 = 0.5$; $\lambda_0 = 0, \lambda_1 = 1$ then from the equation above we've $m_{\alpha_1} = 2$; and the kernels with Hammerstein terms are $\mathcal{K}_0(t,s) = 0$; with $N_0(u(s)) = \mathcal{H}_0(s, u(s)) = 0; \mathcal{K}_1(t,s) = ts$; with $N_1(u(s)) = \mathcal{H}_1\left(s, {}_a^C D_s^{\beta_1} u(s)\right) = [{}_0^C D_s^{0.5} u(s)]^2$;and the inhomogeneous term$f(t)$.

Applying the SADM for solving our problem, the recursive formula (6) with initial condition conditions leads to the following scheme:

$$u_0(t) = {}_0J_t^{1.4}\left[\frac{6}{\Gamma(2.6)} t^{1.6} - \frac{36}{7\Gamma^2(3.5)} t^8 + t^4 - t\right] - 1$$
$$= t^3 - \frac{36\Gamma(9)}{7\Gamma^2(3.5)\Gamma(10.4)} t^{9.4} + \frac{\Gamma(5)}{\Gamma(6.4)} t^{5.4} - \frac{1}{\Gamma(3.4)} t^{2.4} - 1$$

Before finding $u_1(t)$ we must calculate $A_0^\ell[u_0] = N_\ell(u_0)$for all $\ell = 0,1$ using the formula (7):

$A_0^0[u_0(s)] = N_0(u_0(s)) = 0$

$$A_0^1[u_0(s)] = N_1\big(u_0(s)\big) = [{}_0^C D_s^{0.5} u_0(s)]^2$$
$$= \frac{36}{\Gamma^2(3.5)} s^5 - \frac{432\Gamma(9)}{7\Gamma^3(3.5)\Gamma(9.9)} s^{11.4} + \frac{12\Gamma(5)}{\Gamma(3.5)\Gamma(5.9)} s^{7.4}$$
$$- \frac{12}{\Gamma(3.5)\Gamma(2.9)} s^{4.4} + \frac{1296\Gamma^2(9)}{49\Gamma^4(3.5)\Gamma^2(9.9)} s^{17.8}$$
$$- \frac{72\Gamma(9)\Gamma(5)}{7\Gamma^2(3.5)\Gamma(9.9)\Gamma(5.9)} s^{13.8} + \frac{72\Gamma(9)}{7\Gamma^2(3.5)\Gamma(9.9)\Gamma(2.9)} s^{10.8}$$
$$+ \frac{\Gamma^2(5)}{\Gamma^2(5.9)} s^{9.8} - \frac{2\Gamma(5)}{\Gamma(5.9)\Gamma(2.9)} s^{6.8} + \frac{1}{\Gamma^2(2.9)} s^{3.8}$$

Also,

$$\bar{\mathcal{K}}_0(t,s) = {}_sJ_t^{\alpha_1} \mathcal{K}_0(t,s) = 0$$

$$\bar{\mathcal{K}}_1(t,s) = {}_sJ_t^{\alpha_1} \mathcal{K}_1(t,s) = {}_sJ_t^{1.4} st = \frac{1}{\Gamma(3.4)} s(t-s)^{2.4} + \frac{1}{\Gamma(2.4)} s^2 (t-s)^{1.4}$$

And

$${}_0J_t^{1.4}\big(\bar{P}_1(t) u_0(t)\big)$$
$$= {}_0J_t^{1.4}\left(-t\left[t^3 - \frac{36\Gamma(9)}{7\Gamma^2(3.5)\Gamma(10.4)} t^{9.4} + \frac{\Gamma(5)}{\Gamma(6.4)} t^{5.4} - \frac{1}{\Gamma(3.4)} t^{2.4} - 1\right]\right)$$
$$= \frac{-\Gamma(5)}{\Gamma(6.4)} t^{5.4} + \frac{36\Gamma(9)\Gamma(11.4)}{7\Gamma^2(3.5)\Gamma(10.4)\Gamma(12.8)} t^{11.8} - \frac{\Gamma(5)\Gamma(7.4)}{\Gamma(6.4)\Gamma(8.8)} t^{7.4}$$
$$+ \frac{\Gamma(4.4)}{\Gamma(3.4)\Gamma(5.8)} t^{4.8} + \frac{1}{\Gamma(3.4)} t^{2.4}$$

Thus:

$$u_1(t) = {}_0J_t^{1.4}\big(\bar{P}_1(t) u_0(t)\big) + \sum_{\ell=0}^{1} \lambda_\ell \int_0^t \bar{\mathcal{K}}_\ell(t,s) A_0^\ell[u_0(s)] ds$$
$$= -\frac{\Gamma(5)}{\Gamma(6.4)} t^{5.4} + \frac{124.8}{\Gamma(3.5)\Gamma(12.8)}\left[\frac{3\Gamma(9)}{7\Gamma(3.5)} + \frac{\Gamma(5)\Gamma(9.4)}{\Gamma(5.9)}\right] t^{11.8}$$
$$- \frac{\Gamma(5)\Gamma(7.4)}{\Gamma(6.4)\Gamma(8.8)} t^{7.4} + \frac{\Gamma(4.4)}{\Gamma(3.4)\Gamma(5.8)} t^{4.8} + \frac{1}{\Gamma(3.4)} t^{2.4}$$
$$+ \frac{36\Gamma(9)}{7\Gamma^2(3.5)\Gamma(10.4)} t^{9.4} - \frac{6220.8\Gamma(9)\Gamma(13.4)}{7\Gamma^3(3.5)\Gamma(9.9)\Gamma(16.8)} t^{15.8}$$
$$- \frac{88.8\Gamma(6.4)}{\Gamma(3.5)\Gamma(2.9)\Gamma(9.8)} t^{8.8} + \frac{26956.8\Gamma^2(9)\Gamma(19.8)}{49\Gamma^4(3.5)\Gamma^2(9.9)\Gamma(23.2)} t^{22.2}$$
$$- \frac{1209.6\Gamma(9)\Gamma(5)\Gamma(15.8)}{7\Gamma^2(3.5)\Gamma(9.9)\Gamma(5.9)\Gamma(19.2)} t^{18.2}$$
$$+ \frac{993.6\Gamma(9)\Gamma(12.8)}{7\Gamma^2(3.5)\Gamma(9.9)\Gamma(2.9)\Gamma(16.2)} t^{15.2} + \frac{12.8\Gamma^2(5)\Gamma(11.8)}{\Gamma^2(5.9)\Gamma(15.2)} t^{14.2}$$
$$- \frac{19.6\Gamma(5)\Gamma(8.8)}{\Gamma(5.9)\Gamma(2.9)\Gamma(12.2)} t^{11.2} + \frac{6.8\Gamma(5.8)}{\Gamma^2(2.9)\Gamma(9.2)} t^{8.2}$$

The noise terms$\pm\frac{\Gamma(5)}{\Gamma(6.4)}t^{5.4}$; $\pm\frac{1}{\Gamma(3.4)}t^{2.4}$ and $\pm\frac{36\Gamma(9)}{7\Gamma^2(3.5)\Gamma(10.4)}t^{9.4}$ appears in $u_0(t)$ and $u_1(t)$. Cancelling these terms from the zeros component $u_0(t)$ gives the solution which is the exact solution: $u(t) = t^3 - 1$ that satisfies the fractional integro-differential equation above.

Applying the MADM 1 for solving our problem, first from equation (11) with initial conditions$u(0) = -1$; $u'(0) = 0$, given $f(t)$ and using the definition of R-L integral for order $\alpha_1 = 1.4$ so $m_{\alpha_1} = 2$, we get:

$$g(t) = {}_aJ_t^{\alpha_1}f(t) + \sum_{k=0}^{m_{\alpha_1}-1}\frac{u^{(k)}(0)}{k!}t^k$$
$$= t^3 - \frac{36\Gamma(9)}{7\Gamma^2(3.5)\Gamma(10.4)}t^{9.4} + \frac{\Gamma(5)}{\Gamma(6.4)}t^{5.4} - \frac{1}{\Gamma(3.4)}t^{2.4} - 1 \qquad (22)$$

From $g(t)$ we assume that:

$$f_1(t) = t^3 - 1$$
$$f_2(t) = -\frac{36\Gamma(9)}{7\Gamma^2(3.5)\Gamma(10.4)}t^{9.4} + \frac{\Gamma(5)}{\Gamma(6.4)}t^{5.4} - \frac{1}{\Gamma(3.4)}t^{2.4}$$

We next use the recurrence formula (13) to obtain:

$$u_0(t) = f_1(t) = t^3 - 1$$
$${}_aJ_t^{\alpha_1}[\bar{P}_1(t)u_0(t)] = {}_0J_t^{1.4}[-t(t^3-1)] = -\frac{\Gamma(5)}{\Gamma(6.4)}t^{5.4} + \frac{1}{\Gamma(3.4)}t^{2.4}$$

Apply formula (7) to calculate $A_0^{\ell}[u_0] = N_{\ell}(u_0)$for all $\ell = 0,1$; so:

$$A_0^0[u_0(s)] = N_0\big(u_0(s)\big) = 0$$

$$A_0^1[u_0(s)] = N_1\big(u_0(s)\big) = [{}_0^CD_s^{0.5}u_0(s)]^2 = \frac{36}{\Gamma^2(3.5)}s^5$$

$$\int_0^t \bar{\mathcal{K}}_1(t,s)A_0^1[u_0(s)]ds = \int_0^t\left[\frac{1}{\Gamma(3.4)}s(t-s)^{2.4} + \frac{1}{\Gamma(2.4)}s^2(t-s)^{1.4}\right]\left[\frac{36}{\Gamma^2(3.5)}s^5\right]ds$$
$$= \frac{36*8\Gamma(7)}{\Gamma^2(3.5)\Gamma(10.4)}t^{9.4}$$

Thus, $u_1(t)$ formed by using equation (13):

$$u_1(t) = f_2(t) + {}_0J_t^{\alpha_1}[\bar{P}_1(t)u_0(t)] + \sum_{\ell=0}^{1}\lambda_\ell\int_0^t \bar{\mathcal{K}}_\ell(t,s)A_0^{\ell}[u_0(s)]ds$$
$$= -\frac{36\Gamma(9)}{7\Gamma^2(3.5)\Gamma(10.4)}t^{9.4} + \frac{\Gamma(5)}{\Gamma(6.4)}t^{5.4} - \frac{1}{\Gamma(3.4)}t^{2.4} - \frac{\Gamma(5)}{\Gamma(6.4)}t^{5.4}$$
$$+ \frac{1}{\Gamma(3.4)}t^{2.4} + \frac{36*8\Gamma(7)}{\Gamma^2(3.5)\Gamma(10.4)}t^{9.4} = 0$$

It follows immediately that:$u_{r+1}(t) = 0$,$\forall\, r \geq 1$. So $u(t) = t^3 - 1$ is the solution which is the exact expression for our nonlinear IFDE of V-H type.

Example (3): Take the nonlinear IFDE of V-H typewith variable coefficients on the interval [0,1] for multi-higher fractional orders:

$$ {}_{0}^{C}D_{t}^{0.4}u(t) = f(t) + \int_{0}^{t} \frac{t}{s}[\,{}_{0}^{C}D_{s}^{0.5}u(s)]^{2}ds \quad + \int_{0}^{t} (-t)^{3}exp(2s^{0.3})\frac{ds}{exp(\Gamma(1.3)\,{}_{0}^{C}D_{s}^{0.7}u(s))} $$

With one initial condition $u(0) = 5$, where

$$ f(t) = \frac{2}{\Gamma(1.6)}t^{0.6} - \frac{4}{\Gamma^{2}(1.5)}t^{2} + t^{4} $$

Now, from the equation above we have:

$$ \mathcal{K}_{2}(t,s) = -t^{3}ex\,p(2s^{0.3}) \quad ; \mathcal{K}_{1}(t,s) = \frac{t}{s} \;\; ; \mathcal{K}_{0}(t,s) = 0 $$

$$ \alpha_{1} = 0.4 \;\; ; \;\; \beta_{2} = 0.7\,,\; \beta_{1} = 0.5 \;\; , \; \beta_{0} = 0 \;\; ; \;\; m_{\alpha_{1}} = 1 $$

$$ N_{2}\big(u(s)\big) = \frac{1}{exp(\Gamma(1.3)\,{}_{0}^{C}D_{s}^{0.7}u(s))} \quad ; \quad N_{1}\big(u(s)\big) = [\,{}_{0}^{C}D_{s}^{0.5}u(s)]^{2} $$

Using equation (11) we get:

$$ g(t) = 2t + 5 - \frac{8}{\Gamma^{2}(1.5)\Gamma(3.4)}t^{2.4} + \frac{24}{\Gamma(5.4)}t^{4.4} $$

Assume that from $g(t)$, we putting

$$ f_{1}(t) = 2t + 5 \quad ; \quad f_{2}(t) = -\frac{8}{\Gamma^{2}(1.5)\Gamma(3.4)}t^{2.4} + \frac{24}{\Gamma(5.4)}t^{4.4} $$

Using equation (8), with $\ell = 1{,}2$ we get:

$$ A_{0}^{1} = N_{1}(u_{0}) = \frac{4}{\Gamma^{2}(1.5)}s $$

$$ A_{0}^{2} = N_{2}(u_{0}) = exp(-2s^{0.3}) $$

*Applying the MADM 1*recurrence formula (13) to obtain:

$$ u_{0}(t) = f_{1}(t) = 2t + 5 $$

$$ u_{1}(t) = f_{2}(t) + \;{}_{0}J_{t}^{0.4}\int_{0}^{t}\frac{4t}{\Gamma^{2}(1.5)}ds - \;{}_{0}J_{t}^{0.4}\int_{0}^{t}t^{3}ds = 0 $$

It follows immediately that: $u_{k}(t) = 0 \;, \forall\, k \geq 2$. So the exact solution $u(t) = 2t + 5$ readily obtained that satisfies the fractional integro-differential equation above.

5. Conclusion

In this study, we derive a novel technique based totally on Adomian and modified Adomian decomposition method which has been efficaciously and successfully applied to finding the approximate as well as specific solution for multi-higher fractional nonlinear integro-differential equations of the Volterra-Hammerstein type. These methods are very powerful and efficient in finding analytical as well as a numerical solution to our problem. It provides a more realistic series solution that converges very rapidly to the solutions. Sometimes the process of finding a standard Adomian decomposition method is not easy, so we use the Modifications and we will use the truncated collection series for the numerical purpose as in instance example (1).

A Considerable advantage of the method is that if we not obtain the exact solution, then the solution can be written as a shape of truncated collection, after which the required function may be without problems evaluated for arbitrary values. To obtain the best approximation we must use more terms.Sometimes the noise terms in the Adomian method will not appear, so we use modified Adomian decomposition method.

Conflict of Interests.
There are non-conflicts of interest

References

1. I. podldubny,"Fractional Differential Equation", Academic press, San Diego, 1999.
2. R. Hilfer,"Applications of Fractional Calculus in Physics", World Scientific, 2000.
3. Anatoly A.Kilbas, Hari M. Srivastava and Juan J. Trujillo,"Theory and Applications of Fractional Differential Equations", Elsevier B.V. Netherlands, 2006.
4. R. C. Mittal and Ruchi Nigam,"Solution of Fractional Integro-Differential Equations by Adomian Decomposition Method ”Int. J. of Appl. Math. and Mech. 4(42): 87-94, 2008.
5. Rifaat Saad Abdul-Jabbar,“A numerical Scheme for the solution of fractional integro-differential equations using the Adomian decomposition method”, *J.of university of anbar for pure science,Vol. 6: No.1:* 2012.
6. A.H. Ali and A.S.J. Al-Saif,“Adomian decomposion method for solving some models of nonlinear partial differential equations”, *Basrah Journal of Science (A), Vol.26(1), 1-11,*2008.
7. Sennur Somali and Guzin Gokmen,“Adomian Decomposition Method for Nonlinear Sturm-Liouville problems”, *Surveys in Mathematics and its Applications, volume 2, 11-20,* 2007.
8. Abdul-Majid Wazwaz,“Linear and Nonlinear Integral Equations, Methods and Application”, Springer Heidelberg Dordrecht London New York, 2011.
9. Kenneth S. Miller and Bertram Ross,“An Introduction to the Fractional Calculus and Fractional Differential Equations”, John Wiley & Sons (New York), 1993.
10. Shazad Shawki Ahmed,“On System of Linear Volterra Integro- Fractional Differential Equations”, Ph.D. Thesis, Sulaimani University, 2009.
11. Zaid M. odibat and Nabil T. shawagfeh,“Generalized Taylor’s Formula”, *Applied Mathematics and computation, Vol.186 pp. 286-293*, 2007.
12. Marc Weilbeer,“Efficient Numerical Methods for Fractional Differential Equations and their Analytical Background”, US Army Medical Research and Material Command, 2005.
13. George Adomian,“Solving Frontier Problems of Physics: The Decomposition Method”, Kluwer Academic, 1994.
14. Duan, J.S.,“Convenient analytic recurrence algorithms for the Adomian polynomials”, *Appl. Math. Comput. 217, 6337–6348*, 2011.
15. Eman Al Awawdah, “The Adomian Decomposition Method for Solving Partial Differential Equations”, M.Sc. Thesis Birzeit University, Palestine, 2016.
16. Shazad Sh. Ahmed, Shokhan Ahmed Hama Salih and Mariwan R. Ahmed,“Laplace Adomian and Laplace Modified Adomian Decomposition Methodsfor Solving Nonlinear Integro-Fractional Differential Equations of theVolterra-Hammerstein Type”, *Iraqi Journal of Science, Vol.60, No.10, pp: 2207-2222*, 2019.

17. Abdul-Majid Wazwaz, "A reliable modification of Adomian decomposition method", *Applied Mathematics and Computation, 102, pp 77-86,* 1999.
18. Abdul-Majid Wazwaz and Salah M. El-Sayed, "A new modification of the Adomian decomposition method for linear and nonlinear operators", *Applied Mathematics and Computation, 122, pp: 393-405,* 2001.

الخلاصة

هذا البحث يطبق بفعاليه طريقه التحليل الادومياني وطريقه التحليل الادومياني المعدله كتقنيات عددية لتعيين الحل شبه التحليلي او الحل شبه التقريبي للمعادلات التفاضليه التكامليه اللاخطيه للرتب الكسريه (IFDE) من نوع فولتيرا–هاميرشتين (V–H) والتي توصف فيها المشتقه الكسريه المتعدده العليا بنمط كابوتو. في هذا النهج سنغير بشكل جذري ال (IFDE) لنوع (V–H) الى بعض معادلات جبريه تكراريه وان الحل لهذه المعادلات هو بمثابه مجموع من المتتابعات اللاعدديه (Countless) لمركبات متقاربه نوعيا للحل المستند (المعتمد) على الحدود الضوضائيه وذلك في حاله عدم حصولنا على حل من النوع المغلق وان الحدود المقطوعه (المحذوفه) يستخدم للاغراض العدديه. واخيرا تم اعطاء امثله لتوضيح هذه الافكار والاعتبارات.