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On Central Closure of Totally Prime Algebras

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Abstract

Seemingly, it is not easy to finding a norm on the center closure of a normed algebra. Cabrera and Mohammed in [1,2], they defined a norm on the center closure of two classes of algebras namely totally multiplicatively prime and associative totally prime algebras. In this paper, we prove the same result in a general setting by considering the class of non-associative totally prime algebras which have a prime multiplication algebra.

Keyword: central closure, multiplicatively prime algebra, totally prime algebra.

Introduction

The central closure of algebra A is an algebra extension of A, denoted by $Q(A)$ and the eventual fact that $Q(A)$ is reduced to A, when the extended centroid of A equal to the base field (see [3,4]). In [3] Cabrera and Palacios introduced a totally prime algebra as generalization of ultraprime algebra and they proved that a totally prime complex algebra is centrally closed.

Cabrera and Mohammed in [1] introduce totally multiplicatively prime algebra which is a subclass of totally prime algebra. Also, Cabrera and Mohammed proved in [1, Theorem 3.2] the following result which is our aim in this paper:

If A is totally multiplicatively real prime algebra with extended centroid equal to $\mathbb{C}$, then there exists a complex norm algebra on the central closure $Q(A)$ of A. Moreover, $Q(A)$ is totally multiplicatively prime complex algebra and the inclusion of A into $Q(A)$ and $M(A)$ into $M(Q(A))$ are topological, where $M(A)$ and $M(Q(A))$ are the multiplication algebra of A and $Q(A)$ respectively.

Later this result was proved by Cabrera and Mohammed in [2, corollary 2] for a class of associative totally prime algebras. In this paper, we prove above result but in a general setting by considering A to be non-associative totally prime and to be multiplicatively prime algebra.

1. Totally prime algebra and its multiplication algebra are also prime.

Throughout this paper, the algebra A considered to be not necessary associative over K equal $\mathbb{R}$ or $\mathbb{C}$. We denote by $L(A)$ the algebra of all linear operators on A. For $a \in A$, we denote by L_a to be a linear operator from A into A, defined by $L_a(x) = ax$ for all $x \in A$ and R_a to be a linear operator from A into A, defined by $R_a(x) = xa$ for all $x \in A$. The operators L_a and R_a are called left and right multiplication by a respectively. Also, we denote by $M(A)$ to be the multiplication algebra of A, defined as a subalgebra of $L(A)$ generated by the identity operator Id_A and the set $\{L_a, R_a : a \in A\}$. We recall that the algebra A is called prime if, for two ideals I and J of A, $IJ = 0$ implies $I = 0$ or $J = 0$. Recall from [1] that an algebra A is multiplicatively prime, if A and $M(A)$ is prime.

For $x, y \in A$, define $N_{x,y}: M(A) \times M(A) \longrightarrow A$ by $N_{x,y}(F, G) = F(x)G(y)$ for all $F, G \in M(A)$. From [3] the totally prime algebra is a normed algebra $(A, \|\cdot\|)$ with a positive constant c such that $\|N_{x,y}\| \geq c\|x\|\|y\|$ for all $x, y \in A$.

We will summarize two definitions, the extended centroid and the central closure of a prime algebra A. A partially defined centralizer (in short p.d.c.) on A is a linear mapping $f: dom\,(f) \longrightarrow A$, where $dom\,(f)$ is a non-zero ideal of A and satisfying $f(ax) = af(x)$ and $f(xa) = f(x)a$, for all $a \in A$ and $x \in dom(f)$. The relation $\simeq$, defined on the set of all p.d.c.'s on A, by $g \simeq h$ if and only if there is a p.d.c. f on A such that g and h are extensions of f, is an equivalence relation. The extended centroid of A, denoted by $C(A)$, is the set of all equivalence classes of p.d.c.'s, with the operations induced by the sum and the composition of p.d.c.'s, the extended centroid becomes a field when A is prime algebra. If $C(A)$ is equal to the base field, then A is called centrally closed. The central closure of A denoted by $Q(A)$ is define as a prime algebra A, the central closure of A, can be seen as an $Q(A) = A \otimes C(A)$, also $Q(A)$ is the algebra A over the field $C(A)$. For more details, see [5].

Note that a real free non-associative algebra generated by any non-empty set is totally prime algebra and its multiplication algebra is also prime but it is not totally multiplicatively prime algebra. For more details see [1].

Theorem 2.2

Let $(A, \|\cdot\|)$ be totally prime real algebra whose multiplication algebra $M(A)$ is prime and the extended centroid of A equal to $\mathbb{C}$, then there exists a complex algebra norm $\|\cdot\|_c$ on $Q(A)$ such that the inclusions of $(A, \|\cdot\|)$ into $(Q(A), \|\cdot\|_c)$ and $(M(A), \|\cdot\|)$ into $(M(Q(A)), \|\cdot\|_c)$ are topological. Further $Q(A)$ is totally prime and $Q(A)$ is multiplicatively prime complex algebra.

Proof:

From [1, Theorem 3.2], we have $Q(A) = \{x + yi, x, y \in A\}$, where i is the imaginary unit in $\mathbb{C}$. Also, for $i \in C(A)$ we have $dom(i) = idom(i)$ and $dom(i)$ is a non-zero ideal of $Q(A)$. Define the set $D = \{F \in M(A): F(A) \in dom(i)$ and $iF \in M(A)\}$. Also, from [1, Theorem 3.2] D is an ideal of $M(Q(A))$. For any $q \in Q(A)$, then the evaluation operator E_q^D is linear operator from D into A defined by $E_q^D(T) = T(q)$ for all $T \in D$.

It easy to show that the evaluation operator E_q^D is linear operator from D into $dom(i)$, since D is an ideal of $M(Q(A))$.

We are going to prove that the mapping $E_q^D: D \longrightarrow A$ $(q \in Q(A))$ is bounded. First we prove that the p.d.c. $i: dom(i) \longrightarrow A$ is bounded.

Let $x, y \in dom(i)$ and $F, G \in M(A)$ with $\|F\| = \|G\| = \|y\| = 1$, we have.

$\|N_{y,i(x)}(F,G)\| = \|F(y)G(i(x))\|$

$= \|F(y)iG(x)\|$ (taking into account that $i \in C(A)$ and $dom(i) = idom(i)$)

$= \|iF(y)G(x)\|$

$= \|F(i(y))G(x)\|$

$\leq \|F(i(y))\|\|G(x)\|$

$\leq \|F\|\|i(y)\|\|G\|\|x\|$

Since A is a totally prime algebra, there exists $c > 0$, such that

$$c\|y\|\|i(x)\| \leq \|N_{y,i(x)}\|$$

$$= \sup_{F,G\in A} \{\|F(y)G(i(x))\|, \|F\| = \|G\| = 1\}$$

$$\leq \sup_{F,G\in A} \{\|F\|\|i(y)\|\|G\|\|x\|, \|F\| = \|G\| = 1\}$$

$$= \|i(y)\|\|x\|$$

We get that $\|i(x)\| \leq \frac{1}{c}\|i(y)\|\|x\|$ for all $x \in dom(i)$, so i is bounded. The rest proof of E_q^D is bounded is similar to that in [1, Theorem 3.2].

Now, for $q \in Q(A)$, we define $\|q\|_r = \|E_q^D\|$, the prove of $(Q(A), \|\cdot\|_r)$ is real normed algebra is similar to that in [3, Theorem 3.2].

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We are going to prove the equivalent between $\|\cdot\|_r$ and $\|\cdot\|$ on A, let $a \in A$ and $T \in D$,

$$\|E_a^D(T)\| = \|T(a)\|$$

$$\leq \|T\|\|a\|$$

$$\|a\|_r = \|E_a^D(T)\|$$

$$= \sup_{T\in D}\{\|E_a^D(T)\|, \|T\| = 1\}$$

$$\leq \sup_{T\in D}\{\|T\|\|a\|, \|T\| = 1\}$$

$$= \|a\|$$

Now, let $F, G \in M(A)$ such that $\|F\| = \|G\| = 1$, for fixed $T \in D$ and $x \in dom(i)$, such that $\|T(x)\| \neq 0$, then

$$\left\|N_{T(x),a}(F,G)\right\| = \|FT(x)G(a)\|$$

$$= \left\|L_{FT(x)}G(a)\right\|$$

Since $T \in D, F \in M(A)$, from the proof of [4, Theorem 1] D is an ideal of $M(A)$, so $FT \in D$, then $FT(x) \in dom(i)$, for any $z \in A$, we get thtat $L_{FT(x)}G(z) \in dom(i)$ for all $z \in A$, so $L_{FT(x)}G(A) \subseteq dom(i), i\left(L_{FT(x)}G(A)\right) \subseteq i(dom(i)) = dom(i) \subseteq A$, so $L_{FT(x)}G \in D$.

$$\left\|N_{T(x),a}(F,G)\right\| = \left\|E_a^D(L_{FT(x)}G)\right\|$$

$$\leq \|E_a^D\|\left\|L_{FT(x)}G\right\|$$

$$\leq \|a\|_r\|FT(x)\|\|G\|$$

$$\leq \|a\|_r\|F\|\|T(x)\|\|G\|$$

$$\left\|N_{T(x),a}\right\| = \sup_{F,G\in M(A)}\{\|FT(x)G(a)\|, \|F\| = \|G\| = 1\}$$

$$\leq \sup_{F,G\in M(A)}\{\|a\|_r\|F\|\|T(x)\|\|G\|, \|F\| = \|G\| = 1\}$$

$$= \|a\|_r\|T(x)\|$$

Since A is a totally prime algebra, then

$$c\|T(x)\|\|a\| \leq \left\|N_{T(x),a}\right\| \leq \|a\|_r\|T(x)\|$$

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$c\|a\| \leq \|a\|_r$

We get that for all $a \in A$, $c\|a\| \leq \|a\|_r \leq \|a\|$------(1)

Hence the inclusion of $(A, \|\cdot\|)$ into $(Q(A), \|\cdot\|_r)$ is topological. Note that, the proof of the inclusion of $(M(A), \|\cdot\|)$ into $(M(Q(A)), \|\cdot\|_r)$ is topological, similar to that in [1, Theorem 3.2].

For proving $Q(A)$ is a totally prime algebra, let $q_1, q_2 \in Q(A), G_1, G_2 \in M(A), F_1, F_2 \in M(Q(A)), T_1, T_2 \in D$. We denoted by $N^D_{q_1,q_2}$ the linear operator from $D \times D$ into A, given by $N^D_{q_1,q_2}(T_1, T_2) = T_1(q_1)T_2(q_2)$ for all $T_1, T_2 \in D$. We denoted by N_{q_1,q_2} the linear operator from $M(Q(A)) \times M(Q(A))$ into $Q(A)$, given by $N_{q_1,q_2}(F_1, F_2) = F_1(q_1)F_2(q_2)$ for all $F_1, F_2 \in M(Q(A))$. Now

$$c^2\left\|E^D_{q_1}(T_1)\right\|\left\|E^D_{q_2}(T_2)\right\| = c^2\|T_1(q_1)\|\|T_2(q_2)\|$$

$$\leq c\left\|N_{T_1(q_1),T_2(q_2)}\right\|$$

$$= c \sup_{G_1,G_2 \in M(A)} \{\left\|N_{T_1(q_1),T_2(q_2)}(G_1, G_2)\right\|, \|G_1\| = \|G_2\| = 1\}$$

$$= \sup_{G_1,G_2 \in M(A)} \{c\left\|N_{T_1(q_1),T_2(q_2)}(G_1, G_2)\right\|, \|G_1\| = \|G_2\| = 1\}$$

$$= \sup_{G_1,G_2 \in M(A)} \{c\left\|N_{q_1,q_2}(G_1T_1, G_2T_2)\right\|, \|G_1\| = \|G_2\| = 1\}$$

$$\leq \sup_{G_1,G_2 \in M(A)} \{\left\|N^D_{q_1,q_2}(G_1T_1, G_2T_2)\right\|_r, \|G_1\| = \|G_2\| = 1\}$$

$$\leq \sup_{G_1,G_2 \in M(A)} \{\left\|N^D_{q_1,q_2}\right\|_r \|G_1T_1\|\|G_2T_2\|, \|G_1\| = \|G_2\| = 1\}$$

$$\leq \sup_{G_1,G_2 \in M(A)} \{\left\|N^D_{q_1,q_2}\right\|_r \|G_1\|\|T_1\|\|G_2\|\|T_2\|, \|G_1\| = \|G_2\| = 1\}$$

$$= \left\|N^D_{q_1,q_2}\right\|_r \|T_1\|\|T_2\|$$

$$\leq \left\|N_{q_1,q_2}\right\|_r \|T_1\|\|T_2\|$$

We get that $c^2\left\|E^D_{q_1}(T_1)\right\|\left\|E^D_{q_2}(T_2)\right\| \leq \left\|N_{q_1,q_2}\right\|_r \|T_1\|\|T_2\|$, Now

$$c^2\|q_1\|_r\|q_2\|_r = c^2\left\|E^D_{q_1}\right\|\left\|E^D_{q_2}\right\|$$

$$= c^2 \sup_{T_1,T_2 \in D} \{\left\|E^D_{q_1}(T_1)\right\|\left\|E^D_{q_2}(T_2)\right\|, \|T_1\| = \|T_2\| = 1\}$$

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$$= \sup_{T_1,T_2\in D}\{c^2\|E^D_{q_1}(T_1)\|\|E^D_{q_2}(T_2)\|, \|T_1\| = \|T_2\| = 1\}$$

$$\leq \sup_{T_1,T_2\in D}\{\|N_{q_1,q_2}\|_r\|T_1\|\|T_2\|, \|T_1\| = \|T_2\| = 1\}$$

$$= \|N_{q_1,q_2}\|_r$$

$$c^2\|q_1\|_r\|q_2\|_r \leq \|N_{q_1,q_2}\|_r$$

So $(Q(A), \|\cdot\|_r)$ is totally prime algebra and $M(Q(A))$ is prime by [4, Corollary 1].

Since $\|\cdot\|_r$ is an algebra real norm on $Q(A)$ for which the mapping $iId_{Q(A)}$ from $Q(A)$ into $Q(A)$, defined by $iId_{Q(A)}(q) = iq$ for $q \in Q(A)$ is bounded, it follows from [6, Theorem 1.3.3] that, we can get complex norm defined by $\|q\|_c = \sup_{\theta\in\mathbb{R}}\|(cos\theta + isin\theta)q\|_r$ for all $q \in Q(A)$ and satisfying $\|q\|_r \leq \|q\|_c \leq c_1\|q\|_r$ with $c_1 = 1 + \|iId_{Q(A)}\|_r$

So the inclusions of $(A, \|\cdot\|)$ into $(Q(A), \|\cdot\|_c)$ and $(M(A), \|\cdot\|)$ into $(M(Q(A)), \|\cdot\|_c)$ are to be topological.

Conflict of Interests.
There are non-conflicts of interest

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الخلاصة

على مايبدو، ليس من السهولة إيجاد معيار للأنغلاق المركزي للجبور المعيارية. كابريرا – محمد في [1,2] أوجدا معيار للأنغلاق المركزي على صنفين من الجبور هما الجبور الأولية المضروبة كلياً والجبور الأولية الكلية التجميعية. في هذا البحث ، سنبرهن نفس النتيجة ولكن على صنف أعم من الجبور وهو الجبور الأولية الكلية الغير تجميعية التي لها جبر مضروبات أولي.

الكلمات الدالة: الأنغلاق المركزي ،جبر أولي مضروب ، جبر أولي كلي.