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Determination Force/Source Function Dependent on Space Under the Non-classical Condition Data

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Abstract.

In this present paper deal with an inverse force/source problem for a one-dimension wave equation under the non-classical condition with the right end displacement boundary as the additional measurement. Furthermore, inverse force/source problem comes from splitting wave equation, which is unstable while adding noise under the right end boundary condition. Numerical algorithm is based on finite different method (FDM) combine with separation variable. L-curve method and min norm is applied to reach regularization parameter where zeroth order Tikhonov regularization is proposed. Figures in section 3 are shown how accurate this unusual condition working in the wave equation.

Keywords: Inverse force/source problem; L-curve method with min norm; The right end displacement measurement; Tikhonov regularization; Unusual condition.

1 Introduction

Indeed, the latter have been appearing in current literature with increasing frequency for solving inverse source problem [1]–[3]. Various conditions but not quite a lot have been approached to approximate the solution of inverse source problem [4]–[8]. But still there are many conditions not establishing. The target of this article to get approximate solution of force function dependent in space by using non-classical condition place flux tension of the string at the right end condition and measuring the right end displacement as addition condition.

The solutions explored here will be useful particularly in applied mathematics, theoretical mechanics, and mathematical physics [9]. Moreover, mathematical method setup in this paper improvement of finite difference method, separation variable, Tikhonov regularization, and L-curve method. In addition, theorems for existences and uniqueness for ill-posed inverse force/source problem are provided in [1], [3] remain to check stability and applying examples, which have worked out in this study

The organization of the research is: The mathematical formulation in Section 2, numerical results and discussed in Section 3 and conclusions in Section 4.

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2 Mathematical Formulation

The governing splitting wave equation in boundary $[0, D]$ (i.e $(x,t) \in (0,D) \times (0,T)$) and $T > 0$ for a numerical force function [4]-[8]

$$\phi_{tt}(x,t) = \nabla^2\phi(x,t) + F(x) = \begin{cases} \psi_{tt}(x,t) = \nabla^2\psi(x,t), \\ \varphi_{tt}(x,t) = \nabla^2\varphi(x,t) + F(x) \end{cases} \quad (1)$$

$$\phi(x,0) = \phi_0(x) = \psi(x,0), \quad \phi_t(x,0) = v_0(x) = \psi_t(x,0), \quad x \in [0,D], \quad (2)$$

$$\varphi(x,0) = 0, \quad \varphi_t(x,0) = 0, \quad x \in [0,D], \quad (3)$$

$$\phi_x(D,t) = q_D(t) = \psi_x(D,t), \quad \phi_{xx}(0,t) = qq_0(t) = \psi_{xx}(0,t), \quad t \in (0,T), \quad (4)$$

$$\varphi_x(D,t) = 0, \quad \varphi_{xx}(0,t) = 0, \quad t \in (0,T), \quad (5)$$

$$\phi(D,t) = p_D(t), \quad t \in (0,T), \quad T > 0. \quad (6)$$

Where (6) the right end displacement measurement is extra condition to approximate $\varphi(D,t) = p_D(t) - \psi(D,t)$ in order to accurate approach $F(x)$. Sequentially, $\{\phi(x,t), F(x), \phi_0, v_0\}$ represent {displacement, force, initial displacement, velocity}, respectively. Consequently, unusual boundary data $qq_0(t)$ represent the left end of bar where is restrained with a rotation spring [3], with the flux tension of the string at the right end as extra condition $\phi_x(D,t)$.

Applying FDM to $\psi_{tt}(x,t) = \nabla^2\psi(x,t)$ with equations (2) and (4) to get $\psi(D,t)$ [1], [4]-[9] as $\psi_{i,j} \coloneqq \psi(x_i, t_j)$, where $x_i = ih\ \left(h = \frac{D}{M}\right), t_j = jk\ \left(k = \frac{T}{N}\right)$, for $i = \overline{0,M}, j = \overline{0,N}$ and $r = \frac{ck}{h}$ [5], [6], [10]

$$\psi_{i,j+1} = r^2\psi_{i+1,j} + 2(1 - r^2)\psi_{i,j} + r^2\psi_{i-1,j} - \psi_{i,j-1}, \quad i = \overline{1,(M-1)}, j = \overline{1,(N-1)}, \quad (7)$$

$$\psi_{i,1} = \frac{1}{2}r^2\phi_0(x_{i+1}) + (1 - r^2)\phi_0(x_i) + \frac{1}{2}r^2\phi_0(x_{i-1}) + kv_0(x_i), \quad i = \overline{1,(M-1)}, j = 0 \quad (8)$$

$$\psi_{i,0} = \phi_0(x_i), \quad i = \overline{0,M}, \quad \frac{\psi_{i,1} - \psi_{i,-1}}{2k} = v_0(x_i), \quad i = \overline{1,(M-1)}, \quad (9)$$

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$$\frac{\partial \psi}{\partial x}(D, t_j) = \frac{3\psi_{M,j} - 4\psi_{M-1,j} + \psi_{M-2,j}}{2h},$$

$$qq_0(t_j) = \frac{\psi_{i+1,j} + \psi_{i-1,j} - 2\psi_{i,j}}{h^2}, \quad (10)$$

$j = \overline{0, N}, j = \overline{1, M},$

$$\psi = (D, t_j) = \psi_{m,j}, \quad j = \overline{0, N}. \quad (11)$$

Then substitute (11) in $\varphi(D, t_j) = p_D(t_j) - \psi(L, t_j)$ where $\varphi(D, t_j)$ have obtained from separation variable for $\varphi_{tt}(x,t) = \nabla^2 \varphi(x,t) + F(x)$ with (3), (5), and K is a truncation number

$$\varphi(D, t; \underline{b}) = p_D(t) - \psi(D, t) =: g(t) = \frac{\sqrt{2}}{c^2} \sum_{s=1}^{K} \frac{b_s}{\lambda_s{}^2} (1 - \cos(c\lambda_s t)) \sin(\lambda_s D), \quad (12)$$

$\lambda_s = \frac{(s - \frac{1}{2})\pi}{L}$; $s = \overline{1, K}$ and $t \in [0, T]$.

Equation (12) is linear algebra system that mean $g = bA$, where $A = A_{ns} = \frac{\sqrt{2}(1 - \cos(c\lambda_s t))\sin(\lambda_s D)}{c^2 \lambda_s^2}$, $b = b_s$ and $g = \varphi(D, t; \underline{b})$, where b can approach by $\underline{b}_\lambda = (A^{\mathrm{tr}} A)^{-1} A^{\mathrm{tr}} \underline{g}$, after that put in equation below for getting numerical solution $F(x)$.

$$F_s(x) = \sqrt{2} \sum_{s=1}^{K} b_s \sin(\lambda_s x), \quad x \in (0, D). \quad (13)$$

Next step, add noisy to the $\mathrm{g}^{\epsilon}(\mathrm{t_n}) = \mathrm{p}_{\mathrm{D}}^{\epsilon}(\mathrm{t_n}) - \psi(\mathrm{D}, \mathrm{t_n}) = \mathrm{g}(\mathrm{t_n}) + \epsilon$ where $\mathrm{p}_{\mathrm{D}}^{\epsilon}(\mathrm{t_n}) = \mathrm{p_D}(\mathrm{t_n}) + \epsilon$ in $\mathrm{n} = \overline{1, \mathrm{N}}$, and ϵ is a Gaussian normal distribution as mean zero and standard deviation σ given by the $\sigma = \gamma\% \times \max_{t \in [0,T]} |\mathrm{p_D}(t)|$ [4]-[8] and $\gamma\%$ represents the percentage of noise.

Finally, the method of regularization are used to stable this noise which is affect on approximation solution of $F(x)$. This regularization method is a zeroth-order Tikhonov regularization (for more detail can see [4]–[6])

$$\underline{b}_{\lambda} = (A^{tr}A + \lambda I)^{-1}A^{tr}\underline{g}^{\epsilon}, \quad (14)$$

3 Numerical Results and Discussion

Same example in [4], [6]-[8] with analytic solution for $(\phi(x,t), F(x))$ have applied because of the objective of this study to see how accurate unusual condition are working, such that $c = 1, D = 1$ and $T = 1$

$$\phi(x,t) = sin(\pi x) + t + \frac{t^2}{2}, \quad F(x) = 1 + \pi^2 \, sin(\pi x), \quad x \in [0,1], \quad (15)$$

$$\phi(x,0) = \phi_0(x) = sin(\pi x), \quad \phi_t(x,0) = v_0(x) = 1, \quad x \in [0,1], \quad (16)$$

$$\phi_x(D,t) = q_D(t) = -\pi, \quad \phi_{xx}(0,t) = qq_0(t) = 0, \quad t \in (0,1]. \quad (17)$$

Additional overdetermination is the right end displacement

$$\phi(D,t) = p_D(t) = t + \frac{t^2}{2}. \quad (18)$$

Starting with using finite difference (FDM) (7)-(11) to find $\psi(L,t)$, as clear in Figure 1 the result so converge to each other as $M = N = \{20,40,80\}$ increase, even the exact solution is not known.

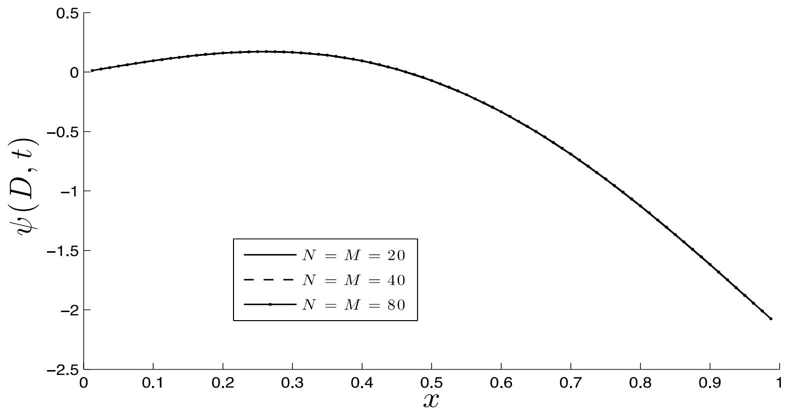

Figure 1 The numerical results for $\psi(D,t)$ using non-classical boundary condition and FDM (7)-(11) with $M = N = \{20, 40, 80\}$.

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Next step finding $\underline{b}$ from equation $\underline{b}_\lambda = (A^{tr}A)^{-1}A^{tr}\underline{g}$ where based on $\psi(L,t)$. Then compare with exact $b_s = \sqrt{2}\int_0^1 F(x)\sin\left((s-\frac{1}{2})\pi x\right)dx$ (see [4,6-8]). Moreover, Figure 2 shows good approach have reached when proposed $\phi_{xx}(0,t) = 0$ as a new work in this study. Sequentially, substituting numerical values $\underline{b}$ in equation (13) where $K \in \{5,10,20\}$ and $M = N = 80$ in order to find numerical approximate force/source function, and it is obvious in Figure 3 converges numerical to exact has obtained.

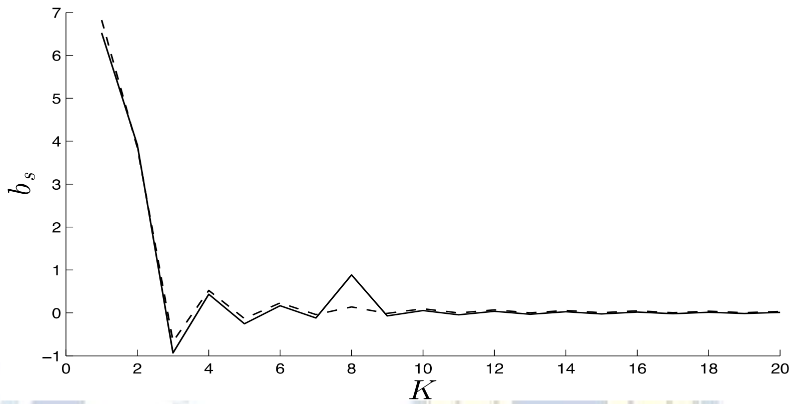

Figure 2 The numerical solution (…) for $\mathbf{\underline{b}}$ for $\mathbf{K = 20, N = 80}$, obtained from equation $\mathbf{\underline{b}_\lambda = (A^{tr}A)^{-1}A^{tr}\underline{g}}$ in compare with the exact solution $\mathbf{b_s = \sqrt{2}\int_0^1 F(x)\sin\left((s-\frac{1}{2})\pi x\right)dx}$ (-)

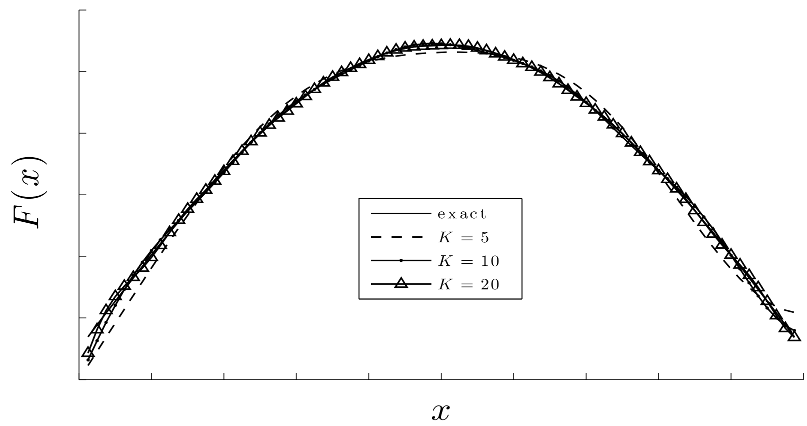

Figure 3 The exact solution (15) for $\boldsymbol{F(x)}$ in comparison with the numerical solution (13) applying unusual condition $\boldsymbol{\phi_{xx}(0,t) = 0}$, for various $\boldsymbol{K \in \{5,10,20\}}$.

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In Figure 4 appears F_{app} oscillations high where amount of noise $\gamma\% = 1\%$ adding in $g^{\epsilon}(t_n) = p_D^{\epsilon}(t_n) - \psi(D, t_n) = g(t_n) + \epsilon$. Consequently, zeroth-order Tikhonov regularization have used as a method for regularizing the solution, several parameters λ test such as $\lambda \in \{10^{-5}, 10^{-4}, 5 \times 10^{-4}, 10^{-3}, \dots, 10^{0}\}$, as clear λ include in equation (14). In addition, for finding which λ is giving a good approximate value, first: minimum norm error $||F_{\text{numerical}} - F_{\text{exact}}|| = \sqrt{\sum_{n=1}^{N}(F_{\text{numerical}}(t_n) - F_{\text{exact}}(t_n))^2}$, second: L-curve criterion [4-8], have applied as shown in Figure 5. In both figures 5(a) and 5(b) receive $\lambda = 5 \times 10^{-4}$ best approach $F(x)$ reaches with exact solution which clear in Figure 6. From Figure 6 one can say accurate and converge solution has gotten in applying unusual boundary condition $\phi_{xx}(0, t)$ which is the main goal in this research.

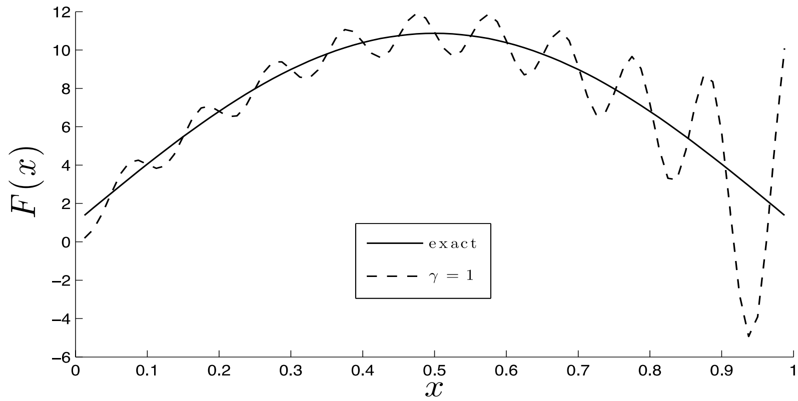

Figure 4 The exact solution (15) for F(x) in comparison with the numerical solution (13), for $\gamma\% = 1\%$ and $K = 20$ using non-classical boundary condition, in noisy data

(a) (b)

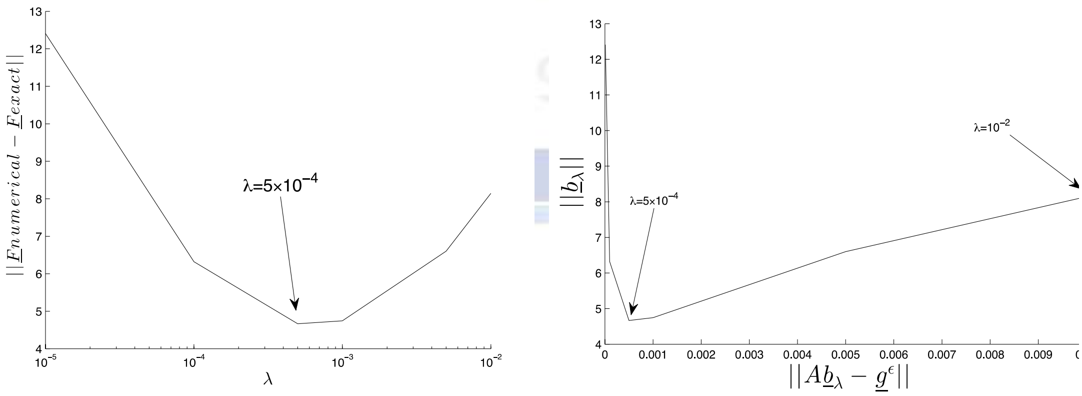

Figure 5: The accuracy (a) $||F_{\text{numerical}} - F_{\text{exact}}||$ (b) in $||\underline{b}_\lambda||$ as a function of λ for

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$\mathrm{K} = 20$ and $\gamma\% = 1\%$, noise, non-classical boundary condition are applied.

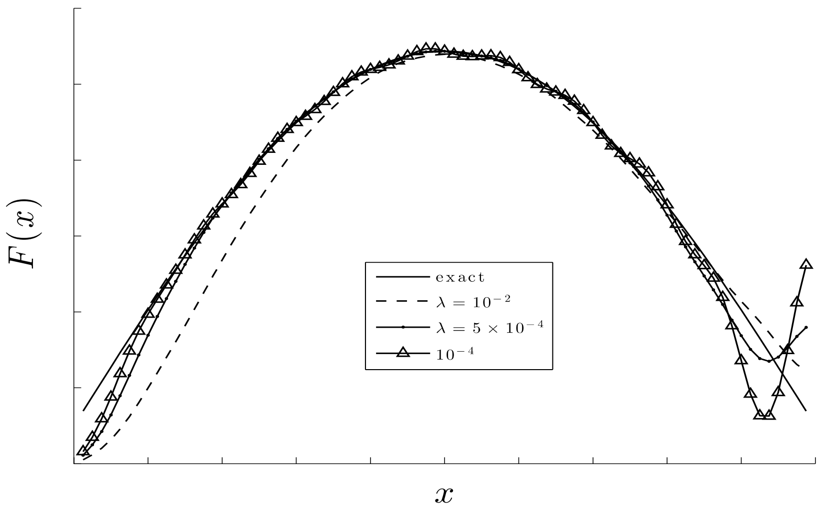

Figure 6 The exact solution (15) for $F(x)$ in comparison with the numerical solution (13), for $\mathrm{K} = 20$ and $\gamma\% = 1\%$, noise, and regularization parameters $\lambda \in \{10^{-2}, 5 \times 10^{-4}, 10^{-4}\}$, where non-classical boundary condition has considered.

4 Conclusions

Considerate on obtaining numerical converge of "force" by non-classical boundary data has researched in this study. Furthermore, additional measurement given by the right end displacement and other condition with unusual condition is the right end flux tension of string. Numerically non-homogenous hyperbolic equation has divided in two part where direct problem comes from first part, and inverse force/source problem (i.e ill-posed problem) obtains in second part, where second one is main part to get analytic solution for force/source function. However, the force/source function becomes unstable after adding noise, zeroth-order Tikhonov regularization method has used to stable it. Meanwhile, L-curve criterion and min norm have proposed to seek best regularization parameter.

Conflict of Interests.

There are non-conflicts of interest

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