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On Modules with Finite Spanning Isodimension

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Abstract

We introduce R −modules with finite spanning isodimension. Let M be an R −module M is called module with finite spanning isodimension, if for every strictly decreasing sequence $X_0 \supseteq X_1 \supseteq \cdots$, there exists a positive integer i such that X_j is isosmall for each $j \geq i$. In the following sense, we define isosmall submodule, a submodule N of an R −module M is called isosmall, if $N + L \cong M$, then $L \cong M$ for any submodule L of M. Some other classes are studied for instances isomaximal and many results are proved. On the other hand, we determine that the ring of endomorphisms of an isosimple module is a local ring.

Keywords. Isosimple modules, Finite spanning isodimension, Isoartinian, Isosmall submodule, Isomaximal submodules.

1. Introduction

Isoartinian, isonoetherian and isosimple module, are three new classes of module which were introduced by Alberto Facchini and Zahra Nazemian [1,2], they studied modules with chain conditions up to isomorphism. A right module M is isoartinian if, for every descending chain $M \geq M_1 \geq M_2 \geq \cdots$ of submodules of M, there exists n, where $n \geq 1$ such that M_n is isomorphic to M_i for every $i \geq n$. Similarly isonoetherian and isosimple modules and rings can be defined. A ring R is a right isoartinian semiprime right noetherian ring if and only if R is a finite direct product of matrix rings over principal right ideal domains. A module M has finite spanning dimension if for every strictly decreasing sequence of submodules $U_0 \supseteq U_1 \supseteq \cdots$, there is i and U_j is small in M for every $j > i$ [3]. Modules investigated in many directions by authors, for instance Rangaswamy, provided the following result: A projective (quasi-projective) module P has finite spanning dimension if and only if P is local (hollow) or Artinian [4]. For more results on module with finite spanning dimension the reader could see [5,6,7,8].

In this paper, modules with finite spanning isodimension are studied. If R is a ring and M is a left R −module. An R −module M is said to be module with finite spanning isodimension, if for every strictly decreasing sequence $X_0 \supseteq X_1 \supseteq \cdots$, there exists a positive integer i such that X_j is I −small for each $j \geq i$. We are studied different classes of modules and submodules up to isomorphism. For instance, we introduce isosmall and isomaximal submodules and some of their properties are studied. A submodule X is called isosmall or I-small submodule if $X + N \cong M$, then $N \cong M$ for any other submodule N of M, and a submodule N of M is called isomaximal or I-max submodule if N is a proper submodule of M and whenever $N \subset X \subseteq M$ then $X \cong M$, where X is a submodule of M. Consequently, a module M is called isohollow, if $Y_1 + Y_2 \cong M$, then $Y_1 \cong M$ or $Y_2 \cong M$(or equivalently, every submodule of M is an isosmall). If A is a submodule of M, then B is called isosupplement of A, if $A + B \cong M$, and $A \cap B$ is isosmall in B.

Now, we mention some new results, which are obtained in this paper, for example: In **Proposition 2.1** we prove that endomorphism of an isosimple module is a local ring. It is not necessary that, any module to be contain an isohollow submodule, but in **Theorem 2.1** we provided that, If M has a finite spanning isodimension and X is a submodule of M which is not isosmall, then X contains an isohollow submodule. Moreover, finite spanning isodimension guarantees that every submodule has an isosupplement as we proved in **Proposition 2.2.** The main result of this paper is and **Theorem 2.2**, which provides the general form of a module with finite spanning isodimention, on the other hand, it is indicating a strong relation between isohollow module and the module's spanning isodimension, this theorem help us to find the isodimention of a module, for example the isodimension of the Z −module Z is 1, whereas the hollow dimension of Z is infinity.

Another aspect of this paper is to provide whether a submodule of a finite spanning isodimension has finite spanning isodimension and isodimension of the sum of two submodule is the same as the sum of the isodimension of the submodules, we obtain under some conditions these properties will be hold, for instance, **Theorem 2.3,** states that if M has finite spanning isodimension and K is an isosupplement in M, then K has finite spanning isodimension and **Theorem 2.4,** states that for a module M which has finite spanning dimension and $M \cong K + L$, where K, L are submodules in M, then $Isd(M) = Isd(K) + Isd(L)$, if and only if K and L are isosupplement of each other in M. An application of **Theorem 2.2** appears in a module which has finite spanning isodimension and isomorphic to a finite direct sum of its submodules, that is $M \cong M_1 \oplus M_2 \oplus \ldots \oplus M_n$, then $Isd(M) = \sum_1^n Isd(M_i)$. From **Theorem 2.5**, we conclude that if M is a finitely generated module over a principal ideal domain in R with finite spanning isodimension, then M has finite spanning isodimension. For this reason , every quotient and finitely generated module over the ring of integers has finite spanning isodimension.

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2. Modules with Finite Spanning Isodimension

In this section, we introduce and investigate the modules with finite spanning isodimention, but first we need some concepts to build our definitions. First of all, we define I-small submodule, then we introduce isohollow and isosupplemented module. Some properties of these classes were provided.

Definition 2.1. Let M be an R-module and X be a submodule of M, X is called isosmall or I-small submodule denoted by $X \ll_i M$, if $X + N \cong M$, then $N \cong M$ for any other nonzero submodule N of M.

Consider Z as Z −module $3Z$ and $2Z$ are isosmall in Z , since $2Z + 3Z \cong Z$ implies that $3Z \cong Z$. In general, if m and n are two relatively prime numbers, then $mZ + nZ = Z$ and it is obvious that $mZ \cong Z$ and $nZ \cong Z$. In fact, every small submodule is an isosmall but the converse is not true, as we illustrate it in the above example. As a basic property of isosmall submodule, if N is an isosmall submodule of L where L is a submodule of an R −module M, then $N$is an isosmall in M. To verify it, let $N + X \cong M$, then $N \cap L + X \cap L \cong M \cap L$ and this implies that $N + X \cap L = L$, but N is an isosmall in L, thus $X \cap L = L$ and hence $L \subseteq X$ which means that $N \subseteq X$ consequnteley $N + X = M$ and hence $X \cong M$.

Definition 2.2. A submodule N of an R −module M is called isomaximal or I-max submodule if N is a proper submodule of M and $N \subset X \subseteq M$ then $X \cong M$, where X is a submodule of M.

In Z as a Z −module $6Z$ is an isomaximal submodul, since $6Z \subset 3Z \subset Z$ implies that $3Z \cong Z$ and this is an illustration to show every maximal submodule is isomaximal, but the converse is not true. In **[1]** $I - Rad$ of an R −module is introduced as $I - Rad(M) = \cap \{N; \frac{M}{N}$ is an isosimple$\}$, so we have an equivalent definition which is $I - Rad = \cap \{N; N$ is an isomaximal$\}$. On the other hand, it is equal to the sum of all isosmall submodules of M, if we consider an element $x \in I - Rad(M)$, this implies that $x \in N$, where N is a submodule of M and $\frac{M}{N}$ is isosimple, then N is an isomaximal submodule, this means that $x \in \cap \{N; N$ is an isomaximal$\}$. The converse is obvious.

Also, if M has no isomaximal submodule, then $I - Rad(M) = M$. Furthermore, if $x \notin I - Rad(M)$, then there exists a submodule N which is isomaximal and $x \notin N$, this means that Rx is not isosmall in M. By using the above concepts, we can verify that $I - Rad(M) = M$ if and only if every finitely generated submodule of M is isosmall.

Definition 2.3. An R −module M is called isohollow, if $Y_1 + Y_2 \cong M$, then $Y_1 \cong M$ or $Y_2 \cong M$(or equivalently, every submodule of M is an isosmall).

Every hollow is an isohollow, but the converse is not true, for instance Z as a Z −module.

In **[1]** isosimple modules are defined, in the following proposition we prove that endomorphism of an isosimple module is a local ring.

Proposition 2.1. Let be M an isosimple module, then $End(M)$ is a local ring.

Proof. Suppose that M be an isosimple module and $f \in End(M)$. Since $M = f(M) + (I - f)(M)$ and M isosimple, then $M = f(M)$ or $M = (I - f)(M)$ in each case we get an isomorphism. Hence $End(M)$ is a local ring.

Definition 2.4. Let M be an R −module and A be a submodule of M, then B is called isosupplement of A, if $A + B \cong M$, and $A \cap B$ is isosmall in B.

If every submodule of M has an isosupplement then M is called isosupplemented module.In the above definition, it is clear that if B is an isosupplement of A, then $A \cap B$ is an I −small in M.

Definition 2.5. An R −module M is said to be module with finite spanning isodimension, if for every strictly decreasing sequence $X_0 \supseteq X_1 \supseteq \cdots$, there exists a positive integer i such that X_j is isosmall for each $j \geq i$.

Theorem 2.1. If M has a finite spanning isodimension and X is a submodule of M which is not isosmall, then X contains an isohollow submodule.

Proof. Suppose that M has finite spanning isodimension. If every submodule N of M is isosmall, then M is isohollow. If M is not isohollow, then there exists a submodule M_1 of M which is not isosmall, that is there exist a submodule X_1 of M with $X_1 \ncong M$ and $X_1 + M_1 \cong M$. If M_1 has a submodule which is not isosmall, say M_2 in M, then there exists asubmodule $X_2 \ncong M$ and $X_2 + M_2 \cong M$. Repeating this process we get a strictly decreasing sequence $M_1 \supseteq M_2 \supseteq \cdots$, then there exists i such that X_j not isosmall in M, for every $j \geq i$. This means that M_i is not isosmall and contains no non isosmall submodules. If M_i is not isohollow, then there exists Two submodule A,B of M_i with $A \ncong M_i$, $B \ncong M_i$ and $A + B \cong M_i$. Since M_i is not isosmall, then there exists a submodule $X_i \ncong M$ with $M_i + X_i \cong M$, then $A + B + X_i \cong M$. This gives that $B + X_i \cong M$, then $X_i \cong M$ which is a contradiction.

It is obvious that, every isohollow has finite spanning isodimension. Moreover every isoartinian module has finite spanning isodimension, to prove that, if $M$is a module which has no finite spanning isodimension , so there exists $M_1 \supseteq M_2 \supseteq \cdots$, such that no i with M_i is an isosmall in M, which is acontradiction because M is an isoartinian module.

Proposition 2.2. Suppose that M has a finite spanning dimensions, then every submodule of M has an isosupplement.

Proof. Let N be a submodule of M, if N is isosmall, then M itself is isosupplement of N. If not , then there exists a submodule $X \ncong M$ with $N + X \cong M$.If X is isosupplement of N, we are done. If not, then there exists a submodule $X_1 \subset X$ such that $N + X_1 \cong M$ .If X_1 is isosupplement of N, then we are done. If not , we get a submodule X_2 of X_1 , by repeating this process, we get a descending chain of submodules $X \supseteq X_1 \supseteq X_2 \supseteq \cdots$, since M has finite spanning isodimension, then there exists j such that X_i is isosmall for every $i > j$, and $N + X_i \cong M$, then $X_i \cap N \subseteq X_i$, so $X_i \cap N$ is isosmall by our hypothesis. We can note that if N and X are two submodules of M with $N + X \cong M$ and X is not isosupplement of N, then $N \cap X$ is not isosmall in X, that is there exists a submodule $X_1 \ncong X$ and $N \cap X + X_1 \cong X$. Hence $N + X_1 \cong M$. Suppose a sequence of submodule $A \subseteq B \subseteq M$, it is obvious that if A is isosmall in B, then it is isosmall in M, but the converse need not be true in general, unless B is an isosupplement submodule in M. Since if B is an isosupplemented for a submodule L of M, then $M \cong B + L$ and $B \cap L$ is an isosmall in B.

Now, for any submodule N of B, which is isosmall in $M$and $B \cong N + Y$, for some $Y \subseteq B$, then we obtain $M \cong N + Y + L$, since N is isosmall in M, then $M \cong Y + L$, which means $B \cong Y + L \cap B$, this implies that $Y \cong B$. Hence N is also isosmall submodule of B. This can provide that the isosupplemented property is asuffecient and necessary condition for A to be isosmall in B, whenever A is isosmall in M.

For two submodules N and K of a module M such that K is an isosupplement submodule of M with $M \cong K + N$ and $K \cap N$ is an isosmall in M, then K is an isosupplement of N, this can be viewed as an implication of the above property of an isosupplement submodule. For any two submodule N and L of an R −module M with $\cong N + L$, then N has an isosupplement in M which is contained in L, in this case M, is called amply isosupplemented module which implies that there exists $N_1 \subseteq N$ and $L_1 \subseteq L$ that are isosupplements for each other. According to this notation, we have the following results:

Proposition 2.3. Suppose that K is an isosupplement of L in a module M and G, H are submodules of M contained in K such that G is an isosupplement of H in K, then G is an isosupplement of $H + L$ in M.

Proof. Let K be an isosuupplement of L in M, and G be an isosupplement of H in K, then $M \cong K + L$, $K \cap L$ is isosmall in K, $K \cong G + H$ and $G \cap H$ is I −small in G, then it is clear that $M \cong G + H + L$. If $M \cong G' + H + L$, then $K \cong G' + H + (L \cap K)$, so $K \cong G' + H$. But G is isosupplement of H in K, then $G' \cong G$. Hence, G is isosupplement of $H + L$ in M.

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Theorem 2.2. Suppose that M has finite spanning isodimension, then there exists an integer p such that $M \cong N_1 + \cdots + N_p$, where N_i is isohollow for $i = 1, \dots, p$. Moreover, $M \ncong N_1 + \cdots + \widehat{N}_\iota + \cdots + N_p$. If $M \cong N_1' + \cdots + N_q'$ and this satisfy the first two condition, then $p = q$.

Proof. Since M has finite spanning isodimension, then by **Proposition 2.3,** we can find an ishollow submodule N_1. If $M \cong N_1$, then we are done, if $M \ncong N_1$, then it has an isosupplement X_1, then $N_1 + X_1 \cong M$, and whenever $N_1 + Y_1 \cong M$,for any submodule Y_1 of M , then $Y_1 \cong X_1$. Now, if all submodules of X_1 is isosmall, then X_1 is isohollow, this means that M is isomorphic to the sum of two isohollow modules. These two isohollwo modules cannot be deleted. If X_2' has asubmodule N_2 which is not isosmall in M, then N_2 has an isosupplement, say, X_2' in M, then $M \cong N_2 + X_2$, this gives that $X_1 \cong N_2 + (X_1 \cap X_2)$, then we can select a supplement for N_2 in X_1, say X_2. Now, since $X_1 \cap X_2' \ncong X_1$ (If $X_1 \cap X_2' \cong X_1$, then N_2 is isosmall), then it is clear that $X_1 \cap X_2' \neq X_1$. If $X_2 = X_1$ since $X_1 \cong N_2 + X_2$, and $N_2 + Y_2 \ncong X_1$, for $Y_2 \ncong X_2$, then we get that $X_1 \cap X_2' \cong X_2 = X_1$, so $X_1 \cap X_2' \cong X_1$ which is a contradiction, this means that X_2 properly contained in X_1 and $M \cong N_1 + N_2 + X_2$, by repeating this process, we arrive a sequence $X_1 \supseteq X_2 \supseteq \cdots$ and after a certain point the submodules of this sequence must be isosmall, then $M \cong N_1 + \cdots + N_p$, where each N_i is isohollow, and we not rid of any of N_i's. If $M \cong N_1' + N_2' + \cdots + N_q'$, where N_i' is isohollow and none of them can be deleted. Suppose that $q > p$,then the submodule $N_2 + \cdots + N_p$ is not isomorphic to M, then we want to show that, there exists $1 \leq i \leq q$, such that $N_i' + N_2 + \cdots + N_p \cong M$ and none of them can be deleted. If $N_1' + N_2 + \cdots + N_p \ncong M$, then there exist a submodule $U$of N_1 such that $N_1' + N_2 + \cdots + N_p = U + N_2 + \cdots + N_p$, so $N_2' + \cdots + N_q' + U + N_2 + \cdots + N_p = N_1' + N_2' + \cdots + N_q + N_2 + \cdots + N_p \cong M$. This means $N_2' + \cdots + N_q' + U + N_2 + \cdots + N_p \cong M$, but U is isosmall, then $N_2' + \cdots + N_q' + N_2 + \cdots + N_p \cong M$. If $N_2' + N_2 + \cdots + N_p \ncong M$, then we add $N_3' + \cdots + N_q'$ and we get $N_3' + \cdots + N_q' + N_2 + \cdots + N_p \cong M$. By repeating this process, we find out that, if there is no i such that $1 \leq i \leq q - 1$,with $N_i' + N_2 + \cdots + N_p \cong M$, then $N_q' + N_2 + N_3 + \cdots + N_p \cong M$. This means that if we remove N_1 from the summation, then there is exactly one N_i', for $1 \leq i \leq q$, such that $M \cong N_i' + N_2 + \cdots + N_p$. Now, it remains to show that we cannot remove each N_i, N_j, for $2 \leq j \leq p$ from the summation. It is clear that N_i' cannot be deleted, if we delete N_2, then $M \cong N_i' + N_3 + \cdots + N_p$ and consider $N_2 + \cdots + N_p = U + N_3 + \cdots + N_p$, for a submodule U of N_i', then $N_1 + U + N_3 + \cdots + N_p \cong M$, U is isosmall, then we obtain $M \cong N_1 + N_3 + \cdots + N_p$, thus we delete N_2 form the summation which is a contradiction. Note that we choose N_2 arbitrary, so for any other N_i's we can do the same process and we find out that each N_i cannot be deleted from the summation $N_i' + N_2 + \cdots + N_p \cong M$. Thus, we replaced N_1 by N_i', for exactly one i, also N_2 can be replaced by N_j' for exactly one j, by the same way by continuing this process we can replace each N_k by N_l', but since $q > p$, then some of the N_i' deleted from the summation $N_1' + N_2' + \cdots + N_q' \cong M$ which is a contradiction. Hence, q must be equal to p.

We will define the integer p in **Theorem 2.2** as isodimension of an R −module M. It can easy to determine that the Z-module Z has a finite spanning isodimension and $Isd(Z) = 1$. In general, if M is an isohollow module, then $Isd(M) = 1$. As a conclusion if M is any infinite cyclic Z −module , then M has finite spanning isodimension and $Isd(M) = 1$.

In the following theorem, we deal with the question whether a submodule of a finite spanning isodimension module, has finite spanning isodimension? We provided that every isosupplemented submodule of a finite spanning isodimension, has finite spanning isodimension.

Theorem 2.3. Suppose that M has finite spanning isodimension and K is an isosupplement in M, then K has finite spanning isodimension . Furthermore, if $Isd(M) = Isd(K)$, then $K \cong M$.

Proof. Let K be an isosupplement submodule for some submodule L in M. If we suppose that K contains a sequence $X_1 \supseteq X_2 \supseteq \cdots$, then there exists $i \in Z^+$, such that X_j is an isosmall submodule in M for any $j \geq i$. If X_j is not isosmall submodule in K, then there exists a submodule $L_j \neq K$ and $X_j + L \cong K$, then $X_j + L_j + L \cong M$.

Since X_j is isosmall in M, then $L_j + L \cong M$ which is a contradiction. Hence X_j must be isosmall in K for every $j \geq i$.

For the second part, if $Isd(M) = Isd(K)$ and $K \ncong M$, then there exists a submodule L of M such that K is an isosupplement of L, then $K + L \cong M$ and there exists a submodule L_1 of L which is isosupplement of K. We can easily verify that K is also an isosupplement of L_1. Suppose that $Isd(L_1) > 0$, then $Isd(K + L_1) > Isd(M)$ which is a contradiction. Hence $L_1 = 0$ and $K \cong M$.

Theorem 2.4. Let M be an R-module which has finite spanning dimension and $M \cong K + L$, where K, L are submodules in M, then $Isd(M) = Isd(K) + Isd(L)$, if and only if K and L are isosupplement of each other in M.

Proof . Suppose that K and L are isosupplement of each other in M, then by **Theorem 2.2** K and L have finite spanning isodimension , then $K \cong K_1 + K_2 + \cdots + K_n$ and $L \cong L_1 + L_2 + \cdots + L_j$, where each of K_i's and L_m's are isohollow submodules and they cannot remove from the summation.

Now, we want to show that $M \cong K_1 + K_2 + \cdots + K_n + L_1 + L_2 + \cdots + L_j$ and non of K_i's and L_m's can be deleted from the summation. The first condition holds , for the second part, suppose that $M \cong K_2 + \cdots + K_n + L_1 + L_2 + \cdots + L_j = K_2 + \cdots + K_n + L$, then by Modular Law we get $K \cong K_2 + \cdots + K_n + (L \cap K)$, but K is isosupplement of L, then we get $K \cong K_2 + \cdots + K_n$ thus we deleted K_1 from the summation which is a contradiction. We can repeat this process for each of K_i ,$2 \leq i \leq n$ until we get a contradiction. Hence none of $K_i's$ can be deleted. The same thing is true for $L_m's$. Finally, we obtain that $Isd(M) = n + j = Isd(K) + Isd(L)$. For the necessary part, suppose that $Isd(M) = Isd(K) + Isd(L)$ this means that $M \cong K_1 + K_2 + \cdots + K_n + L_1 + L_2 + \cdots + L_j$, where each of $K_i's$ and $L_m's$ are isohollow submodules, then it is easy to see that $L \cap L$ is an isosmall in $K$and L, since each $K_i's$ and $L_m's$ are isohollow. Hence $K$and L are mutual isosupplement.

As a sequence of Theorem 2.2 we can note that for a finite spanning isodimension module M, if there exists a submodules K_i for $i = 1, \ldots, n$, and $M \cong K_1 + K_2 + \cdots + K_n$, then $Iso(M) = \sum_1^n Isd(K_i)$ if and only if K_i and K_j are mutual isosupplemente submodules , where $1 \leq i, j \leq n$. Furthermore, if Mamply isosupplemented module, then $Isd(M) = \sum_1^n Isd(K_i)$ if and only if K_i and $K_1 + K_2 + \cdots + K_{i-1} + K_{i+1} + \cdots + K_n$ are mutual supplement in M for each i.

The most significant result of Theorem 2.2 appears in a module which has finite spanning isodimension and isomorphic to a finite direct sum of its submodules, that is $M \cong M_1 \oplus M_2 \oplus \ldots \oplus M_n$,then $Isd(M) = \sum_1^n M_i$.

In the following proposition we investigate the relationship between hollow dimension (dual of goldie dimension) and isohollow isodimension.

Proposition 2.4. Let Mbe a module with finite spanning dimension, then it has finite spanning isodimension. Moreover, $Isd(M) \leq Sd(M)$.

Proof . Suppose that M has finite spanning dimension and consider a descending chain of submodules $X_1 \supseteq X_2 \supseteq \cdots$, then there exsits $i \in Z$ such that X_j is a small submodule for each $j \geq i$ which implies that X_j is isosmall, hence M has finite spanning isodimension. Moreover, it is possible to have a positive integer $m \leq j$ for which every X_m is isosmall. In this case, we conclude that $Isd(M) \leq Sd(M)$.

The converse of the above proposition need not be true in general, for example, consider Z as a Z −module which has finite spanning isodimension, but has no finite spanning dimension, since if we take the sequence $< 2 > \supseteq < 4 > \supseteq < 8 > \supseteq \cdots$ there is no integer j such that $< j >$ small in Z.On the other hand, if we take $< 6 >$ a submodule of Zas a Z −module, it is obvious that $\frac{Z}{<6>} \cong Z_6$, for Z_6 as a Z −module and $Z_6 = \{0, 3\} + \{0, 2, 4\}$ which are hollow modules, this implies that $Sd\left(\frac{Z}{<6>}\right) = 2$ as a Z −module. Hence $\frac{Z}{<6>}$ has finite spanning dimension and consequently it has finite spanning isodimension. In general, for any positive integer n, $\frac{Z}{<n>}$ as a Z −module has finite spanning isodimension. The aim of the next theorem is to prove this property.

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Theorem 2.5. Suppose that R is a principal ideal domain with finite spanning isodimension, then $\frac{R}{I}$ is also has finite spanning isodimension.

Proof. Consider the sequence $\frac{X_1}{I} \supseteq \frac{X_2}{I} \supseteq \frac{X_3}{I} \supseteq \cdots$ in $\frac{R}{I}$, then $X_1 \supseteq X_2 \supseteq \cdots$ is a sequence in R, but has finite spanning isodimension, then there exists a positive integer i such that X_j is isosmall for every $j \geq i$. If there exists $n \geq j$ such that $\frac{X_n}{I} + \frac{L}{I} \cong \frac{R}{I}$ for some submodule L of R which contains I, then we have $\frac{X_n+L}{I} \cong \frac{R}{I}$, since R is a principal ideal domain, then we get $X_n + L \cong R$, this implies that $L \cong R$. Hence, for every $j \geq i$, we obtain that $\frac{X_j}{I}$ is isosmall in $\frac{R}{I}$, thus means that it has finite spanning isodimension.

We can note that from Theorem 2.5 , if M is a finitely generated module over a principal ideal domain in R with finite spanning isodimension ,then there exists an ideal $I$of R such that $M \cong \frac{R}{I}$, but $\frac{R}{I}$ has finite spanning isodimension , hence M has finite spanning isodimension. For this reason , every quotient and finitely generated module over the ring of integers has finite spanning isodimension . Unfortunately, we cannot obtain the result for an arbitrary modules and rings. If M is an R −module with finite spanning isodimension and N is a direct summand of M, then $M = N \oplus L$, for some submodule L of M, hence by the First Isomophism Theorem $\frac{M}{N} \cong L$ and we have the direct summand of is also has finite spanning isodimension , thus we conclude that $\frac{M}{N}$ has finite spanning isodimension. That is the quotient module has finite spanning isodimension and in this case $Isd\left(\frac{M}{N}\right) = Isd(M) - Isd(N)$. Consider the Z_{24} as a Z −module, then $N = \{0, 8, 16\}$ is a direct summand for Z_{24}, and $\frac{Z_{24}}{N} \cong < 3 >$, then $Isd\left(\frac{Z_{24}}{N}\right) = 1$, $Isd(Z_{24}) = 2$ and $Isd(N) = 1$. Furthermore, in Z as a Z −module and N is adirect summand of Z, then the result need not be true in general.

A module M is called isosemisimple if it is a direct sum of isosimple modules. In the following theorem we provide that an isosemisimple module with finite spanning isodimension it is a finite direct summand of isosimple modules.

Theorem 2.6. Suppose that M be an isosemi-simple module with finite spanning isodimension, then M is a finite direct sum of its isosimple submodules.

Proof. If $Isd(M) = p$, then there exist isohollow submodules of M, say $N_1, N_2, \dots, N_p$ such that $M \cong N_1 + N_2 + \cdots + N_p$. If K_i is a submodule of N_i then K_i is an isosmall in N_i, means that $N_i \cong K_i \oplus L_i$ which implies that $L_i \cong N_i$ thus $K_i = \{0\}$,thus N_i is an isosimple module. Now, if the sum is not direct, then for every pair of submodules N_i and N_j we have $N_i \cap N_j \neq \{0\}$, hence $N_i \cap N_j$ isomorphic to N_i or N_j , then one of them must be deleted from the sum which is a contradiction, so the sum is direct.

Conflict of Interests.

There are non-conflicts of interest .

References

[1] Facchini, Alberto, and Zahra Nazemian. "Modules with chain conditions up to isomorphism." *Journal of Algebra* 453 (2016): 578-601.

[2] Facchini, Alberto, and Zahra Nazemian. "Artinian dimension and isoradical of modules." *Journal of Algebra* 484 (2017): 66-87.

[3] Fleury, Patrick. "A note on dualizing Goldie dimension." *Canadian Mathematical Bulletin* 17, no. 4 (1974): 511-517.

[4] Rangaswamy, K. M. "Modules with finite spanning dimension." *Canadian Mathematical Bulletin* 20, no. 2 (1977): 255-262.

[5] Varadarajan, Kalathoor. "Dual goldie dimension." *Communications in Algebra* 7, no. 6 (1979): 565-610.

[6] Varadarajan, K. "Dual Goldie dimension of certain extension rings." *Communications in Algebra* 10, no. 20 (1982): 2223-2231.

[7] Sarath, B., and K. Varadarajan. "Dual goldie dimension-II." *Communications in Algebra* 7, no. 17 (1979): 1885-1899.

[8] Rim, Seog Hoon, and Kazunari Takemori. "On dual Goldie dimension." *Communications in Algebra* 21, no. 2 (1993): 665-674.

الخلاصة

فى هذا البحث نقدم المقاسات من النمط (R) المنتهية الامتداد المتكافئات بعديا (finite spanning isodimension) . ليكن (M) مقاسا من النمط (R) و ان (N) هو مقاس جزئى من (M) نقول ان (N) هو مقاس جزئى متكافئ صغريا (isosmall) اذا كان (L) اى مقاس جزئى من (M)بحيث ان $(N + L \cong M)$ فان $(L \cong M)$ و نقول ان (M) مقاس منتهى الامتداد متكافئ بعديا اذا كان لكل متتابعة متناقصة تامة $X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$, من المقاسات الجزئية (X_j) من (M) يوجد عدد صحيح (i) بحيث ان (X_j) يكون متكافئ صغريا لكل $(j \geq i)$.

وتمت كذلك تعريف و دراسة بعض الاصناف الاخرى من المقاسات الجزئية و منها المقاسات الجزئية المتكافئة عضما (isomaximal) حيث تمت البرهنة على نتائج عديدة تتعلق بهذا النوع من المقاسات الجزئية. و من بين النتائج التى تمت البرهنة عليها ايضا فى هذا العمل هى ان حلقة التشاكلات الذاتية(ring of endomorphisms) على مقاس متكافئ ببساطة (isosimple) تكون حلقة محلية (موضعية) (local ring).