# The Periodicity of Two Disconnecting Arc Spaces

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## Abstract

Let $S$ be two disconnecting arc and $f: S \to S$ be a continuous map . Suppose $f$ has a fixed point and a periodic point of period n for some odd integer $n > 1$.Then for every integer $m > n$ , $f$ has a periodic point of period $m$ . If $f$ have periodic points of period $r$ and $s$ where $r \neq s$ are odd integer numbers , then the set of periodic points are infinite . Finally , if the set of periodic points is finite , then there are two integers $r$ and $s$ with $r \geq 1$ and $s \geq 0$ such that the set of period is $\{2^i . r | i \geq 0\}$.

**Keywords:** Periodic point , Periodic orbit, Two disconnecting arc.

## الخلاصة

لتكن S قوس المفصول باثنين و $f: S \to S$ دالة مستمرة . ولتكن $f$ لها نقطة ثابتة و نقطة دورية دوارها $n$ لبعض الاعداد الصحيحة الفردية $n > 1$ . فانه لكل الاعداد الصحيحة $m > n$ , $f$ تمتلك نقطة دورية لها دوارها $m$ . اذا كانت $f$ لها نقاط دورية دوراها $r$ و $s$ حيث $r \neq s$ اعداد صحيحة , فان مجموعة النقاط الدورية تكون غير منتهية . اخيرا , اذا مجموعة النقاط الدورية تكون منتهية , فانه يوجد عددان صحيحان $r$ و $s$ مع $r \geq 1$ و $s \geq 0$ بحيث ان مجموعة دوراتها $\{2^i . r | i \geq 0\}$ .

**الكلمات المفتاحية** : النقطة الدورية , المسار الدوري , قوس المفصول باثنين.

## 1.Introduction

In the recent years, many papers (and even some books) studied the periodic points on one dimensional spaces , for examples Real interval and circle etc.) .

( Sarkovskii ,1964) proved that if $f: R \to R$ continuous map and $f$ has periodic point of period $m$ then $f$ has a periodic points of $n$ when $m \rhd n$ ( in Sharkovskii's ordering). (Bolck 1980,1981) improved the theorem sarkovski to maps of the circle . (Bolck, 1980) proves that if has a fixed point and $f$ has periodic point of period $n$ , then for every integer $m > n$ , $f$ has a periodic point of period $m$ .(Alseda *et al.*,1990) introduced new space which called it disconnecting interval .

In our work , we define a new space $S$ , we call it two disconnecting arc . The set of periodic points studied on the new space .

## 2.Preliminary

In this section , we introduce the definitions and notations which we use in this work. Frist , we define a new space .

Let $X$ be a topological space. We say that X has two disconnecting arc if $\exists J \subset X$ such that $J$ is an open subset of $X$ homeomorphic with an open interval of $R$ and for every connected component of $X$ which contains J then for all $x \neq y \in J$ the set $Y - \{x, y\}$ has at least two connected components.

**Definition 2.1**

Let $a, b \in S$ with $a \neq b$ .We write $[a, b], (a, b), (a, b]$, or $[a, b)$ to denote the closed , open , or half-open arc from a counterclockwise to b.

**Definition 2.2**

Let $L$ and $K$ be proper closed arcs on $S$ and let $f: S \to S$ be a continuous map. We say $L$ f-covers $K$ if for some closed arc $J \subset L$, $f(J) = K$.

**Definition 2.3**

The orbit of $p$ is the set of points $p, f(p), f^2(p), \dots$ , and is denoted by $orb(p) = \{f^n(p) | n \in N_\circ\}$ where $N_\circ = N \cup \{0\}$. If $p$ is a periodic point of period n ,then we say that orbit of $p$ is a periodic orbit of period n. We denoted the set of periodic points by $p(f)$.

Now we define the fixed point and periodic point on $S$.

**Definition 2.4**

We say that $p \in S$ is a fixed point of $f$ if $f(p) = p + \beta\omega$, where $\omega$ is the length of $S$ and $\beta \in Z$. And $p$ is a periodic point if $f^n(p) = p + \beta\omega$.

**Definition 2.5**

Let $a, b \in S$, we say that $a \propto b$ if $a$ follows counterclockwise to $b$ .

**Definition 2.6:**

We say that $f: S \to S$ is monotone if $f$ has one of these properties : increasing or decreasing.

1)We say that $f: S \to S$ is increasing map if $\forall a_1, a_2 \in S$ such that $a_1 \propto a_2$, then $f(a_1) \propto f(a_2)$.

2) We say that $f: S \to S$ is decreasing map if $\forall a_1, a_2 \in S$ such that $a_1 \propto a_2$, then $f(a_2) \propto f(a_1)$.

**Definition 2.7:**

Let $f: S \to S$ and let $P = \{p_1, \dots, p_n\}$ be a periodic orbit of $f$ of period n. We say $P$ is labeled in order if for $k = 1, \dots, n-1$, $P \cap (p_k, p_{k+1}) = \emptyset$ and $P \cap (p_n, p_1) = \emptyset$ . In this case we define the arcs determined by $P$ to be the n closed arcs $A_1 = [p_1, p_2], A_2 = [p_2, p_3], \dots, A_{n-1} = [p_{n-1}, p_n], A_n = [p_n, p_1]$.

## 3.Main Result:

In this section , we prove some propositions and a theorem which study the periodic points on the two disconnecting arc space.

**Proposition 3.1 :**

Let $f: S \to S$ be a continuous map and let L be proper closed arc on $S$ with endpoints $e_1$and $e_2$.Suppose that $f(e_1) = e_3$ and $f(e_2) = e_4$ and $e_3 \neq e_4$.Then if $f$ is increasing then $L$ $f$ −covers $[e_3, e_4]$ and if $f$ is decreasing then $L$ $f$ −covers $[e_4, e_3]$.

Proof: Let $C = \{x \in D | f(x) = e_3\}$. Since $e_1 \in C$. Then $C \neq \emptyset$. Let a be a closed element in $C$ such that $[a, e_2] \cap C = \{a\}$. Let $D = \{x \in [a, e_2] | f(x) = e_4\}$. Since $e_2 \in B$. Then $B \neq \emptyset$. Let b be closed element in $D$ such that $[a, b] \cap D = \{b\}$.We have $f(a) = e_3, f(b) = e_4$ and if $x \in (a, b)$ ,then $f(x) \neq e_3$ and $f(x) \neq e_4$. If $J = [a, b]$ .Since $f$ is increasing, then $f(J) = [e_3, e_4]$ ,and hence $L$ $f$ −covers $[e_3, e_4]$ or $f$ is decreasing ,then $f(J) = [e_4, e_3]$ ,and hence $L$ $f$ −covers $[e_4, e_3]$.

**Proposition 3.2**

Let $f: S \to S$ be a continuous map . Let $L$ and $K$ be proper closed arcs on $S$ such that $L$ f-covers $K$. Suppose $J$ is a closed arc with $J \subset K$. Then $L$ f-covers $J$.

Proof. By definition 2.2 , there is a closed arc $I \subset L$ with $f(I) = K$. Let $J = [e_3, e_4]$ .There are points $e_1 \in I$ and $e_2 \in I$ such that $f(e_1) = e_3$ and $f(e_2) = e_4$ and $e_3 \neq e_4$. Let $I_1$ be the closed arc with endpoints $e_1$and $e_2$ such that $I_1 \subset I$.By proposition 3.1 either $I_1$ $f$ −covers $[e_3, e_4]$ or $I_1$ $f$ −covers $[e_4, e_3]$. Since $I_1 \subset I$ and $f(I) = K$, $I_1$ cannot $f$-cover $[e_4, e_3]$. Hence $I_1$ $f$ −covers $[e_3, e_4]$.Since $I_1 \subset I \subset L$, $L$ $f$ −covers $[e_3, e_4]$.

**Proposition 3.3**

Let $f: S \to S$ be a continuous map . Suppose $L$ is a proper closed arc on $S$ such that $L$ f-covers $L$ . Then $f$ has a fixed point in $L$.

Proof: By definition 2.2 , for some closed arc $J \subset L$ with $f(J) = L$. There are points $a \in J$ and $b \in J$ such that $f(a)$and $f(b)$ are the two endpoints of $L$. Let $K$ be the closed arc with endpoints a and b such that $K \subset J$. By continuity $f$ has a fixed point in $K$.

**Proposition 3.4**

Suppose that $f: S \to S$ be a continuous map. and suppose $E_1, E_2, \ldots, E_n$ are proper closed arcs on S such that $E_k$ $f$-covers $E_{k+1}$for $k = 1, \ldots, n-1$, and $E_n$ $f$-covers $E_1$ .Then there is a periodic point y of $f$ such that $y \in E_1, f(y) \in E_2, \ldots, f^{n-1}(y) \in E_n$.

Proof: Since $E_k$ $f$ − covers $E_{k+1}$.Then there is $L_k \in E_k$ such that $f(L_k) = E_{k+1}$ for $k = 1, \ldots, n-1$. Then if $y \in L$, then $f(y) \in E_2$ and so $f^2(y) \in E_2$ , thus by induction on n, $f^{n-1}(y) \in E_n$. Since $E_n$ $f$-covers $E_1$, $f^n(y) \in E_n$ .Then by proposition 3.3, $E_1$ $f$ −covers $E_1$.Thus $f^n$ has a fixed point on $E_1$, and then $f$ has a fixed point $f^n$.That is $f$ has a periodic point of period $n$ .

**Proposition 3.5**

Let $f: S \to S$ be a continuous map. Let $P = \{p_1, p_2, \ldots, p_n\}$ a periodic orbit of $f$ of odd period $n \geq 3$. Suppose $P$is labeled in order and let $A_1, A_2, \ldots, A_n$ be the arcs determined by $P$. Suppose that for some $s$ and $k$ with $s \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, n\}$, $A_l$ does not f-cover $A_s$ for all $l \in \{1, \ldots, n\}$ with $l \neq s$, and $A_i$ does not $f$-cover $A_i$ for all $l \in \{1, \ldots, n\}$ with $l \neq i$. Then $s = i$.

Proof: Suppose that $s \neq i$. Let $x \in A_i^\circ$ ,then $A_i^\circ \neq \emptyset$ and let $y \in A_s^\circ$ , then $A_s^\circ \neq \emptyset$. Then $x \neq y$. Let $C = P \cap (x, y)$ and $D = P \cap (y, x)$ . Then $C \neq \emptyset, D \neq \emptyset, C \cap D = \emptyset$ and $C \cup D = P$.

If $f(a) \in C$ for some $a \in C$ then by hypothesis and proposition(3.1), $f(C) = f\big(P \cap (x, y)\big) \subseteq f(p) \cap f\big((x, y)\big) = (x, y) = C$ ,and hence $f(C) \subset C$.Which contradiction the fact that $P$ is a periodic orbit . This implies that $f(a) \notin C$ for all $a \in C$. Thus $f(C) \subset D$.In the same way we can prove that $f(D) \subset C$. Since $f$ maps $P$ onto $P$ we have $f(C) = D$and $f(D) = C$.Thus $C$ and $D$ have the same number of elements. Since by assumption $C \cap D = \emptyset$ and $C \cup D = P$ this contradicts the fact that $P$ has an odd number of elements. From this we get $s = i$.

**Proposition 3.6**

Suppose $f: S \to S$ be a continuous map and $f$ aperiodic orbit $P = \{p_1, p_2, \dots, p_n\}$ of odd period $n \geq 3$. Suppose $P$ is labeled in order and let $A_1, A_2, \dots, A_n$ be the arcs determined by $P$. Suppose also that $f$ has a fixed point $x$. Then $f$ has a fixed point y with the property that if $A_i$ is the arc determined by $P$ with $y \in A_i$ , there is some $s \in \{1, \dots, n\}$ with $s = i$ such that $A_s$ f-covers $A_i$.

Proof: Suppose that $x \in A_n$. Since the conclusion of the proposition holds with $x = y$ .So suppose that for each $s \in \{1, \dots, n\}$ $A_s$ does not f-covers $A_n$. Let $m$ be the smallest positive integer such that if $f(p_j) = p_k$ ,then $k < j$. Note that $2 \leq j \leq n$, so $1 \leq j - 1 \leq n - 1$. In particular $j - 1 \neq n$. Since $A_{j-1}$ does not $f$-covers $A_n$, it follows from Proposition3.1 and Proposition3.2 $A_{j-1}$ $f$-covers $A_{j-1}$. By Proposition3.3 $f$ has a fixed point $x \in A_{j-1}$ . Since $A_s$ does not f-covers $A_n$ for all $s \in \{1, \dots, n\}$ ,it follows from proposition3.5 that for some $s \in \{1, \dots, n\}$ with $j \neq m - 1$ , $A_s$ $f$-covers $A_{j-1}$.

**Proposition 3.7**

Let $f: S \to S$ be a continuous map and let $P$ be a periodic orbit of $f$ of period $k$ where $k \geq 3$. Suppose that $\{E_1, \dots, E_s\}$ is a collection of closed arcs with $2 \leq s \leq k$ such that

1) For each $j \in \{1, \dots, s\}$ ,there are no elements of P in $E_j^{\circ}$,
2) If $i \neq j$ , $E_i^{\circ} \cap E_j^{\circ} = \emptyset$,
3) If $j \in \{2, \dots, s\}$ the endpoints of $E_j$ are in $P$,
4) if b is an endpoint of $E_1$, either $b \in P$or $f(b) = b + \beta\omega$, where $\omega$ is the length of $S$ and $\beta \in Z$,
5) for each $j \in \{1, \dots, s-1\}$, $E_j f$-covers $E_{j+1}$,
6) $E_1$ $f$-covers $E_1$ and $M_s$ f-covers $E_1$.

Then for any positive integer n > s, $f$ has a periodic point of period n.

Proof: Let $n > s$.Suppose that $n \neq k$ , since $P$ is a periodic orbit of $f$ of period $k$.Then $f$ has a periodic points of period k . Let $F_1 = E_1, F_2 = E_1, \dots, F_{n-s} = E_1, F_{n-s+1} = E_2, F_{n-s+2} = E_3, \dots, F_{n-s+s} = F_n = E_s$. By proposition (3.4 ) there is a periodic point y of $f$ such that $y \in F_1, f(y) \in F_2, \dots, f^{n-1}(y) \in F_n$.Since $y \in E_1, f(y) \in E_1, \dots, f^{n-s+1}(y) \in E_2$ and since if $i \neq j$ ,$E_i^{\circ} \cap E_j^{\circ} = \emptyset$ and $j \in \{2, \dots, s\}$ the endpoints of $E_j$ are in $P$, we get that y is not a fixed point of $f$ .

We may assume that $y \notin P$.To prove this, we assume that $n \geq s + 2$. Then $F_1 = F_2 = F_3 = E_1$. Hence $y, f(y), f^2(y) \in E_1$ . Since $P$ is a periodic orbit of period $k \geq 3$ and by the hypothesis for each $j \in \{1, \dots, s\}$ ,there are no elements of P in $M_j^{\circ}$,we get that $y \notin P$.Suppose that $n < s + 2$.Then $n < k + 2$.Since $n \neq k$ and $k \geq 3$, n is not multiple of k .Since $f^n(y) = y + \beta\omega$, where $\omega$ is the length of $S$ and $\beta \in Z$, we get $y \notin P$. Since $f(y) \neq y + \beta\omega$, where $\omega$ is the length of $S$ and $\beta \in Z$. and $y \notin P$ and $y \in E_1$.Then from(4) $f(y) \in E_1^{\circ}$. Also since $f^n(y) = y + \beta\omega \notin P$ for any positive integer $r < n, f^r(y) \notin P$, and $f^r(y) \neq y + \beta\omega$, where $\omega$ is the length of $S$ and $\beta \in Z$.Therefore by($j \in \{2, \dots, s\}$ the endpoints of $E_j$ are in $P$ ,if b is an endpoint of $E_1$, either $b \in P$or $f(b) = b + \beta\omega$ where $\omega$ is the length of $S$ and $\beta \in Z$) for any positive integer $r < n$ , $f^r(y)$ is not an endpoint of any of the arcs $E_1, \dots, E_s$ . It follow from this and $E_i^\circ \cap E_j^\circ = \emptyset$ for $i \neq j$ ,and from the fact,

$y \in E_1, f(y) \in E_1, f^2(y) \in E_1, \dots, f^{n-s}(y) \in E_1, f^{n-s+1}(y) \in E_2, \dots, f^{n-1}(y) \in E_s$

that y is a periodic point of period n.

**Theorem 3.8**

Suppose $f: S \to S$ be a continuous map . Suppose f has a fixed point and a periodic point of period n for some odd integer $n > 1$.Then for every integer m>n , f has a periodic point of period m.

Proof: Since $f$ has a periodic points of period n .Then $f$ has a periodic orbit $P = \{p_1, p_2, \dots, p_n\}$ of $f$ of period $n$. Suppose $P$is labeled in order and let $A_1, A_2, \dots, A_n$ be the arcs determined by $P$. Since $f$ has a fixed point q. Suppose that $q \in A_n$.By proposition 3.6 ,we suppose that for some $s \in \{1,2, \dots, n-1\}$, $A_s$ $f$-covers $A_n$.

Let $f(p_1) = p_i$ and $f(p_n) = p_j$.We have two cases .

Case 1: either $[q, p_1]f$ −covers $[q, p_i]$ or $[p_n, q]f$ −covers $[p_j, q]$.

Suppose that $[q, p_1]f$ −covers $[q, p_i]$. Since $[q, p_1] \subset [q, p_1]$and the arc $A_s \subset [q, p_s]$ with $s \in \{1,2, \dots, k-1\}$.Therefore by proposition 3.2 $[q, p_1]f$ −covers$[q, p_1]$ and $[q, p_1]$ $f$ −covers each arc $A_s$ with $s \in \{1,2, \dots, i-1\}$.We will use proposition 3.7 from now to prove the conclusion of the theorem and by induction on $n$ .

Now suppose that for some $s \in \{1,2, \dots, i-1\}$, $A_s$ $f$ −covers $A_n$ . For $k = 2$ , $E_1 = [q, p_1]$ and $E_2 = A_s$ such that (1) $\forall s \in \{1,2\}$, there are no element of P in $E_s^\circ$ ,2) $E_1^\circ \cap E_2^\circ = \emptyset$, 3) the endpoint of $E_2$ is in $P$ , 4) Since q is an endpoints of $E_1$ , then $f(q) = q + \beta\omega$ , 5) for each $s \in \{1\}$, $E_1 f$-covers $E_2$, 6)$E_1$ $f$-covers $E_1$ and $E_1$ $f$-covers $E_2$.Therefore by proposition 3.7 $\forall m > k$, $f$ has a periodic point of period m. Thus we suppose that for all $s \in \{1, \dots, i-1\}$, $A_s$does not f-covers $A_n$ .Since $A_s$ $f$ −covers $A_n$ for some $s \in \{1, \dots, n-1\}$, this implies that $i - 1 < n - 1$and hence $i < n$. For k=3. Since $i < n$. Let v be the smallest integer such that $f(p_v) \notin \{p_1, p_2, \dots, p_i\}$ for some integer v with $2 \leq v \leq i$ and if necessary that $f(p_{v-1}) \in \{p_1, p_2, \dots, p_i\}$ . Let $f(p_v) = p_l$. Since $A_{v-1}$ does not $f$-covers $A_n$, by proposition3.1 ,proposition3.2 , $A_{v-1}$ f-covers $[f(p_{v-1}), p_l]$ . Hence for each integer s with $i \leq s \leq l-1, A_{v-1}$ f-covers $A_s$ .Note that by choice of $p_v$ and $p_l$, $i \leq l - 1$.Suppose that for some positive integer s with $i \leq s \leq l-1, A_s$ $f$-covers $A_n$. Let $F_1 = [q, p_1]$ , $F_2 = A_{v-1}$and $F_3 = A_s$.Then by proposition 3.7 $\forall m > k$, $f$ has a periodic point of period m. In the same way the theorem hold for $k = n$ , by using the fact that for some $s \in \{1, \dots, n-1\}$, $A_s$ $f$-covers $A_n$, eventually we get a collection of closed arcs $\{F_1, F_2, \dots, F_k\}$ with $k \leq n$, such that (1),(2),(3)and (4) from proposition3.7 .Then $\forall m > k$, $f$ has a periodic point of period m.

Case (2): $[q, p_1]$ does not $f$ −covers $[q, p_i]$ or $[p_n, q]$ does not $f$ − covers $[p_j, q]$. Then by proposition3.1 $[q, p_1]$ $f$ −covers $[p_i, q]$ and $[p_n, q]$ $f$ −covers $[q, p_j]$.We claim that $A_n = [p_n, p_1]$ $f$ −covers$A_n$.To see that , note that since $[q, p_1]$ $f$ −covers $[p_s, q]$ , there is a point $e \in (q, p_1]$ such that $f(e) = p_n$ but $f(x) \neq p_n$ ,$\forall x \in (q, e)$.Since $f(q) = q$ and $f(e) = p_n$.Then by proposition3.1 $[q, e]$ $f$ −covers $[q, p_n]$ or $[q, e]$ $f$ −covers $[p_n, q]$ .Since by the assumption $[q, p_1]$ does not $f$ −covers $[q, p_s]$.By proposition3.2 $[q, e]$ does not $f$ −covers $[q, p_n]$ and hence $[q, e]$ $f$ −covers $[p_n, q]$ .Then by the definition 2.2 , there is $K \subset [q, e]$ such that $f(K) = [p_n, q]$ and so $[p_n, q] \subset f([q, e])$.Suppose that $e_1 \in (q, e)$, $f(e_1) \notin A_n$.Since $f(q) = q + \beta\omega$ where $\omega$ is the length of $S$ and $\beta \in Z$, then $f(q) \in A_n$, by continuity , for some $y \in (q, e)$ ,either $f(y) = p_1$or $f(y) = p_n$ .Since $e_1 \in (q, e)$, it follows from the choice of $e$ that $f(y) \neq p_n$. Hence $f(y) = p_1$. It follows from the choice of e, that $f([q, e])$ is a proper closed arc on S and $p_n$ is an endpoint of $f([q, e])$. Also $q \in f([q, e])$ and $p_1 \in f([q, e])$. Then either $A_n \subset f([q, e])$ or $[q, p_n] \subset f([q, e])$.

If $[q, p_n] \subset f([q, e])$.Then by the prove of proposition3.2 and using the fact $f([q, e]) \neq S$. We have $[q, e]$ $f$ −covers $[q, p_n]$ . Since $[q, p_i] \subset [q, p_n]$.Thus by proposition3.2 $[q, e]$ $f$-covers $[q, p_i]$ which contradiction the assumption . Hence $A_n \subset f([q, e])$.Since $f([q, e]) \neq S$. Then by proposition3.2 $[q, e]$ $f$-covers $A_n$.Therefore $A_n f$- covers $A_n$.

We shall prove that our claim holds if $f(y) \notin A_n$ for some $y \in (q, e)$. Hence suppose that $f([q, e]) \subset A_n$.We prove that $A_n$ $f$- covers $A_n$ ,if $(y) \notin A_n$ for some $y \in (q, e)$. Hence suppose that $f([q, e]) \subset A_n$.

Since $[p_n, q]$ $f$ −covers $[q, p_j]$, there is a point $z \in [p_n, q]$ such that $f(z) = p_1$,but $f(x) \neq p_1$ ,$\forall x \in (z, q)$.In the same way of the proof of $A_n \subset f([q, e])$, we can proof that $[p_n, q] \subset f([q, e])$.Also we suppose that $f([z, q]) \subset A_n$( by the same way) to see that we may assume that $f([q, e]) \subset A_n$. Thus $f([z, q]) = A_n$ .Since $[z, e]) \subset A_n$.Therefore $A_n f$- covers $A_n$. Since $[q, p_1]$ $f$ −covers $[p_i, q]$, $A_n f$- covers$[p_i, q]$.We use the prove of proposition 3.7 and induction on n .Since $[p_n, q]$ $f$ −covers $[q, p_j]$, $A_n f$-covers$[q, p_j]$.Then by proposition 3.2 for each integer $s$ with $1 \leq s \leq j - 1$ or $i \leq s \leq n - 1$, $A_n f$-covers $A_s$.

Suppose that for some integer $s$ with $1 \leq s \leq j - 1$ or $i \leq s \leq n - 1$, $A_s f$-covers $A_n$. For $k = 2$ , let $F_1 = A_n$ and $F_2 = A_s$. Thus from (1),(2),(3) and (4) from proposition 3.7 hold . So suppose that for each integer $s$ with $1 \leq s \leq j - 1$ or $i \leq s \leq n - 1$, $A_s$ does not $f$ –covers $A_n$. Since $A_s f$-covers $A_n$ , for some integer $s$ with $1 \leq s \leq j - 1$ or $i \leq s \leq n - 1$,this implies that $j < i$.

Since either $f(\{p_1, \dots, p_j\} \not\subset \{p_i, \dots, p_n\}$ or $f(\{p_i, \dots, p_n\}) \not\subset \{p_1, \dots, p_j\}$ . Then $\{p_1, \dots, p_n\}$ is a periodic orbit and $j < i$ and use the fact that n is odd in the case $j = i - 1$ .We suppose that $f(\{p_1, \dots, p_j\} \not\subset \{p_i, \dots, p_n\}$ . Let r be the smallest positive integer such that $f(p_r) \notin \{p_i, \dots, p_n\}$. Since $A_{r-1}$ does not $f$-covers $A_n$.Then $2 \leq r \leq j$ and $A_{r-1} f$-covers the arc $[f(p_r), f(p_{r-1})]$ . Hence $A_{r-1} f$-covers the arc $[f(p_r), p_i]$. Let $f(p_r) = p_t$. Then $t \leq i-1$ and $A_{r-1} f$-covers each arc $A_s$ with $t \leq s \leq i-1$.

Suppose for some integer with $q \leq s \leq j-1$, $A_s$ $f$-covers $A_n$. For $k=3$, $F_1 = A_n, F_2 = A_{r-1}$ and $F_3 = A_s$. Thus (1),(2),(3) and (4) of proposition 3.7 hold . By the same way we can prove that for $k = n$ ,using the fact that for some $s \in \{1, \dots n-1\}, A_s$ $f$-covers $A_n$, we eventually obtain a collection of closed arcs $\{F_1, \dots, F_k\}$ with $k \leq n$ such that (1),(2),(3)and (4) of proposition 3.7 hold . Then $\forall m > k$, $f$ has a periodic point of period $m$.

**Proposition 3.9**

Let $f: S \to S$ be a continuous map . Suppose that $f$ have periodic points of period r and s where r and s are odd ,$r \neq s$ .Then $p(f)$ is infinite .

Proof: Let $r, s \in Z$ such that $r \neq s$ .Suppose $r < s$. Let $H = f^r$.Then H has a fixed point say p and for some integer $m > 1$ , H has a periodic point of period m say q .Then $H(p) = f^r(p) = p + \beta\omega$, where $\omega$ is the length of $S$ and $\beta \in Z$ and $H^m(q) = (f^r)^k(q) = q + \beta\omega$ , where $\omega$ is the length of $S$ and $\beta \in Z$.Then by theorem3.8 the set of period of H is infinite, and hence $p(f)$ is infinite.

**Proposition 3.10**

Suppose $f: S \to S$ be a continuous map and $p(f)$ is finite .Then for some integers $r, s$ with $r \geq 1$and $s \geq 0$, $p(f) \subset \{2^i.r | i \geq 0\}$.

Proof: Let r be the smallest period of $f$ such that $f^r(p) = p + \beta\omega$, where $\omega$ is the length of $S$ and $\beta \in Z$. Since the set of period $f$ is finite , it enough to prove that if $f$ has periodic point q of period k of $f$ such that $f^k(q) = q + \beta\omega$, where $\omega$ is the length of $S$ and $\beta \in Z$.Then $k = 2^i.r$ for some $i \geq 0$.

Let q be a periodic point of period k . Let $r = 2^m.l$ where $l$ is odd ,$l \geq 1$ and $m \geq 0$, and let $k = 2^j.t$ where $t$ is odd ,$t \geq 1, j \geq 0$ .Then $f^r(p) = f^{2^m.l}(p) = p + \beta\omega$, where $\omega$ is the length of $S$ and $\beta \in Z$ and $f^k(q) = f^{2^j.t}(q) = q + \beta\omega$ .Let v be the largest element of $\{2^m, 2^j\}$ and $H = f^v$. Then by theorem 3.8 $p(H)$is finite and $l, t \in p(H)$ and $l$ and $t$ are odd .By proposition 3.9 $l = t$.Since r is the smallest period of $f$ and $l = t$, we have $j \geq m$.Let $i = j - m$.Then $i \geq 0$ and $k = 2^j.t = 2^{j-m+m}.l = 2^{j-m}.2^m.l = 2^i.r$.

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