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Behavior of Solution of Singularly Perturbed Difference Equations with Quadratic Small Parameter

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Abstract

In this paper we introduced a new asymptotic solution of system of singularly perturbed difference equations with quadratic small parameter . We study a behavior of asymptotic solution with certain cases , also we constructed the general solution of this problem. We obtained a formula for the terms of expansion solution, finally illustrated examples are given in this paper.

Key words:
Difference equations, asymptotic, singularly perturbed , small parameter, eigenvalues.

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حول سلوك حل معادلات الفروقات المنفردة المضطربة مع معلمة مربعة صغيرة

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الخلاصة:

في هذا البحث قدمنا تقريب جديد لحل نظام معادلات الفروقات للاضطراب المنفرد مع معلمة تربيعية. كذلك درسنا تقريب الحلول في حالات معينة، كذلك كونا الحل العام لهذه المسألة. حصلنا على صيغة الحدود للحل التقريبي، وأخيرا قدمنا أمثلة التي تتعلق بموضوع البحث.

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I. Introduction

many studies that related with our subject in this paper like [1,2,3,4,5] but all these studies take on state , that is the problem contains one parameter of power one , in this paper we will study a singular perturbed difference equations with quadratic parameter and we will introduce formula of asymptotic solution as following:

$$\bar{x}(n,\varepsilon) = \sum_{i,j=0}^{N} \varepsilon^{i+2j}\, \bar{x}_{i,j}(n)$$

$$\bar{y}(n,\varepsilon) = \sum_{i,j=0}^{N} \varepsilon^{i+2j}\, \bar{y}_{i,j}(n)$$

We will study the relation between the eigenvalues of matrices and the solution of a system of singular perturbed difference equations and we will find the estimation of the asymptotic solution of system.

II. View Problem:

Consider the system:

$$\left.\begin{aligned} \varepsilon x(n+1) &= ax(n)+by(n) \\ \varepsilon^2 y(n+1) &= cx(n)+dy(n) \end{aligned}\right\} \quad , \quad 0 \le n \le N-1 \ldots\ldots\ldots (1)$$

$$\left.\begin{aligned} x(0) &= x_0 \\ y(0) &= y_0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Such that a,b,c,d are constants, where ε > 0 is small parameter.

We can describe this system by using the matrices as following:

$$z(n+1) = Az(n) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Where

$$z(n) = \begin{bmatrix} x(n) \\ y(n) \end{bmatrix} \quad , \quad A = \begin{bmatrix} \frac{a}{\varepsilon} & \frac{b}{\varepsilon} \\ \frac{c}{\varepsilon^2} & \frac{d}{\varepsilon^2} \end{bmatrix}, \quad z(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

III. The asymptotic solution of the problem:

The asymptotic expansion for the solution of problem (1),(2) will be constructed as type:

$$\left.\begin{aligned} x(n,\varepsilon) &= \bar{x}(n,\varepsilon) \\ y(n,\varepsilon) &= \bar{y}(n,\varepsilon) \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$$\bar{x}(n,\varepsilon) = \sum_{i,j=0}^{N} \varepsilon^{i+2j}\, \bar{x}_{i,j}(n)$$

$$\bar{y}(n,\varepsilon) = \sum_{i,j=0}^{N} \varepsilon^{i+2j}\, \bar{y}_{i,j}(n)$$

By substituting the expansoin (4) in the equations (1,2) and equating coefficients which the same power of ε we can determine the coffcients of series (4). In partcular, the system for finding $x_{0,0}, y_{0,0}$ coincides with a degenrate system of the system (1) (at ε=0).

The remaining coefficients of the expansion (4) are found from the following expressions :

$$x_{ij}(n+1) = a \sum_{k=0}^{\left[\!\left[\frac{2i+j+1}{2}\right]\!\right]} x_{k,j+1-2(k-i)}(n) + b \sum_{k=0}^{\left[\!\left[\frac{2i+j+1}{2}\right]\!\right]} y_{k,j+1-2(k-i)}(n) \ldots (5)$$

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$$y_{ij}(n+1) = c \sum_{k=0}^{\left[\!\left[\frac{2i+j+2}{2}\right]\!\right]} x_{k,j-2(k-i-1)}(n) + d \sum_{k=0}^{\left[\!\left[\frac{2i+j+1}{2}\right]\!\right]} y_{k,j-2(k-i-1)}(n) \ldots\ldots\ldots (6)$$

IV. Estimate of e terms of internal expansion:

4.1.Definition:

The system (3) can rewritten as :

$$z(n+1) = Az(n)$$

$$z(n) = \begin{bmatrix} x(n) \\ y(n) \end{bmatrix} \quad , \quad A = \begin{bmatrix} \frac{a}{\varepsilon} & \frac{b}{\varepsilon} \\ \frac{c}{\varepsilon^2} & \frac{d}{\varepsilon^2} \end{bmatrix} = \frac{1}{\varepsilon^2}\begin{bmatrix} \varepsilon a & \varepsilon b \\ c & d \end{bmatrix} = \frac{1}{\varepsilon^2} B \, ,$$

$$where\ B = \begin{bmatrix} \varepsilon a & \varepsilon b \\ c & d \end{bmatrix}$$

We note that the exact solution of a system (3) is:

$$z(n) = A^n z(0) = \frac{1}{\varepsilon^{2n}} B^n z(0)$$

Therefore our estimate will be about a matrix B.

Without loss of generality we suppose that a=d, we will study the cases of eigenvalues of matrix B.

Suppose that α, β are eigenvalues of B such that:

$$\alpha = \frac{a\left[(\varepsilon+1) + \sqrt{(\varepsilon+1)^2 - 4\varepsilon\left(1 - \frac{bc}{a^2}\right)}\right]}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

$$\beta = \frac{a\left[(\varepsilon+1) - \sqrt{(\varepsilon+1)^2 - 4\varepsilon\left(1 - \frac{bc}{a^2}\right)}\right]}{2} \ldots\ldots\ldots\ldots\ldots\ldots.. (8)$$

According values of b and c we will study the following cases:

1) α, β are distinct real numbers

In this case we can get the following relations:

a) bc > 0 , $\varepsilon > 0$
b) bc=0 , $\varepsilon > 1$
2) $\alpha = \beta$

In this case we get :

bc=0 and $\varepsilon = 1$

3) α, β are complex numbers

We note that the real part of α, β is:

$$\frac{a(1+\varepsilon)}{2} < \frac{a\varepsilon}{1+\varepsilon}$$

$$|\alpha| = \left| \frac{a\left[(\varepsilon+1) + \sqrt{(\varepsilon+1)^2 - 4\varepsilon\left(1-\frac{bc}{a^2}\right)}\right]}{2} \right| > \frac{a(1+\varepsilon)}{2} > |\beta|$$

$$= \left| \frac{a\left[(\varepsilon+1) - \sqrt{(\varepsilon+1)^2 - 4\varepsilon\left(1-\frac{bc}{a^2}\right)}\right]}{2} \right|$$

$$= \left| \frac{2\varepsilon a^2\left(1-\frac{bc}{a^2}\right)}{a\left[(\varepsilon+1) + \sqrt{(\varepsilon+1)^2 - 4\varepsilon\left(1-\frac{bc}{a^2}\right)}\right]} \right| > \frac{\varepsilon a\left(1-\frac{bc}{a^2}\right)}{(\varepsilon+1)} = \left(1-\frac{bc}{a^2}\right)R,$$

$$such\ that\ \ R = \frac{\varepsilon a}{(\varepsilon+1)}$$

4.2 Proposition:

The solution of the system

$$z(n+1) = Az(n)$$

$z(n) = \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}$, $A = \begin{bmatrix} \frac{a}{\varepsilon} & \frac{b}{\varepsilon} \\ \frac{c}{\varepsilon^2} & \frac{d}{\varepsilon^2} \end{bmatrix} = \frac{1}{\varepsilon^2}\begin{bmatrix} \varepsilon a & \varepsilon b \\ c & d \end{bmatrix} = \frac{1}{\varepsilon^2} B\ , where\ B = \begin{bmatrix} \varepsilon a & \varepsilon b \\ c & d \end{bmatrix}$ and that α, β are eigenvalues of B which given in (7) and (8) is:

$$z(n)=\begin{cases}\dfrac{1}{\varepsilon^{2n}}\begin{bmatrix}\dfrac{\alpha^n(\alpha-d)-\beta^n(\beta-d)}{\alpha-\beta} & \dfrac{(\alpha-d)(\beta-d)(\beta^n-\alpha^n)}{c(\alpha-\beta)}\\ \dfrac{c(\alpha^n-\beta^n)}{\alpha-\beta} & \dfrac{c\beta^n(\alpha-d)-c\alpha^n(\beta-d)}{\alpha-\beta}\end{bmatrix}\begin{bmatrix}x(0)\\y(0)\end{bmatrix} & ; \quad \alpha\neq\beta,\\ \dfrac{\alpha^{n-1}}{\varepsilon^{2n}}\begin{bmatrix}(n+1)\alpha-nd & \dfrac{(2-n)\alpha^2-2(1-n)\alpha+nd^2}{c}+\dfrac{2\alpha}{c(\alpha-d)}\\ nc & (1-n)\alpha+n\end{bmatrix}\begin{bmatrix}x(0)\\y(0)\end{bmatrix} & ; \ \alpha=\beta, b\neq 0 \text{ and } c\neq 0,\\ \dfrac{1}{\varepsilon^{2n}}\begin{bmatrix}\alpha^n & 0\\ 0 & \alpha^n\end{bmatrix}\begin{bmatrix}x(0)\\y(0)\end{bmatrix} & ; \alpha=\beta, b=0 \text{ or } c=0\,.\end{cases}$$

Proof:

Since $z(n)=A^n\begin{bmatrix}x(0)\\y(0)\end{bmatrix}$ *and* $A=\frac{1}{\varepsilon^2}B$ thus the proof is complete if we compute B^n .

1) If $\alpha\neq\beta$ then be using the diagonalizable methods we have:

$$P=\begin{bmatrix}1 & 1\\ \dfrac{c}{\alpha-d} & \dfrac{c}{\beta-d}\end{bmatrix},\quad P^{-1}=\begin{bmatrix}\dfrac{\alpha-d}{\alpha-\beta} & \dfrac{(\alpha-d)(\beta-d)}{c(\beta-\alpha)}\\ \dfrac{\beta-d}{\beta-\alpha} & \dfrac{(\alpha-d)(\beta-d)}{c(\alpha-\beta)}\end{bmatrix},$$

$$D=\begin{bmatrix}\alpha & 0\\ 0 & \beta\end{bmatrix}, D^n=\begin{bmatrix}\alpha^n & 0\\ 0 & \beta^n\end{bmatrix}$$

Now , by application the equation

$$B^n=PD^nP^{-1}$$

We get :

$$B^n=\begin{bmatrix}\dfrac{\alpha^n(\alpha-d)-\beta^n(\beta-d)}{\alpha-\beta} & \dfrac{(\alpha-d)(\beta-d)(\beta^n-\alpha^n)}{c(\alpha-\beta)}\\ \dfrac{c(\alpha^n-\beta^n)}{\alpha-\beta} & \dfrac{c\beta^n(\alpha-d)-c\alpha^n(\beta-d)}{\alpha-\beta}\end{bmatrix}$$

Therefore

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$$A^n = \frac{1}{\varepsilon^{2n}} B^n = \frac{1}{\varepsilon^{2n}} \begin{bmatrix} \frac{\alpha^n(\alpha-d)-\beta^n(\beta-d)}{\alpha-\beta} & \frac{(\alpha-d)(\beta-d)(\beta^n-\alpha^n)}{c(\alpha-\beta)} \\ \frac{c(\alpha^n-\beta^n)}{\alpha-\beta} & \frac{c\beta^n(\alpha-d)-c\alpha^n(\beta-d)}{\alpha-\beta} \end{bmatrix}$$

2) If $\alpha = \beta$ then be using the Jordan form method we have:

$$P = \begin{bmatrix} 1 & \frac{2}{\alpha-d} \\ \frac{c}{\alpha-d} & \frac{c}{(\alpha-d)^2} \end{bmatrix}, \quad P^{-1} = -\frac{1}{c} \begin{bmatrix} c & -2(\alpha-d) \\ -c(\alpha-d) & (\alpha-d)^2 \end{bmatrix}, \quad J = \begin{bmatrix} \alpha & \frac{-4\varepsilon bc}{(\varepsilon a-d)^2} \\ 0 & \alpha \end{bmatrix}$$

Now , we have two cases as the following:

i) If $b \neq 0$ and $c \neq 0$

$$J = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix}, \quad J^n = \begin{bmatrix} \alpha^n & n\alpha^{n-1} \\ 0 & \alpha^n \end{bmatrix}$$

$$B^n = PJ^nP^{-1}$$

We get :

$$B^n = \alpha^{n-1} \begin{bmatrix} (n+1)\alpha - nd & \frac{(2-n)\alpha^2 - 2(1-n)\alpha + nd^2}{c} + \frac{2\alpha}{c(\alpha-d)} \\ nc & (1-n)\alpha + n \end{bmatrix}$$

Therefore

$$A^n = \frac{1}{\varepsilon^{2n}} B^n = \frac{\alpha^{n-1}}{\varepsilon^{2n}} \begin{bmatrix} (n+1)\alpha - nd & \frac{(2-n)\alpha^2 - 2(1-n)\alpha + nd^2}{c} + \frac{2\alpha}{c(\alpha-d)} \\ nc & (1-n)\alpha + n \end{bmatrix}$$

ii) If $b = 0$ and $c = 0$

$$J = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}, \quad J^n = \begin{bmatrix} \alpha^n & 0 \\ 0 & \alpha^n \end{bmatrix}, P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, P^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B^n = \begin{bmatrix} \alpha^n & 0 \\ 0 & \alpha^n \end{bmatrix},$$

Therefore

$$A^n = \frac{1}{\varepsilon^{2n}} B^n = \frac{1}{\varepsilon^{2n}} \begin{bmatrix} \alpha^n & 0 \\ 0 & \alpha^n \end{bmatrix}$$

4.3 Example:

Consider the system of singular perturbed difference equations:

$$\left.\begin{aligned} \varepsilon x(n+1) &= y(n) \\ \varepsilon^2 y(n+1) &= x(n) \end{aligned}\right\}$$

Also we can describe by using the matrices as following:

$$z(n+1) = Az(n)$$

$z(n) = \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}$, $A = \begin{bmatrix} 0 & \frac{1}{\varepsilon} \\ \frac{1}{\varepsilon^2} & 0 \end{bmatrix} = \frac{1}{\varepsilon^2}\begin{bmatrix} 0 & \varepsilon \\ 1 & 0 \end{bmatrix} = \frac{1}{\varepsilon^2} B$, $where\ B = \begin{bmatrix} 0 & \varepsilon \\ 1 & 0 \end{bmatrix}$ and α, β are eigenvalues of B such that:

$$\alpha = \sqrt{\varepsilon}, \beta = -\sqrt{\varepsilon}$$

we note that:

$$P = \begin{bmatrix} 1 & 1 \\ \frac{1}{\sqrt{\varepsilon}} & -\frac{1}{\sqrt{\varepsilon}} \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{\varepsilon}}{2} \\ \frac{1}{2} & -\frac{\sqrt{\varepsilon}}{2} \end{bmatrix}, \quad D = \begin{bmatrix} \sqrt{\varepsilon} & 0 \\ 0 & -\sqrt{\varepsilon} \end{bmatrix}, D^n = \begin{bmatrix} (\sqrt{\varepsilon})^n & 0 \\ 0 & (-\sqrt{\varepsilon})^n \end{bmatrix}$$

Now , by application the equation $B^n = PD^nP^{-1}$

We get : $B^n = \begin{bmatrix} \frac{(\sqrt{\varepsilon})^n + (-\sqrt{\varepsilon})^n}{2} & \frac{(\sqrt{\varepsilon})^{n+1} + (-\sqrt{\varepsilon})^{n+1}}{2} \\ \frac{(\sqrt{\varepsilon})^{n-1} + (-\sqrt{\varepsilon})^{n-1}}{2} & \frac{(\sqrt{\varepsilon})^n + (-\sqrt{\varepsilon})^n}{2} \end{bmatrix}$

Therefore $A^n = \frac{1}{\varepsilon^{2n}} B^n = \frac{1}{\varepsilon^{2n}} \begin{bmatrix} \frac{(\sqrt{\varepsilon})^n + (-\sqrt{\varepsilon})^n}{2} & \frac{(\sqrt{\varepsilon})^{n+1} + (-\sqrt{\varepsilon})^{n+1}}{2} \\ \frac{(\sqrt{\varepsilon})^{n-1} + (-\sqrt{\varepsilon})^{n-1}}{2} & \frac{(\sqrt{\varepsilon})^n + (-\sqrt{\varepsilon})^n}{2} \end{bmatrix}$

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4.4 Theorem:

If $A = \begin{bmatrix} \frac{a}{\varepsilon} & \frac{b}{\varepsilon} \\ \frac{c}{\varepsilon^2} & \frac{a}{\varepsilon^2} \end{bmatrix} = \frac{1}{\varepsilon^2}\begin{bmatrix} \varepsilon a & \varepsilon b \\ c & a \end{bmatrix} = \frac{1}{\varepsilon^2}B\ , where\ B = \begin{bmatrix} \varepsilon a & \varepsilon b \\ c & a \end{bmatrix}$ which given in (1) and let α, β be the eigenvalues of B such that:

$$\alpha = \frac{a(\varepsilon+1) + \sqrt{a^2(\varepsilon+1)^2 - 4\varepsilon(a^2 - bc)}}{2}, \beta = \frac{a(\varepsilon+1) - \sqrt{a^2(\varepsilon+1)^2 - 4\varepsilon(a^2 - bc)}}{2}$$

Then

$$\|(B - \beta I_{2\times 2})D\| < \varepsilon M$$

$where\ D = \begin{bmatrix} \varepsilon & 0 \\ 0 & 1 \end{bmatrix}$, $\|C\| = \sum_{i,j=1}^{2}|c_{ij}|$ stands for the norm of the matrix C .

Proof:

We note that

$$(B - \beta I_{2\times 2})D = \begin{bmatrix} \varepsilon(\varepsilon a - \beta) & \varepsilon b \\ \varepsilon c & a - \beta \end{bmatrix}$$

Now we will search on boundedness the elements of this matrix.

$$|a - \beta| = \left|\frac{a(\varepsilon-1) - \sqrt{a^2(\varepsilon-1)^2 + 4\varepsilon bc}}{2}\right| = |a|\varepsilon \left|\frac{\left(1-\frac{1}{\varepsilon}\right) - \sqrt{\left(1-\frac{1}{\varepsilon}\right)^2 + \frac{4bc}{\varepsilon}}}{2}\right|$$

Put $\lambda = \frac{1}{\varepsilon}$ then we have if $\rightarrow 0\ then\ \lambda \rightarrow \infty$, thus

$$\lim_{\lambda\rightarrow\infty} \frac{(1-\lambda) - \sqrt{(1-\lambda)^2 + 4\lambda bc}}{2} = 4bc$$

Now we have the following cases:

1) If $bc > 0$ then $\left|\frac{(1-\lambda) - \sqrt{(1-\lambda)^2 + 4\lambda bc}}{2}\right| < M$

Therefore $|a - \beta| < |a|\varepsilon M < \varepsilon M_1\ , where\ M_1 = |a|M.$

Similarly we can proof that $|\varepsilon(\varepsilon a - \beta)| < \varepsilon M_2$

Thus by taking $M = M_1 + M_2 + |b| + |c|$ we get:

$$\|(B - \beta I_{2\times 2})D\| < \varepsilon M$$

2) If $bc > 0$ then we can use the same method to get the result.

3) If $bc = 0$ then $\beta = \frac{a(\varepsilon+1)-|a||\varepsilon-1|}{2}$

I) $a > 0$

I.1) $\varepsilon > 1 \rightarrow \beta = a \rightarrow (\varepsilon a - \beta) < a\varepsilon$

I.2) $\varepsilon < 1 \rightarrow \beta = \varepsilon a \rightarrow (\varepsilon a - \beta) = 0 < a$

II) $a < 0$

II.1) $\varepsilon > 1 \rightarrow \beta = \varepsilon a \rightarrow (\varepsilon a - \beta) = 0 < \varepsilon$

II.2) $\varepsilon < 1 \rightarrow \beta = a \rightarrow (\varepsilon a - \beta) = |a|(1-\varepsilon) < |a|$

Thus we can take $M = |c| + |b| + |a|$ to get that:

$$\|(B - \beta I_{2\times 2})D\| < \varepsilon M$$

Therefore the proof is complete.

Conflict of interests.

There are non-conflicts of interest.

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