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Spline Fractional Polynomial for Computing Fractional Differential Equations

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متعددة حدود سبلاين الكسرية لحساب معادلات تفاضلية

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ABSTRACT

We propose a fractional spline method for solving fractional differential equations subject to initial conditions. Using the Caputo fractional integral and derivative have to construct the spline interpolation with polynomial coefficients. For the given spline function, error bounds were studied and a stability analysis was completed. The numerical explanation for the provided method was considered using three examples. The results show that the fractional spline function which interpolates data is productive and profitable in solving unique problems.

Key words: Spline model, fractional polynomial, fractional derivative, stability analysis.

الخلاصة

نقترح طريقة سبلاين الكسرية للمعادلات التفاضلية الكسرية الخاضعة للشروط الأبتدائية. ان استخدام تكامل واشتقاق كابوتو(Caputo) الكسرية يؤدي الى انشاء استيفاء سبلاين بمعاملات متعددة الحدود. تمت دراسة حدود الخطأ لدالة سبلاين المفروضة و كان تحليل الأستقرار مكتملا. تم النظر في التفسير العددي للطريقة المقدمة بأستخدام ثلاثة أمثلة. أظهرت النتائج أن طريقة سبلاين الكسرية التي تقحم المعامل مع متعددة الحدود الكسرية فريدة من نوعها.

الكلمات المفتاحية: نموذج سبلاين ، متعددة الحدود الكسري ، المشتقة الكسرية ، تحليل الثبات.

INTRODUCTION

Spline functions can be applied to numerical solutions of ordinary partial differential equations and integral equations as well as lacunary interpolation, interpolation and data fitting.[1] Initial value problems occur in many fields of sciences and engineering and therefore a numerical solution is needed rather than an analytical one. (See [2], [3] and [4]) Various spline functions have been used previously by authors as a solution for initial value problems. This paper further develops these ideas in order to solve fractional initial value problems. A spline is a particular function defined piecewise using polynomials in mathematics. Spline interpolation is generally preferred to polynomial interpolation in interpolating problems because it provides similar results

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even when using low degree polynomials while avoiding Rung's effect for higher degrees. Fractional derivative with Caputo and Reimann-Liouville formula is a derivative of any noninteger order, real or complex, are valuable for modeling dynamic phenomena science and technology fields. However, for some difficult the fractional differential equations cannot provide results. Therefore, instead of fractional order methods, analytical or numerical methods are used to obtain the best approximations, see[5, 7 and 8].

The current article can be considered to repeat the constructed spline polynomial in the first section, the error bounded in the second section, and stability analysis in the third section. The third section, which presents the collations of our numerical results with exact solutions, has been studied. This section shows the results and conclusions of the study. (See [1], [9], [10], and [11])

A new type of class $C^{\frac{7}{2}}$- Approximation method with the lacunary spline is constructed and used to find a numerical solution to a fractional initial value problems see [5, 6].

$$D^{(2\alpha)}y(x) + D^{(\alpha)}y(x) = f(x,y), x \in [0,1] \quad (1)$$

2. Preliminaries and assumptions

There are various definitions of fractional derivative and Taylor's Theorem, which we used in our work, will be presented in this section. Fractional derivatives are defined in different of methods, the most common of which are the Riemann-Liouville and Caputo derivatives (see [6, 7 and 8]).

Definition 2.1:[6, 7]Suppose that $\lambda > 0, x > b, \lambda, b, x \in \mathbb{R}$. Then the Caputo fractional derivative of order $\lambda > 0$ is defined by the following fractional operator

$$D_b^{\lambda} g(x) = \begin{cases} \frac{1}{\Gamma(n-\lambda)} \int_b^x \frac{g^{(n)}(t)}{(x-t)^{\lambda+1-n}} ds, n-1 < \lambda < n, n \in N, \\ \frac{d^m}{dx^n} g(x), \lambda = n, n \in N. \end{cases}$$

Definition 2.2: [6, 7] Suppose that$\lambda > 0, x > b, \lambda, b, x \in \mathbb{R}$. Then the Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined by the following fractional operator

$$D_b^{\lambda} g(x) = \begin{cases} \frac{1}{\Gamma(n-\lambda)} \frac{d^m}{dx^n} \int_b^x \frac{g(t)}{(x-t)^{\lambda+1-n}} ds, n-1 < \lambda < n, n \in N, \\ \frac{d^m}{dx^n} g(x), \text{ for } \lambda = n, n \in N. \end{cases}$$

Definition 2.3:[5] Suppose that $D_a^{z\lambda} G(x) \in \mathbb{C}[a,b]$ for $z = 0,1, \dots, n+1$ and $0 < \lambda \leq 1$ then we have the Taylor series expansion about $x = \tau$

$$g(x) = \sum_{i=0}^{n} \frac{(x-\tau)^{i\lambda}}{\Gamma(i\lambda+1)} D_a^{i\lambda} g(\tau) + \frac{\left(D_a^{(n+1)\lambda} g\right)(\xi)}{\Gamma((n+1)\lambda+1)} (x-\tau)^{(n+1)\lambda} \text{ with } a \leq \xi \leq x,$$

for all $x \in [a,b]$ where $D_a^{z\lambda} = D_a^{\lambda} \cdot D_a^{\lambda} \dots D_a^{\lambda}$ (z times).

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3. Theoretical of the spline method

In this section, construct the fractional spline model for solving the FDEs; by spline function with the coefficients fractional polynomial, stability and error bounds must be regarded with respect the following theorems.

Theorem 3.1:

Given the real numbers $D^{\left(\frac{3}{2}\right)}s_j, j = 0,1,\dots,\mathrm{N}, s_0, D^{\left(\frac{1}{2}\right)}s_0(x)$ and $D^{(1)}s(x)$, then there exist a unique spline $s(x) \in S_{\left(n,\frac{7}{2}\right)}$ Such that

$$D^{\left(\frac{1}{2}\right)}s(x_0) = D^{\left(\frac{1}{2}\right)}f(x_0)\ ,\ D^{(1)}s(x_0) = D^{(1)}f(x_0)$$

$$D^{(\alpha)}s_j(x) = D^{(\alpha)}f_j(x)\ ,\quad j = 0,1,\dots,n \text{ and } \alpha = \frac{3}{2} \tag{2}$$

$$D^{(\alpha)}s(x_j) = D^{(\alpha)}f(x_j)\ \text{, where}\quad j = 0,1,\dots,n\ ,\ \alpha = \frac{1}{2}$$

Proof:

The spline function with fractional polynomial $s(x) \in S\left(n,\frac{7}{2}\right)$ in the interval$(0,1]$, we developed the construction spline function from [1, 12, 13], as follows:

$$s(x) = A(x)S_j + B(x)(x-x_j)^{\frac{1}{2}}D^{\left(\frac{1}{2}\right)}S_{j+1}(x) + C(x)(x-x_j)^{\frac{3}{2}}D^{\left(\frac{3}{2}\right)}S_{j+1}(x) + (x - x_j)^2\left[D(x)D^{(2)}S_j(x) + E(x)D^{(2)}S_{j+1}(x)\right] + F(x)(x-x_j)^{\frac{5}{2}}D^{\left(\frac{5}{2}\right)}S_{j+1}(x)$$

Where

$$A(x) = 1$$

$$B(x) = \frac{\sqrt{\pi}}{2}\ x$$

$$C(x) = \frac{-\sqrt{\pi}}{2}\ x + \frac{4}{3\sqrt{\pi}}\ x^{\frac{3}{2}} \tag{3}$$

$$D(x) = \frac{\pi x}{8} - x^{\frac{3}{2}} + \frac{4x^{\frac{5}{2}}}{5} - \frac{8x^{\frac{7}{2}}}{35}$$

$$E(x) = \left(\frac{-3\pi+8}{24}\right)x + \left(\frac{3\pi-8}{3\pi}\right)x^{\frac{3}{2}} + \frac{x^2}{2} - \frac{4x^{\frac{5}{2}}}{5} + \frac{8x^{\frac{7}{2}}}{35}$$

$$F(x)=\frac{2x^{\frac{3}{2}}}{3\sqrt{\pi}}-\frac{16x^{\frac{5}{2}}}{15\sqrt{\pi}}+\frac{16x^{\frac{7}{2}}}{35\sqrt{\pi}}$$

$x_j+t\lambda h$,$0\le t\le 1$, h is the step size and λbe a constant, with the same expression for $s(x)$ in $[x_{j-1},x_j]$. Since $s(x)\in C^{\frac{7}{2}}$ and $s(x_j{}^{+})=s(x_j{}^{-})$to $D^{(\frac{5}{2})}s(x_j{}^{+})=D^{(\frac{5}{2})}s(x_j{}^{-})$ respectively, for $j=0,1,2,\dots,N$, leads to the following linear system of equations:

$$S_j=\left(\frac{6\sqrt{\pi}(2-\sqrt{\pi}\pi h^{\frac{5}{2}}\lambda^{\frac{3}{2}})}{2(6\sqrt{\pi}-3\pi h^{\frac{5}{2}}\lambda^{\frac{3}{2}}+8h^3\lambda^{\frac{3}{2}})}\right)s_{j-1}+\left(\frac{3\pi h^{\frac{3}{2}}}{6\sqrt{\pi}-3\pi h^{\frac{5}{2}}\lambda^{\frac{3}{2}}+8h^3\lambda^{\frac{3}{2}}}\right)s_{j-1}^{\frac{1}{2}}+\left(\frac{6\sqrt{\pi}h^2}{6\sqrt{\pi}-3\pi h^{\frac{5}{2}}\lambda^{\frac{3}{2}}+8h^3\lambda^{\frac{3}{2}}}\right)s'_{j-1}+\left(\frac{\sqrt{\pi}(6\pi h^3-16h^{\frac{7}{2}}+3\pi h^4}{\pi(6\sqrt{\pi}-3\pi h^{\frac{5}{2}}\lambda^{\frac{3}{2}}+8h^3\lambda^{\frac{3}{2}})}\right)s''_{j-1}+\left(\frac{280\pi h^{\frac{7}{2}}+1120\pi h^4-2240h^4+420\pi h^{\frac{9}{2}}-1120\pi h^5 80h^4+384\pi h^6}{70\pi(6\sqrt{\pi}-3\pi h^{\frac{5}{2}}\lambda^{\frac{3}{2}}+8h^3\lambda^{\frac{3}{2}})}\right)s_{j-1}^{\frac{5}{2}} \quad (4)$$

$$hD^{(\frac{1}{2})}S_j=\left(\frac{-3\sqrt{\pi}h^2\lambda^{\frac{3}{2}}+3\sqrt{\pi}h^{\frac{5}{2}}\lambda^{\frac{3}{2}}+4h^{\frac{7}{2}}\lambda^3-4h^4\lambda^3}{3\sqrt{\pi}}\right)s_{j-1}-\left(\frac{2h^{\frac{5}{2}}\lambda^{\frac{3}{2}}-2h^3\lambda^{\frac{3}{2}}}{\sqrt{\pi}}\right)s_{j-1}^{\frac{1}{2}}-\left(h^3\lambda^{\frac{3}{2}}-h^{\frac{7}{2}}\lambda^{\frac{3}{2}}\right)s'_{j-1}-\left(\frac{3\pi\sqrt{\pi}h^4\lambda^{\frac{3}{2}}-3\pi\sqrt{\pi}h^{\frac{9}{2}}\lambda^{\frac{3}{2}}+4\pi h^{\frac{5}{2}}+8\pi h^{\frac{7}{2}}}{6\pi\sqrt{\pi}}\right)s''_{j-1}-\left(\frac{48\pi h^{\frac{9}{2}}\lambda^{\frac{3}{2}}-48\pi h^5\lambda^{\frac{3}{2}}-45\pi\sqrt{\pi}h^3+120\sqrt{\pi}h^3+180\pi\sqrt{\pi}h^{\frac{7}{2}}-360\sqrt{\pi}h^{\frac{7}{2}}+240\sqrt{\pi}h^4-225\pi\sqrt{\pi}h^{\frac{9}{2}}+90\pi\sqrt{\pi}h^{\frac{11}{2}}}{90\pi\sqrt{\pi}}\right)s_{j-1}^{\frac{5}{2}} \quad (5)$$

$$hD^{(1)}S_j=\left(\frac{-3\pi\sqrt{\pi}h^{\frac{7}{2}}\lambda^{\frac{3}{2}}-12\sqrt{\pi}h^2\lambda^{\frac{3}{2}}+4\pi h^3\lambda^{\frac{3}{2}}-16h^{\frac{7}{2}}\lambda^{\frac{3}{2}}-3\pi\sqrt{\pi}h^{\frac{3}{2}}}{3(2\pi+\pi\sqrt{\pi}h^{\frac{3}{2}}\lambda^{\frac{3}{2}}-4\sqrt{\pi}h^2\lambda^{\frac{3}{2}})}\right)s_{j-1}-\left(\frac{2\pi h^2-8h^{\frac{5}{2}}\lambda^{\frac{3}{2}}+\pi\sqrt{\pi}h^{\frac{1}{2}}}{2\pi+\pi\sqrt{\pi}h^{\frac{3}{2}}\lambda^{\frac{3}{2}}-4\sqrt{\pi}h^2\lambda^{\frac{3}{2}}}\right)s_{j-1}^{\frac{1}{2}}-\left(\frac{3\pi\sqrt{\pi}h^{\frac{7}{2}}\lambda^{\frac{3}{2}}-12\sqrt{\pi}h^4\lambda^{\frac{3}{2}}+8\pi h^2-48h^{\frac{5}{2}}+12\pi h^3}{6\left(2\pi+\pi\sqrt{\pi}h^{\frac{3}{2}}\lambda^{\frac{3}{2}}-4\sqrt{\pi}h^2\lambda^{\frac{3}{2}}\right)}\right)s''_{j-1}-\left(\frac{16\pi^2h^4\lambda^{\frac{3}{2}}-64\pi h^{\frac{9}{2}}\lambda^{\frac{3}{2}}+40\pi\sqrt{\pi}h^{\frac{5}{2}}+240\pi\sqrt{\pi}h^3-48\sqrt{\pi}h^3+120\pi\sqrt{\pi}h^{\frac{7}{2}}-400\pi\sqrt{\pi}h^4+192\pi\sqrt{\pi}h^5}{30\pi(2\pi+\pi\sqrt{\pi}h^{\frac{3}{2}}\lambda^{\frac{3}{2}}-4\sqrt{\pi}h^2\lambda^{\frac{3}{2}})}\right)S_{j-1}^{\frac{5}{2}} \quad (6)$$

$$h^2D^{(2)}S_j=-\left(\frac{3h^{\frac{5}{2}}-2h^{\frac{7}{2}}}{1-3h^{\frac{1}{2}}+2h^{\frac{3}{2}}}\right)s''_{j-1}-\left(\frac{-4h^3+4h^4}{\sqrt{\pi}\left(1-3h^{\frac{1}{2}}+2h^{\frac{3}{2}}\right)}\right)s_{j-1}^{\frac{5}{2}} \quad (7)$$

$$h^{\frac{5}{2}}D^{(\frac{5}{2})}S_j=\left(\frac{6\sqrt{\pi}h^{\frac{5}{2}}-6\sqrt{\pi}h^{\frac{7}{2}}}{6\sqrt{\pi}h-4\sqrt{\pi}}\right)s_{j-1}^{\frac{5}{2}},,j=1,\dots,\mathrm{N} \quad (8)$$

The proof of the theorem completed,

Hint: From the above theorem we can prove that the model of the spline method are existence and unique. Moreover, the following theorems can be used to demonstrate that the convergence analysis of the spline method construction is correct.

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Theorem 3.2: [12-13] let $f \in C^{2n}[0,h]$be the unique Hermite interpolation polynomial of degree $2n-1$that matches f and its first $n-1$ derivatives f^r at 0 and h, and p_{2n-1}be the unique Hermite interpolation polynomial of degree $2n-1$ that matches f and its first $n-1$ derivatives f^r at 0 and h. Then

$$\left|e^{(r)}\right| \le \frac{h^r[x(h-x)]^{n-r}w}{r!(2n-2r)!}, r = 0,1,\dots,n, 0 \le x \le h \tag{9}$$

Where

$\left|e^{(r)}\right| = \left|f^r(x) - p_{2n-1}^{(r)}(x)\right|$ and $w = \max_{0\le x\le h}\left|f^{(2n)}(x)\right|$.

The bounds in (9) are the best possible for only $r = 0$. (10)

Theorem 3.3: Assume that $s(x)$ be the fractional spline defined in Theorem 3.1, $D^{\left(\frac{1}{2}\right)}f, D^{\left(\frac{3}{2}\right)}f \in C^{\frac{7}{2}}[0,1]$ and that $D^{(j)}f(0) = 0, j = 1,2$ then for any $x \in [0,1]$ we have $|D^{(\alpha)}s(x) - D^{(\alpha)}f(x)| \le \frac{h^2}{2\sqrt{\pi}}f^{\alpha}$

Proof: (See [1])

4.Stability Analysis

The given methods (4), (5), (6), (7) and (8) are being studied for evaluating its stability analysis and providing the method to test the equation.

$$y^{\left(\frac{3}{2}\right)}(x) = -\lambda^{\frac{3}{2}}y(x), \lambda \in \mathbb{R}, y(x_0) = y_0, y'(x_0) = y' \tag{11}$$

Can be written as the following linear system

$S_j = DS_{j-1}, j = 1,\dots,\mathbb{N}$

Where $S_j = \begin{bmatrix} S_j \\ D^{\left(\frac{1}{2}\right)}s_j \\ D^{(1)}s_j \\ D^{(2)}s_j \\ D^{\left(\frac{5}{2}\right)}s_j \end{bmatrix}, S_{j-1} = \begin{bmatrix} S_{j-1} \\ D^{\left(\frac{1}{2}\right)}s_{j-1} \\ D^{(1)}s_{j-1} \\ D^{(2)}s_{j-1} \\ D^{\left(\frac{5}{2}\right)}s_{j-1} \end{bmatrix}$ and

$$\mathbf{D} = \begin{pmatrix} \alpha1 & \alpha2 & \alpha3 & \alpha4 & \alpha5 \\ \beta1 & \beta2 & \beta3 & \beta4 & \beta5 \\ \gamma1 & \gamma2 & 0 & \gamma3 & \gamma4 \\ 0 & 0 & 0 & \delta1 & \delta2 \\ 0 & 0 & 0 & 0 & \varepsilon1 \end{pmatrix} \tag{12}$$

Where

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$$\alpha 1 = \left(\frac{6\sqrt{\pi}(2-\sqrt{\pi}h^{\frac{5}{2}}\lambda^{\frac{3}{2}})}{2(6\sqrt{\pi}-3\pi h^{\frac{5}{2}}\lambda^{\frac{3}{2}}+8h^3\lambda^{\frac{3}{2}})}\right),\ \alpha 2 = \left(\frac{3\pi h^{\frac{3}{2}}}{6\sqrt{\pi}-3\pi h^{\frac{5}{2}}\lambda^{\frac{3}{2}}+8h^3\lambda^{\frac{3}{2}}}\right),\ \alpha 3 = \left(\frac{6\sqrt{\pi}h^2}{6\sqrt{\pi}-3\pi h^{\frac{5}{2}}\lambda^{\frac{3}{2}}+8h^3\lambda^{\frac{3}{2}}}\right)$$

$$\alpha 4 = \left(\frac{\sqrt{\pi}(6\pi h^3-16h^{\frac{7}{2}}+3\pi h^4)}{\pi(6\sqrt{\pi}-3\pi h^{\frac{5}{2}}\lambda^{\frac{3}{2}}+8h^3\lambda^{\frac{3}{2}})}\right) \text{ And } \alpha 5 = \left(\frac{280\pi h^{\frac{7}{2}}+1120\pi h^4-2240h^4+420\pi h^{\frac{9}{2}}-1120\pi h^5+384\pi h^6}{70\pi(6\sqrt{\pi}-3\pi h^{\frac{5}{2}}\lambda^{\frac{3}{2}}+8h^3\lambda^{\frac{3}{2}})}\right)$$

$$\beta 1 = \left(\frac{-3\sqrt{\pi}\lambda^{\frac{3}{2}}h^2+3\sqrt{\pi}h^{\frac{5}{2}}\lambda^{\frac{3}{2}}+4h^{\frac{7}{2}}\lambda^3-4h^4\lambda^3}{3\sqrt{\pi}}\right),\ \beta 2 = -\left(\frac{2h^{\frac{5}{2}}\lambda^{\frac{3}{2}}-2h^3\lambda^{\frac{3}{2}}}{\sqrt{\pi}}\right),\ \beta 3 = -(h^3\lambda^{\frac{3}{2}} - h^{\frac{7}{2}}\lambda^{\frac{3}{2}})$$

$$\beta 4 = -\left(\frac{3\pi\sqrt{\pi}h^4\lambda^{\frac{3}{2}}-3\pi\sqrt{\pi}h^{\frac{9}{2}}\lambda^{\frac{3}{2}}+4\pi h^{\frac{5}{2}}-12\pi h^3+8\pi h^{\frac{7}{2}}}{6\pi\sqrt{\pi}}\right) \text{ And}$$

$$\beta 5 = -\left(\frac{48\pi h^{\frac{9}{2}}\lambda^{\frac{3}{2}}-48\pi h^5\lambda^{\frac{3}{2}}-45\pi\sqrt{\pi}h^3\lambda^{\frac{3}{2}}+120\sqrt{\pi}h^3+180\pi\sqrt{\pi}h^{\frac{7}{2}}-360\sqrt{\pi}h^{\frac{7}{2}}+240\sqrt{\pi}h^4-225\pi\sqrt{\pi}h^{\frac{9}{2}}+90\pi\sqrt{\pi}h^{\frac{11}{2}}}{90\pi\sqrt{\pi}}\right)$$

$$\gamma 1 = \left(\frac{-3\pi\sqrt{\pi}h^{\frac{3}{2}}\lambda^{\frac{3}{2}}-12\sqrt{\pi}h^2\lambda^{\frac{3}{2}}+4\pi h^3\lambda^{\frac{3}{2}}-16h^{\frac{7}{2}}\lambda^{\frac{3}{2}}+3\pi\sqrt{\pi}h^{\frac{3}{2}}}{3\left(2\pi+\pi\sqrt{\pi}h^{\frac{3}{2}}\lambda^{\frac{3}{2}}-4\sqrt{\pi}h^2\lambda^{\frac{3}{2}}\right)}\right),\ \gamma 2 = -\left(\frac{2\pi h^2\lambda^{\frac{3}{2}}-8h^{\frac{5}{2}}\lambda^{\frac{3}{2}}+\pi\sqrt{\pi}h^{\frac{1}{2}}}{2\pi+\pi\sqrt{\pi}h^{\frac{3}{2}}\lambda^{\frac{3}{2}}-4\sqrt{\pi}h^2\lambda^{\frac{3}{2}}}\right)$$

$$\gamma 3 = -\left(\frac{3\pi\sqrt{\pi}h^{\frac{7}{2}}\lambda^{\frac{3}{2}}-12\sqrt{\pi}h^4+8\pi h^2+4\pi h^2-48h^{\frac{5}{2}}+12\pi h^3}{6\left(2\pi+\pi\sqrt{\pi}h^{\frac{3}{2}}\lambda^{\frac{3}{2}}-4\sqrt{\pi}h^2\lambda^{\frac{3}{2}}\right)}\right)$$

$$\gamma 4 = -\left(\frac{16\pi^2h^4\lambda^{\frac{3}{2}}-64\pi h^{\frac{9}{2}}\lambda^{\frac{3}{2}}+15\pi\sqrt{\pi}h^{\frac{5}{2}}+240\pi\sqrt{\pi}h^3-480\sqrt{\pi}h^3+120\pi\sqrt{\pi}h^{\frac{7}{2}}-400\pi\sqrt{\pi}h^4+192\pi\sqrt{\pi}h^5}{30\pi(2\pi+\pi\sqrt{\pi}h^{\frac{3}{2}}\lambda^{\frac{3}{2}}-4\sqrt{\pi}h^2\lambda^{\frac{3}{2}})}\right)$$

$$\delta 1 = = -\left(\frac{3h^{\frac{5}{2}}-2h^{\frac{7}{2}}}{1-3h^{\frac{1}{2}}+2h^{\frac{3}{2}}}\right),\ \delta 2 = -\left(\frac{-4h^3+4h^4}{\sqrt{\pi}\left(1-3h^{\frac{1}{2}}+2h^{\frac{3}{2}}\right)}\right) \text{ And } \varepsilon 1 = \left(\frac{6\sqrt{\pi}h^{\frac{5}{2}}-6\sqrt{\pi}h^{\frac{7}{2}}}{6\sqrt{\pi}h-4\sqrt{\pi}}\right)$$

Theorem 4.1: The spline model with fractional equations of the system of equation (12) is stable.

Proof: Let the matrix D be a complex conjugate matrix, and$|v_3| \leq 1$. The characteristic equation will be stable as the characteristic polynomial **[14]**, if all complex eigenvalues have negative real parts.

Theorem 4.2: If D be a $n \times n$ Matrix with$\| D \|_\infty < 1$, then the matrix $(I - D)$ is invertible. In addition to $\|(I - D)^{-1}\|_\infty \leq \frac{1}{1-\|D\|_\infty}$.

Proof: see **[15]**

Theorem 4.3: let D is non-singular if it has n independent columns, D^{-1}exists, and $Du = f$ has a unique solution u.

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Proof: Since from linear system of the equation (12), we have a matrix D, if $|D| \neq 0$ then D^{-1} is exists and the system is unique solution by using the [16].

Theorem 4.4: The eigenvalues which satisfies the matrix D then there is necessary and a sufficient condition D is convergence.

Proof: From the (theorem 3.1 [15]), the spectral radius of the matrix D is less than one, assume that all the eigenvalues of D are distinct, and sufficiently $|\lambda_i| < 1$, then the matrix D is converges.

5. Numerical Results

The method is used to three numerical examples in this section is used to complete all calculations three fractional initial value problems are considered in order to define the class $C^{\frac{7}{2}}$ of fractional interpolation spline and verify the computational applicability of the given method. The application of the outcome in such two parts demonstrates the usefulness of the proposed technique. The tables 1, 2 and 3 are as follows.

The term $e, e^{\left(\frac{1}{2}\right)}$ and $e^{(1)}$ represent the maximum magnitude errors $|e(x)| = |s(x) - y(x)|$, $\left|D^{\left(\frac{1}{2}\right)}e(x)\right| = \left|D^{\left(\frac{1}{2}\right)}s(x) - D^{\left(\frac{1}{2}\right)}y(x)\right|$, $\lambda = 1$ and $|\, e'(x)| = |S'(x) - y'(x)|$ respectively.

Example 1: Consider the fractional differential equation as [17]

$$D^2y(x) - x^2D^{(3/2)}y(x) - \sqrt{x}D^{(1/2)}y(x) - x^{1/3}y(x) = 6\sqrt{\pi}x - \frac{16}{5}x^3 - x^{10/3}\sqrt{\pi},$$
$$y(0) = y'(0) = 0,\ 0 \leq x \leq 1,$$

Table 1. Absolute error of S(x) and its derivative of example

h	$\lvert s(x) - f(x)\rvert$	$\left\lvert s^{\left(\frac{1}{2}\right)} - f^{\left(\frac{1}{2}\right)}\right\rvert$	$\lvert s' - f'\rvert$	Exact solution	Approximation solution
0.001	1.9610×10^{-11}	1.9620×10^{-7}	1.6010×10^{-5}	1.7724×10^{-9}	1.7920×10^{-9}
0.05	1.2209×10^{-4}	3.5270×10^{-3}	3.6273×10^{-2}	2.2155×10^{-4}	3.4365×10^{-4}
0.02	3.1230×10^{-6}	3.4816×10^{-4}	6.1017×10^{-3}	1.4179×10^{-5}	1.7302×10^{-5}
0.1	1.9581×10^{-3}	2.0829×10^{-2}	0.1337	1.7724×10^{-3}	3.7306×10^{-3}

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Example 2: Consider the following nonlinear FIVP as [1]

$$D^{\frac{3}{2}}y(t) + y^2(t) = \frac{\Gamma(6)}{\Gamma(4.5)}t^{\frac{7}{2}} - \frac{3\Gamma(5)}{\Gamma(3.5)}t^{\frac{5}{2}} + \frac{\Gamma(4)}{\Gamma(2.5)}t^{\frac{3}{2}} + [t^5 - 3t^4 + 2t^3]^2$$

with initial condition $y(0) = 0, y'(0) = 0$ and the exact solution is $y(t) = t^5 - 3t^4 + 2t^3$

Table 2. Absolute error of S(x) and its derivative of example 2.

h	$\lvert s(x) - f(x)\rvert$	$\left\lvert s^{\left(\frac{1}{2}\right)} - f^{\left(\frac{1}{2}\right)}\right\rvert$	$\lvert s' - f'\rvert$	Exact solution	Approximation solution
0.001	2.2062×10^{-11}	2.2089×10^{-7}	1.8017×10^{-5}	1.99700×10^{-9}	2.0190×10^{-9}
0.05	1.1743×10^{-4}	3.4897×10^{-3}	3.6147×10^{-2}	2.3156×10^{-4}	3.4899×10^{-4}
0.02	3.3143×10^{-6}	3.7418×10^{-4}	6.5400×10^{-3}	1.5523×10^{-5}	1.8837×10^{-5}
0.1	1.5704×10^{-3}	1.7629×10^{-2}	0.119350	1.7100×10^{-3}	3.2804×10^{-3}

Table 3. In comparison to the method of [1], the absolute error in Example 2 is shown.

h	Our method	Ref[1]
0.1	1.5704×10^{-3}	7.6×10^{-3}
0.02	3.3143×10^{-6}	1.5×10^{-4}
0.05	1.1743×10^{-4}	1.6×10^{-4}

Example 3: Consider the fractional differential equation as [5]

$D^{\alpha}y(t) = t^4 - \frac{1}{2}t^3 + \frac{24}{\Gamma(4-\alpha)}t^{3-\alpha} + \frac{3}{\Gamma(5-\alpha)}t^{4-\alpha} - y(t),\ 0 < \alpha < 1$,With the initial condition $y(0) = 0$. The exact solution is $y(t) = t^4 - \frac{1}{2}t^3$.

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Table 4. Absolute error of S(x) and its derivative of example 3.

h	$\lvert s(x) - f(x) \rvert$	$\left\lvert s^{\left(\frac{1}{2}\right)} - f^{\left(\frac{1}{2}\right)} \right\rvert$	$\lvert s' - f' \rvert$	Exact solution	Approximation solution
0.001	5.5100×10^{-12}	5.5183×10^{-8}	4.500×10^{-6}	-4.99×10^{-10}	-5.0451×10^{-10}
0.05	2.7462×10^{-5}	8.27190×10^{-4}	8.5983×10^{-3}	-5.6250×10^{-5}	-8.3712×10^{-5}
0.02	8.10299×10^{-7}	9.19284×10^{-5}	1.6050×10^{-3}	-3.8400×10^{-6}	-4.65029×10^{-6}
0.1	3.2618×10^{-4}	3.8049×10^{-3}	2.6718×10^{-2}	-4×10^{-4}	-7.2618×10^{-4}

Table 5. In comparison to the method of [5], the absolute error in Example 3 is shown.

h	Our method	Ref[5]
0.1	5.2901×10^{-8}	35.9352×10^{-10}
0.02	8.10299×10^{-7}	57.4963×10^{-9}
0.001	5.5100×10^{-12}	35.9352×10^{-14}

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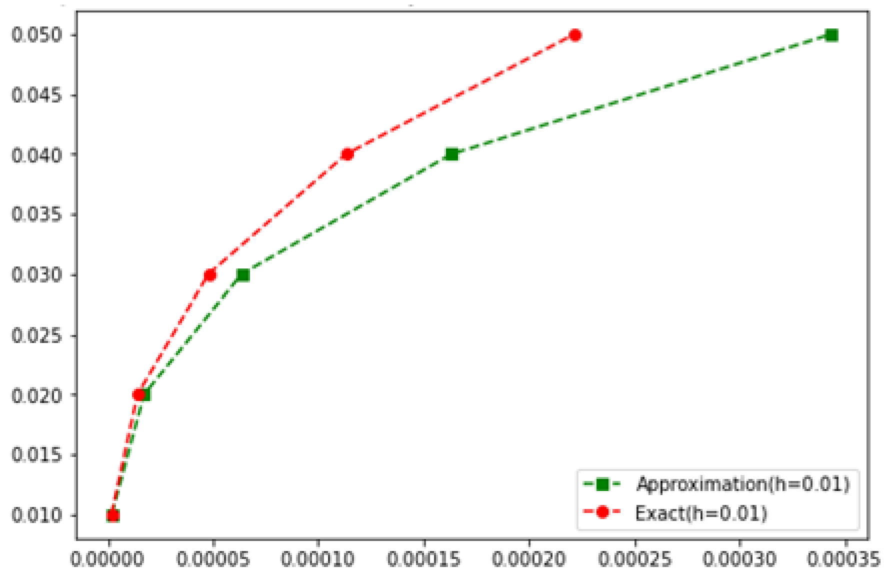

Figure 1. Comparison of the solution obtained by the spline method with exact solution of h=0.01, for example 1.

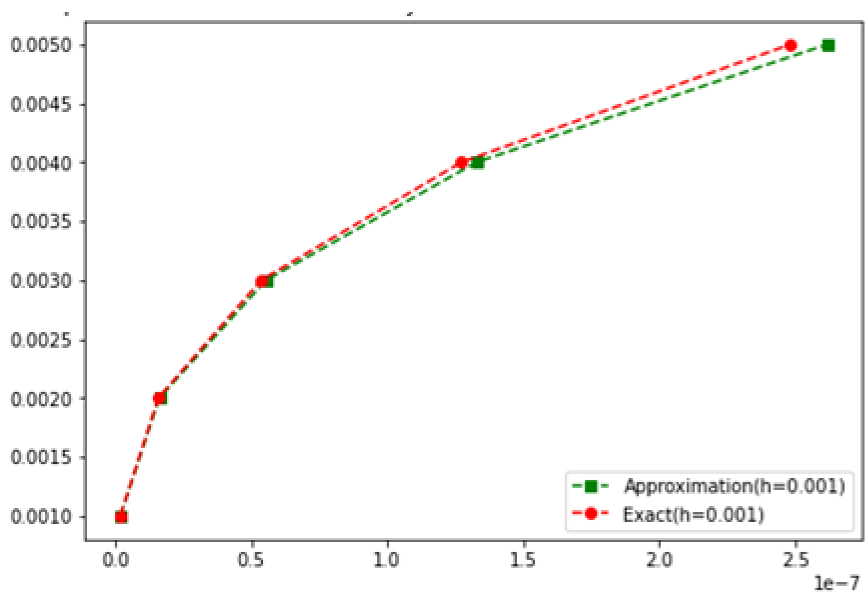

Figure 2. Comparison of the solution obtained by the spline method with exact solution of h=0.001, for example 2

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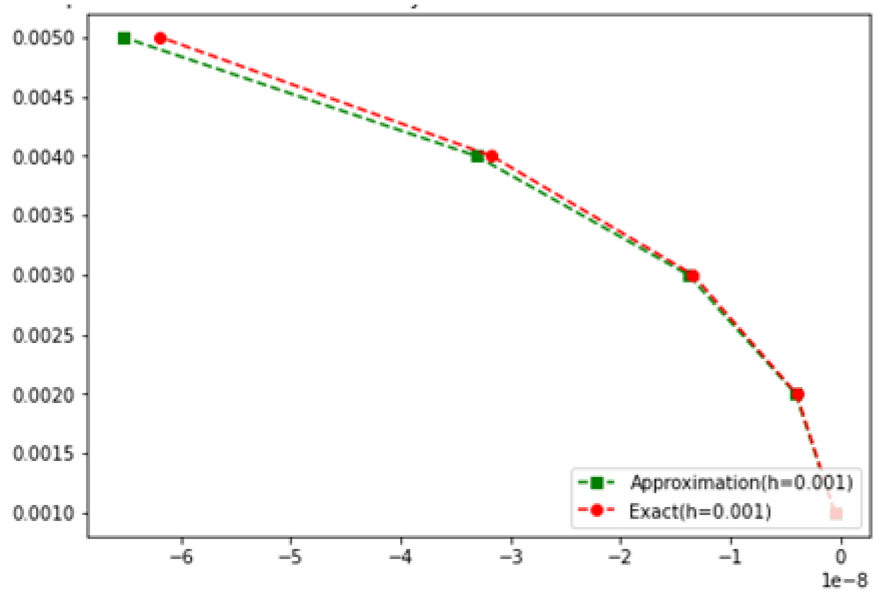

Figure 3. Comparison of the solution obtained by the spline method with exact solution of h=0.001, for example 3

6. Conclusions

The fractional spline interpolation function was proposed for solving fractional initial value problems because the application of the spline function is rarely studied. The three examples show that a given method can approximate the solution successfully; however a minimum step size h should be used. The new presented method approximates the higher order derivative s as well as giving an approximation to the solution of fractional initial value problems. In further, a spline method is created based on the use of coefficient of the method and convergence.

Conflict of interests.

There are non-conflicts of interest.

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