Review

JOURNAL OF UNIVERSITY OF BABYLON
For Pure and Applied Sciences (JUBPAS)

Vol.30; No.2.| 2022

ISSN: 2312-8135 | Print ISSN: 1992-0652

info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

مجلة جامعة بابل للعلوم الصرفة والتطبيقية

Existence and Uniqueness Solution of Fractional Order Regge Problem

Hozan Hilmi[1] and Karwan H.F Jwamer[2*]

1 College of Science, University of Sulaimani hozan.mhamadhilmi@univsul.edu.iq sulaymaniyah-Department of Mathematics, Kurdistan Region, sulaymaniyah-Iraq.

2 College of Science, University of Sulaimani karwan.jwamer@univsul.edu.iq sulaymaniyah-Department of Mathematics, Kurdistan Region, sulaymaniyah-Iraq

***Corresponding author email:** karwan.jwamer@univsul.edu.iq; **mobile: 07719706002**

وجود ووحدانية حل للمشكلة رجي من رتبة الكسرية

هۆزان حلمی[1] كاروان حمه فرج جوامير[2*]

1 كلية العلوم، جامعة السليمانية **hozan.mhamadhilmi@univsul.edu.iq** ، السليمانية، العراق

2 كلية العلوم، جامعة السليمانية **karwan.jwamer@univsul.edu.iq**، السليمانية، العراق

Received:	7 2/2022	Accepted:	16 /6 /2022	Published:	30 /6 /2022

ABSTRACT

In this paper, we look into a group of fractional boundary value problem equations involving fractional derivative fractional orders $\alpha \in (1,2]$ and $t \in (0,a], a \in R^{+}$ there are two boundary value criteria in this equation. The existence and uniqueness solutions are obtained using the Banach fixed point theorem (Contraction mapping theorem) and the Schauder fixed point theorem. based on the method of fractional integral and integral operator, our primary findings are illustrated using examples.

Keywords: Regge Problem; Fractional differential; Fractional Integral; Fractional Boundary problem.

الخلاصة

في هذا البحث، ننظر في مجموعة من معادلات مشكلة القيمة الحدية الكسرية التي تتضمن أوامر كسور مشتقة جزئية α∈ (1 ، [2و0) t∈ ، a]، a∈R ^ + هناك معياران للقيمة الحدية في هذه المعادلة يتم الحصول على حلول الوجود والتفرد باستخدام نظرية فيBanach في النقطة الثابتة (نظرية رسم خرائط الانكماش) ونظرية Schauder ذات النقطة الثابتة. استنادًا إلى طريقة عامل التكامل الجزئي والتكامل ، تم توضيح النتائج الأولية باستخدام الأمثلة.

الكلمات المفتاحية: مشكلة ريج؛ تفاضل كسري، تكامل كسرين مشكلة كسور الحدود.

1. INTRODUCTION

When there is a limited support for interaction, the Regge issue occurs in the development of quantum scattering. The S-wave radial Schrödinger equation in physics is essentially the Sturm–Liouville equation on the semiaxis, which results following the separation of variables in the three-dimensional Schrödinger equation with radial symmetric potential. (see Reference for further information on the Regge problem).[1]):

$$-y''(x,\lambda) + q(x)y(x,\lambda) = \lambda^2 y(x,\lambda). \qquad 1.1$$

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

At $x = 0$, with the boundary condition: $y(0, \lambda) = 0.$ 1.2

Various models have been proposed, despite the fact that the sort of interaction in nuclear physics is unknown. Regge claimed that the potential has limited support, and that the boundary condition for a positive integer an is:

$y'(a, \lambda) - i\lambda y(a, \lambda) = 0,$ 1.3

T. Regge, an Italian physicist, was the first to investigate this topic in[1][2], demonstrating that the system of eigenfunctions in problem (1.1)–(1.3) was complete and studying the asymptotic eigenvalues of this problem.

We investigate fractional boundary value problem solutions in this work. As previously stated[1][3], The investigations of the Schrodinger operator with potential q compactly supported on the interval on the half-axis R+. [0,a] is linked to the study of the Regge spectral problem on this interval.This issue takes of the form

$-{}_{0}^{C}D_{x}^{\alpha} y(x) + q(x)y(x) = \lambda^2 p(x)y(x), \quad x \in [0, a], \quad 1 < \alpha \leq 2$ 1.4

$y(0) = 0, \quad y'(a) - i\lambda y(a) = 0.$ 1.5

Such that $q(x), p(x) \in L_+ [0, a], where\ L_+ [0, a]$ is the set of all integrable function

$f(x)$ $on\ [0, a]$ $and\ 0 < m \leq f(x) \leq M < \infty, and\ \alpha \in (1,2]$ $, and\ \lambda$ Is a spectral parameter

Fractional calculus is a strong tool for the purpose of describing the memory and inherited features of different materials and procedures. [4], [5] It has applications in biology, chemistry, viscoelasticity, anomalous diffusion, fluid mechanics, acoustics, control theory, as well as other scientific and technical domains. Fractional differential equations were implicated in a family of integro-differential equations with singularities in these applications.[6]

For fractional ordinary differential equations, the existence and uniqueness theorems were introduced. [4], [7].

Several analytical or numerical approaches for solving fractional differential equations were proposed previously, such as [5], [8], [9]

1.1. Preliminaries

We offer several definitions, lemmas, and theorems that are essential for our theorems in this section.

Definition 1.1 [7]The Gamma function is defined by the integral formula

$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$

The integral converges absolutely for $Re(z) > 0$.

Definition 1.2 [10]**(Fractional Integral of Order α)** for every $\alpha > 0$ and a local integrable function $h(t)$, the right FI of order α is defined:

$$ {}_aI_t^{\alpha}h(u) = \int_a^u \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds, \qquad -\infty \le a < u < \infty $$

Alternately, the left FI can really be defined as follows:

$$ {}_tI_b^{\alpha}h(u) = \int_u^b \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds, \qquad -\infty < u \le b \le \infty $$

The following examples are known for certain values of the a and b parameters:

- Riemann $a = 0,\ b = +\infty$
- Liouville $a = -\infty, b = 0$

Properties 1.1 [10]If $f(x)$ and $h(x)$ are continuous functions $a, b \in R$, and $n, m > 0$, then:

i) $I_a^n\left(I_a^m f(x)\right) = I_a^m\left(I_a^n f(x)\right) = I_a^{n+m} f(x)$
ii) $I_\alpha^n\left(af(x) + bh(x)\right) = aI_\alpha^n f(x) + bI_\alpha^n h(x)$

Definition1.3 [6](Fractional Derivative of Order α) for every α, and $m = [\alpha]$ the The derivative for order α Riemann-Liouville is defined as follows:

$$ {}_aD_t^{\alpha}h(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-s)^{m-\alpha-1} f(s)ds $$

Definition 1.4 [4], [6], [7], [10]Let $\alpha > 0,\ m = [\alpha]$. The Caputo derivative operator of order α and $f(t)$ be n −times differentiabl function, $t > a$ is defined as

$$ {}_a^CD_t^{\alpha}h(u) = \frac{1}{\Gamma(m-\alpha)} \int_a^u (u-s)^{m-\alpha-1} \left(\frac{d}{ds}\right)^m h(s)ds $$

Or
$$ {}_a^CD_t^{\alpha}h(u) = \frac{1}{\Gamma(m-\alpha)} \int_a^u \frac{h^m(s)}{(u-s)^{\alpha-m+1}} ds $$

For $a = 0$, we introduce the notation:

$$ {}^CD_t^{\alpha}h(u) = {}^C\mathrm{D}^{\alpha}h(u) $$

Remark: the Fractional differential and integration operators is linear.

Let $f(x), g(x)$ be two functions such that both ${}^CD_a^{\alpha}f(t),\ {}^CD_a^{\alpha}g(t)$ exist for $\alpha \in [m-1, m)$ and $a, b \in \mathbb{C}$.

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

Then $^{C}D_a^{\alpha}(af(t)+bg(t)) = a^{C}D_a^{\alpha}f(t) + b\,^{C}D_a^{\alpha}g(t)$

The following is the relationship between the integration and differentiation of the Caputo operator of order:

- Caputo derivative of the fractional integral is
$$^{C}D_a^{\alpha}(I_a^{\alpha}f(u)) = f(u)$$

- fractional integral of the Caputo derivative is
$$I_a^{\alpha}(\,^{C}D_a^{\alpha}f(u)) = f(u) - \sum_{p=0}^{m-1}\frac{(u-a)^p}{p!}f^{(p)}(a)$$

From the above we got $^{C}D_a^{\alpha}(I_a^{\alpha}f(u)) \neq I_a^{\alpha}(\,^{C}D_a^{\alpha}f(u))$

Remark[7]**:** from the above definitions and properties we have The fractional derivative of Caputo is not the same as the fractional derivative of (Riemann-Liouville). but their fractional integral is equivalent

1.2. (The derivatives order α of Caputo and Riemann-Liouville are related.). [7], [10]

Let $m \in \mathbb{N}, \alpha \in [m-1, m)$. And let $f(u)$ be a function such that $^{c}D_a^{\alpha}f(u)$ and $D_a^{\alpha}f(u)$ exist. The following is the relationship between the (R-L) and Caputo derivatives:

$$^{c}D_a^{\alpha}f(u) = D_a^{\alpha}f(u) - \sum_{p=0}^{m-1}\frac{(u-a)^{p-\alpha}}{\Gamma(k+1-\alpha)}f^{(p)}(a)$$

1.3. The Laplace Transformation [9]

A function $f(t)$ is called original function if

1. $f(t) \equiv 0$ for $t < 0$,
2. $|f(t)| < Me^{s_0 t}$ For $t > 0$ with $M > 0, s_0 \in \mathbb{R}$.
3. The function satisfies the Dirichlet requirements for any closed interval [a,b]: (a) is bounded.
(b) Or it is continuous, or it has a finite number of first-order discontinuities, or it has a finite number of extremes.
$s \in \mathbb{C}$, then the Laplace transformation (LT) defined as
$$F(s) = L\{f(t)\} = \int_0^{\infty} e^{-st}f(t)dt$$

- **Fractional Derivatives [9]**
- The Laplace Transformation of Caputo FD is

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

$$L[D^\alpha f(t)] = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0)$$

*** Inverse of the Laplace Transformation [9]**

If $f(t)$ is an origin function and $F(s) = L\{f(t)\}$ the corresponding inverse Laplace transform is

$$f(t) = \frac{1}{2\pi i} \lim_{t\to\infty} \int_{\gamma-it}^{\gamma+it} F(s)e^{st}dt = L^{-1}[F(s)], \qquad \gamma \in R\,, i = \sqrt{-1}$$

1.4 Materials and Methods:

Lemma 1.1: [11]
The vector space $C[e,d]$ Among the continuous complex-valued functions defined on a closed interval$[e,d]$ is Banach space with respect to the following norm

$$\|v\|_{C[e,d]} = max_{x\in[e,d]} |v(x)| \quad , v \in C[e,d]$$

Lemma 1.2 [11]**:**
Let H and S be two normed spaces and $T: H \to S$ the operator T if there is an actual boundary, it is said to be bounded.z such that $\|Tx\| \le z\|x\|, x \in H.$

Lemma 1.3 :[12]
Let $\beta > 0$ and $n = [\beta]$, then the solutions to the equation ${}_0^C D_t^\beta h(t) = 0$ is given by $h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$, where $c_i \in R\,, i = 0,1,2, \ldots, n-1$ If we assume that, there are some constants. $h \in C^n[0,a]$, then $I^\beta\; {}_0^C D_t^\beta h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$ for some constants $c_i \in R\,, i = 0,1,2, \ldots, n-1$

Lemma 1.4 Let $y(x) \in C(0,a)$ with $1 < \alpha \le 2$. Then the solution of the boundary value problem (1.4)-(1.5) is

$$y(x) = x\left[\frac{y(a)}{a} - \frac{1}{a\Gamma(\alpha)} \int_0^a (a-t)^{\alpha-1} \big(q(t) - \lambda^2 p(t)\big) y(t)dt\right] + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \big(q(t) - \lambda^2 p(t)\big) y(t)dt$$

Or $y(x) = x\vartheta + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \big(q(t) - \lambda^2 p(t)\big) y(t)dt$

Where $\vartheta = \frac{y(a)}{a} - \frac{1}{a\Gamma(\alpha)} \int_0^a (a-t)^{\alpha-1} \big(q(t) - \lambda^2 p(t)\big) y(t)dt$

Proof: The Fractional Boundary Value Problem given from equation 1.4 and 1.5 we have
$-{}_0^C D_x^\alpha y(x) + q(x)\; y(x) = \lambda^2 p(x)\; y(x)$
${}_0^C D_x^\alpha y(x) = q(x) y(x) - \lambda^2 p(x) y(x)$

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

Since $1 < \alpha \leq 2$ we get $I^{\alpha}\, {}_{0}^{C}D_{t}^{\alpha} y(x) = y(x) + c_0 + c_1 x$

$\rightarrow I^{\alpha}\, {}_{0}^{C}D_{x}^{\alpha} y(x) = I^{\alpha}\big(q(x)y(x) - \lambda^2 p(x)y(x)\big)$

$$y(x) + c_0 + c_1 x = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\big(q(t) - \lambda^2 p(t)\big)y(t)dt \qquad *$$

Now to find $c_0 and\ c_1$

From Boundary conditions we have $y(0) = 0$

So $y(0) + c_0 + c_1(0) = \frac{1}{\Gamma(\alpha)} \int_0^0 (x-t)^{\alpha-1}\big(q(t) - \lambda^2 p(t)\big)y(t)dt$

$\rightarrow c_0 = 0$

Putting $x = a$ we get

$$y(a) + c_1 a = \frac{1}{\Gamma(\alpha)} \int_0^a (a-t)^{\alpha-1}\big(q(t) - \lambda^2 p(t)\big)y(t)dt$$

$$\rightarrow c_1 = \frac{-1}{a}\left[y(a) - \frac{1}{\Gamma(\alpha)} \int_0^a (a-t)^{\alpha-1}\big(q(t) - \lambda^2 p(t)\big)y(t)dt\right]$$

Substituting c_0 $and\ c_1$ in equation $(*)$ we get

$$y(x) = x\left[\frac{y(a)}{a} - \frac{1}{a\Gamma(\alpha)} \int_0^a (a-t)^{\alpha-1}\big(q(t) - \lambda^2 p(t)\big)y(t)dt\right] + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\big(q(t) - \lambda^2 p(t)\big)y(t)dt$$

$$\rightarrow y(x) = x\vartheta + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\big(q(t) - \lambda^2 p(t)\big)y(t)dt$$

Where $\vartheta = \frac{y(a)}{a} - \frac{1}{a\Gamma(\alpha)} \int_0^a (a-t)^{\alpha-1}\big(q(t) - \lambda^2 p(t)\big)y(t)dt$

Now, the fractional operator related to the fractional boundary value problem given by equation 1.4 and 1.5

$T: C[0,a] \rightarrow C[0,a]$ Is

$$\rightarrow Ty(x) = x\vartheta + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\big(q(t) - \lambda^2 p(t)\big)y(t)dt$$

Where $\vartheta = \left[\frac{y(a)}{a} - \frac{1}{a\Gamma(\alpha)} \int_0^a (a-t)^{\alpha-1}\big(q(t) - \lambda^2 p(t)\big)y(t)dt\right].$

Lemma 1.5 Let $y(x) \in C(0,a]$ with $1 < \alpha \leq 2$ Then the solution of the boundary value problem (1.4)-(1.5) is

$$y(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\big(q(t) - \lambda^2 p(t)\big)y(t)dt + x\frac{1}{\Gamma(\alpha)(ai\lambda-1)} \int_0^a (\alpha - 1 - i\lambda a + i\lambda t)(a-t)^{\alpha-2}\big(q(t) - \lambda^2 p(t)\big)y(t)dt$$

Or $y(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\big(q(t) - \lambda^2 p(t)\big)y(t)dt + x\omega$

Where $\omega = \frac{1}{\Gamma(\alpha)(ai\lambda-1)} \int_0^a (\alpha - 1 - i\lambda a + i\lambda t)(a-t)^{\alpha-2}\big(q(t) - \lambda^2 p(t)\big)y(t)dt$

Proof:

$-{}_{0}^{C}D_{x}^{\alpha} y(x) + q(x)y(x) = \lambda^2 p(x)y(x); \qquad x \in [0,a], \qquad 1 < \alpha \leq 2$

$$y(0) = 0, \qquad y'(a) - i\lambda y(a) = 0,$$

Now ${}_{0}^{C}D_{x}^{\alpha} y(x) = q(x)y(x) - \lambda^2 p(x)y(x) = \big(q(x) - \lambda^2 p(x)\big)y(x)$

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

مجلــة جامعــة بابــل للعلــوم الصــرفة والتطبيقيــة مجلــة جامعــة بابــل للعلــوم الصــرفة والتطبيقيــة مجلــة جامعــة بابــل للعلــوم الصــرفة والتطبيقيــة

Take I^{α} for both sides we get

$$I^{\alpha}\,{}_0^C D_x^{\alpha} y(x) = I^{\alpha}\big(q(x) - \lambda^2 p(x)\big)y(x)$$

$$y(x) + c_0 + c_1 x = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \big(q(t) - \lambda^2 p(t)\big)y(t)dt$$

$$y(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \big(q(t) - \lambda^2 p(t)\big)y(t)dt - c_0 - c_1 x$$

From the condition $y(0) = 0$

$$y(0) = \frac{1}{\Gamma(\alpha)} \int_0^0 (0-t)^{\alpha-1} \big(q(t) - \lambda^2 p(t)\big)y(t)dt - c_0 - c_1(0)$$

So $c_0 = 0$

Now $y(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \big(q(t) - \lambda^2 p(t)\big)y(t)dt \; - c_1 x$ $\qquad (**)$

$$y'(x) = \frac{\alpha-1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-2} \big(q(t) - \lambda^2 p(t)\big)y(t)dt - c_1$$

$$y(a) = \frac{1}{\Gamma(\alpha)} \int_0^a (a-t)^{\alpha-1} \big(q(t) - \lambda^2 p(t)\big)y(t)dt \; - c_1 a$$

$$i\lambda y(a) = \frac{i\lambda}{\Gamma(\alpha)} \int_0^a (a-t)^{\alpha-1} \big(q(t) - \lambda^2 p(t)\big)y(t)dt \; - c_1 a i\lambda$$

And $y'(a) = \frac{\alpha-1}{\Gamma(\alpha)} \int_0^a (a-t)^{\alpha-2} \big(q(t) - \lambda^2 p(t)\big)y(t)dt - c_1$

From second condition we have $y'(a) = i\lambda y(a)$

$$\frac{\alpha-1}{\Gamma(\alpha)} \int_0^a (a-t)^{\alpha-2} \big(q(t) - \lambda^2 p(t)\big)y(t)dt - c_1 = \frac{i\lambda}{\Gamma(\alpha)} \int_0^a (a-t)^{\alpha-1} \big(q(t) - \lambda^2 p(t)\big)y(t)dt \; - c_1 a i\lambda$$

$$\rightarrow c_1 - c_1 a i\lambda = \frac{\alpha-1}{\Gamma(\alpha)} \int_0^a (a-t)^{\alpha-2} \big(q(t) - \lambda^2 p(t)\big)y(t)dt - \frac{i\lambda}{\Gamma(\alpha)} \int_0^a (a-t)^{\alpha-1} \big(q(t) - \lambda^2 p(t)\big)y(t)dt$$

$$\rightarrow c_1 \Gamma(\alpha)(1 - a i\lambda) = \int_0^a \big((\alpha-1)(a-t)^{\alpha-2} - i\lambda(a-t)^{\alpha-1}\big)\big(q(t) - \lambda^2 p(t)\big)y(t)dt$$

$$\rightarrow c_1 = \frac{1}{\Gamma(\alpha)(1 - a i\lambda)} \int_0^a (\alpha - 1 - i\lambda a + i\lambda t)\,(a-t)^{\alpha-2}\big(q(t) - \lambda^2 p(t)\big)y(t)dt$$

Put c_1 in equation $**$

$$y(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \big(q(t) - \lambda^2 p(t)\big)y(t)dt + \; x\frac{1}{\Gamma(\alpha)(a i\lambda - 1)} \int_0^a (\alpha - 1 - i\lambda a + i\lambda t)\,(a-t)^{\alpha-2}\big(q(t) - \lambda^2 p(t)\big)y(t)dt$$

Now, the fractional operator related to the fractional boundary value problem given by equation 1.4 and 1.5

$T: C[0,a] \rightarrow C[0,a]$ Is $Ty(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \big(q(t) - \lambda^2 p(t)\big)y(t)dt + \; x\omega$

Where $\omega = \frac{1}{\Gamma(\alpha)(a i\lambda - 1)} \int_0^a (\alpha - 1 - i\lambda a + i\lambda t)\,(a-t)^{\alpha-2}\big(q(t) - \lambda^2 p(t)\big)y(t)dt$

Lemma 1.6 Let $y(x) \in C([0,a]$ with $1 < \alpha \leq 2$. then $y(x)$ is a solution of the boundary value problem (1.4)-(1.5) if and only if

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

$$y(x) = \int_0^a H(x,t)\big(q(t) - \lambda^2 p(t)\big)y(t)dt$$

Where

$$H(x,t) = \begin{cases} \dfrac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} + \dfrac{x(\alpha - 1 - i\lambda a + i\lambda t)(a-t)^{\alpha-2}}{\Gamma(\alpha)(ai\lambda - 1)}, & 0 \le t \le x \le a \\ \dfrac{x(\alpha - 1 - i\lambda a + i\lambda t)(a-t)^{\alpha-2}}{\Gamma(\alpha)(ai\lambda - 1)}, & 0 \le x \le t \le a \end{cases}$$

Proof: The boundary value problem is solved using the given lemma. 1.4 and 1.5 is

$$y(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\big(q(t) - \lambda^2 p(t)\big)y(t)dt + x\frac{1}{\Gamma(\alpha)(ai\lambda-1)}\int_0^a (\alpha - 1 - i\lambda a + i\lambda t)\,(a-t)^{\alpha-2}\big(q(t) - \lambda^2 p(t)\big)y(t)dt$$

$$\to y(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\big(q(t) - \lambda^2 p(t)\big)y(t)dt + x\frac{1}{\Gamma(\alpha)(ai\lambda-1)}\int_0^x (\alpha - 1 - i\lambda a + i\lambda t)\,(a-t)^{\alpha-2}\big(q(t) - \lambda^2 p(t)\big)y(t)dt + x\frac{1}{\Gamma(\alpha)(ai\lambda-1)}\int_x^a (\alpha - 1 - i\lambda a + i\lambda t)\,(a-t)^{\alpha-2}\big(q(t) - \lambda^2 p(t)\big)y(t)dt$$

$$\to y(x) = \int_0^x \left(\frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} + \frac{x(\alpha-1-i\lambda a+i\lambda t)(a-t)^{\alpha-2}}{\Gamma(\alpha)(ai\lambda-1)}\right)\big(q(t) - \lambda^2 p(t)\big)y(t)dt + \int_x^a \frac{x(\alpha-1-i\lambda a+i\lambda t)(a-t)^{\alpha-2}}{\Gamma(\alpha)(ai\lambda-1)}\big(q(t) - \lambda^2 p(t)\big)y(t)dt$$

$$\to\ y(x) = \int_0^a H(x,t)\big(q(t) - \lambda^2 p(t)\big)y(t)dt$$

Definition 1.5 (Contraction). [13]A mapping T: E ⟶ E is called a contraction on E (E,d) is a complete metric space, if there exist a positive real number $h < 1$ such that for all $x, y \in E$

$$d(Tx,Ty) \le hd(x,y) \qquad .$$

Geometrically this means that any points x and y have images that are closer together than those points x and y more precisely, the ratio $d(Tx,Ty)/d(x,y)$ does not exceed a constant h which is strictly less than 1.

Theorem 1.1, (Banach Fixed Point Theorem) [14]
If $T: S \to S$ is a contraction operator defined on a Banach space S then T has a unique fixed point in S.

Theorem 1.2 Existence and Uniqueness Theorem

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

The Fractional Boundary Value Problem (FBVP) given by 1.4 and 1.5 has a unique solution if the condition is hold < 1 ; where $D = \frac{a^{\alpha}M(1+|\lambda^2|)(2\alpha-\alpha^2+1)}{(\alpha-1)(|ai\lambda-1|)\Gamma(\alpha+1)}$

proof:

The operator $\rightarrow Ty(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\left(q(t)-\lambda^2 p(t)\right)y(t)dt + x\omega$

Where

$$\omega = \frac{1}{\Gamma(\alpha)(ai\lambda-1)}\int_0^a (\alpha-1-i\lambda a+i\lambda t)\,(a-t)^{\alpha-2}\left(q(t)-\lambda^2 p(t)\right)y(t)dt$$

So we have

$$|Tu(x)-Tv(x)| = \left|\frac{x}{\Gamma(\alpha)(ai\lambda-1)}\int_0^a(\alpha-1-i\lambda a+i\lambda t)\,(a-t)^{\alpha-2}\left(q(t)-\lambda^2p(t)\right)u(t)dt + \frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}\left(q(t)-\lambda^2p(t)\right)u(t)dt - \frac{x}{\Gamma(\alpha)(ai\lambda-1)}\int_0^a(\alpha-1-i\lambda a+i\lambda t)\,(a-t)^{\alpha-2}\left(q(t)-\lambda^2p(t)\right)v(t)dt - \frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}\left(q(t)-\lambda^2p(t)\right)v(t)dt\right|$$

$$\leq \left|\frac{x}{\Gamma(\alpha)(|ai\lambda-1|)}\int_0^a(\alpha-1-i\lambda a+i\lambda t)\,(a-t)^{\alpha-2}\left(q(t)-\lambda^2p(t)\right)(u(t)-v(t))dt\right| + \left|\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}\left(q(t)-\lambda^2p(t)\right)(u(t)-v(t))dt\right|$$

$$\leq \frac{|x|}{\Gamma(\alpha)(|ai\lambda-1|)}\int_0^a|\alpha-1-i\lambda a+i\lambda t|\,|(a-t)^{\alpha-2}||q(t)-\lambda^2p(t)||u(t)-v(t)|dt + \frac{1}{\Gamma(\alpha)}\int_0^x|(x-t)^{\alpha-1}||q(t)-\lambda^2p(t)||u(t)-v(t)|\,dt$$

$$\leq \frac{a}{\Gamma(\alpha)(|ai\lambda-1|)}\int_0^a|\alpha-1-i\lambda a+i\lambda t|\,|(a-t)^{\alpha-2}||q(t)-\lambda^2p(t)||u(t)-v(t)|dt + \frac{1}{\Gamma(\alpha)}\int_0^a|(a-t)^{\alpha-1}|\left|\left(q(t)-\lambda^2p(t)\right)\right|(|u(t)-v(t)|)dt$$

$$\leq \frac{a}{\Gamma(\alpha)(|ai\lambda-1|)}\int_0^a|\alpha-1-i\lambda a+i\lambda t|\,|(a-t)^{\alpha-2}||q(t)-\lambda^2p(t)||u(t)-v(t)|dt + \frac{1}{\Gamma(\alpha)}\int_0^a|(a-t)^{\alpha-1}|\left|\left(q(t)-\lambda^2p(t)\right)\right|(|u(t)-v(t)|)dt$$

$$\leq \frac{a}{\Gamma(\alpha)(|ai\lambda-1|)}\int_0^a|\alpha-1-i\lambda a+i\lambda t|\,|(a-t)^{\alpha-2}||q(t)-\lambda^2p(t)||u(t)-v(t)|dt + \frac{1}{\Gamma(\alpha)}\int_0^a|(a-t)^{\alpha-1}|\left|\left(q(t)-\lambda^2p(t)\right)\right||u(t)-v(t)|dt$$

$$\leq \frac{a}{\Gamma(\alpha)(|ai\lambda-1|)}\int_0^a|(\alpha-1-i\lambda a+i\lambda t)(a-t)^{\alpha-2}|\,(|q(t)|+|\lambda^2||p(t)|)\|u-v\|dt + \frac{1}{\Gamma(\alpha)}\int_0^a|(a-t)^{\alpha-1}|\,(|q(t)|+\lambda^2|p(t)|)\|u-v\|dt$$

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com
مجلة جامعة بابل للعلوم الصرفة والتطبيقية

$$\leq \frac{a\|u-v\|}{\Gamma(\alpha)(|ai\lambda-1|)}\int_0^a |(\alpha-1-i\lambda a+i\lambda t)(a-t)^{\alpha-2}|\,(M+|\lambda^2|M)dt$$
$$+\frac{\|u-v\|}{\Gamma(\alpha)}\int_0^a |(a-t)^{\alpha-1}|\,(M+|\lambda^2|M)dt \qquad ,since\ q(x),p(x)\in L_+[0,a]$$

$$\leq \frac{M(1+|\lambda^2|)}{\Gamma(\alpha)}\|u-v\|\int_0^a\left|\left(\frac{a(\alpha-1-i\lambda a+i\lambda t)}{(ai\lambda-1)}\right)(a-t)^{\alpha-2}+(a-t)^{\alpha-1}\right|dt$$
$$\leq \frac{M(1+|\lambda^2|)}{\Gamma(\alpha)}\|u-v\|\int_0^a\left|(a-t)^{\alpha-2}\left(\frac{a(\alpha-1-i\lambda a+i\lambda t)}{(ai\lambda-1)}+a-t\right)\right|dt$$
$$\leq \frac{M(1+|\lambda^2|)}{\Gamma(\alpha)}\|u-v\|\int_0^a\left|(a-t)^{\alpha-2}\left(\frac{a(\alpha-2)}{(ai\lambda-1)}+\frac{t}{(ai\lambda-1)}\right)\right|dt$$
$$=\frac{M(1+|\lambda^2|)}{\Gamma(\alpha)}\|u-v\|\left(\frac{a|\alpha-2|}{(|ai\lambda-1|)}\int_0^a|(a-t)^{\alpha-2}|\,dt+\frac{1}{(|ai\lambda-1|)}\int_0^a|t(a-t)^{\alpha-2}|\,dt\right)$$
$$=\frac{M(1+|\lambda^2|)}{\Gamma(\alpha)}\|u-v\|\left(\frac{a|\alpha-2|}{(|ai\lambda-1|)}\int_0^a(a-t)^{\alpha-2}\,dt+\frac{1}{(|ai\lambda-1|)}\int_0^a t(a-t)^{\alpha-2}\,dt\right)$$

We solve the last integral in above equation by partial fraction and we get

$$=\frac{M(1+|\lambda^2|)}{(|ai\lambda-1|)\Gamma(\alpha)}\|u-v\|\left(\frac{a(\alpha-2)}{(\alpha-1)}(a-t)^{\alpha-1}|_0^a-\frac{1}{(\alpha-1)}t(a-t)^{\alpha-1}|_0^a-\frac{1}{\alpha(\alpha-1)}(a-t)^{\alpha}|_0^a\right)$$
$$=\frac{a^\alpha M(1+|\lambda^2|)}{(|ai\lambda-1|)(\alpha-1)\Gamma(\alpha)}\|u-v\|\left(2-\alpha+\frac{1}{\alpha}\right)$$
$$=\frac{a^\alpha M(1+|\lambda^2|)(2\alpha-\alpha^2+1)}{\alpha(\alpha-1)(|ai\lambda-1|)\Gamma(\alpha)}\|u-v\|=\frac{a^\alpha M(1+|\lambda^2|)(2\alpha-\alpha^2+1)}{(\alpha-1)(|ai\lambda-1|)\Gamma(\alpha+1)}\|u-v\|=D\|u-v\|$$

Now $\quad |Tu(x)-Tv(x)|\leq D\|u-v\|_{C[0,a]}$

$\rightarrow max_{x\in[0,a]}|Tu(x)-Tv(x)|\leq max_{x\in[0,a]}D\|u(x)-v(x)\|_{C[0,a]}$

$\|Tu-Tv\|\leq D\|u-v\|_{C[0,a]}$

Where $D=\frac{a^\alpha M(1+|\lambda^2|)(2\alpha-\alpha^2+1)}{(\alpha-1)(|ai\lambda-1|)\Gamma(\alpha+1)}$; $such\ that\ \lambda\neq\frac{1}{ai}$

Since $D<1$ then T Using the Banach Fixed Point Theorem, is a contraction operator. T possesses a singular fixed point, which is the singular solution to the fractional boundary value issue provided by equation 1.4 and 1.5 ∎

Definition 1.6: [15]An operator $T: H\rightarrow H$ is said to be compact if for each bounded sequence $\emptyset_n\in H$, $T(\emptyset_n)$ has a convergent subsequence

Definition 1.7: [15]A sequence function f_n is said to be equicontinuous if $for\ all\ \varepsilon>0\ there\ exist\ \delta>0\ ,and\ for\ all\ x_1,x_2\in D(f_n)$ such that $|x_2-x_1|<\delta$ then $|f_n(x_2)-f_n(x_1)|<\varepsilon$

Theorem1.3: [13] (Arzela Theorem) Every bounded equicontinuous function has a convergent subsequence. (Every bounded equicontinuous Operator is Compact)

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

Theorem 1.4 (Schauder Fixed Point Theorem)[13], [16]

Let B be a nonempty, convex, closed and bounded set in a Banach space E and let $T: B \longrightarrow B$ be a compact operator. Then T has at least one fixed point in B.

Theorem 1.5. If there exist real number $\mathrm{K} > 0$, such that $\mathrm{K} = \frac{F\Gamma(\alpha+1)}{\Gamma(\alpha+1)-a^{\alpha}M(1+|\lambda^2|)}$ And $F = |a\omega|$ Then the fractional boundary value problem given by equations (1.4)-(1.5) has at least one solution.

1.5 Examples

We'll show you some instances in this section

.**Example 1.5.1:** solve the fractional boundary value problem

$$-{}_0^CD_x^{\frac{3}{2}}y(x) + \frac{1}{10}y(x) = \lambda^2\frac{1}{10}y(x) \;; \qquad x \in [0,1]$$

$$y(0) = 0\,, \qquad y'(1) - i\lambda y(1) = 0$$

Solution: we have $M = \frac{1}{10}$

By Existence and uniqueness theorem the above fractional boundary value problem has solution for $\frac{1+|\lambda^2|}{|i\lambda-1|} < \frac{15\sqrt{\pi}}{7}$ because $\frac{a^{\alpha}M(1+|\lambda^2|)(2\alpha-\alpha^2+1)}{(\alpha-1)(|ai\lambda-1|)\Gamma(\alpha+1)} = \frac{1^{\frac{3}{2}}\frac{1}{10}(1+|\lambda^2|)(2.\frac{3}{2}-\frac{9}{4}+1)}{\left(\frac{3}{2}-1\right)(|i\lambda-1|)\Gamma\left(\frac{3}{2}+1\right)} = \frac{\frac{7}{40}(1+|\lambda^2|)}{\frac{1}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)|i\lambda-1|} = \frac{7}{15\sqrt{\pi}}\frac{1+|\lambda^2|}{|i\lambda-1|} < \frac{7}{15\sqrt{\pi}}\,.\frac{15\sqrt{\pi}}{7} = 1$

Now $-{}_0^CD_x^{\frac{3}{2}}y(x) + \frac{1}{10}y(x) = \frac{1}{10}\lambda^2y(x) \rightarrow {}_0^CD_x^{\frac{3}{2}}y(x) = \frac{1}{10}(1-\lambda^2)y(x)$

We prove it by Laplace transformation take Laplace for both sides we get

$L\left\{{}_0^CD_x^{\frac{3}{2}}y(x)\right\} = L\{\frac{1}{10}(1-\lambda^2)y(x)\}$ By properties of Laplace we have

$$L\left\{{}_0^CD_x^{\frac{3}{2}}y(x)\right\} = \left\{s^{\frac{3}{2}}Y(s) - s^{\frac{1}{2}}y(0) - s^{\frac{-1}{2}}y'(0)\right\}$$

Now $s^{\frac{3}{2}}Y(s) - s^{\frac{1}{2}}y(0) - s^{\frac{-1}{2}}y'(0) = \frac{1}{10}(1-\lambda^2)Y(s)$

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

$$\rightarrow \left(s^{\frac{3}{2}} + \frac{1}{10}(\lambda^2 - 1)\right) Y(s) = s^{\frac{-1}{2}} y'(0) \quad \text{, such that } y'(0) \neq 0$$

$$Y(s) = \frac{y'(0) s^{\frac{-1}{2}}}{s^{\frac{3}{2}} + \frac{1}{10}(\lambda^2 - 1)} \rightarrow \; Y(s) = \frac{g s^{\frac{3}{2} - 2}}{s^{\frac{3}{2}} + \frac{1}{10}(\lambda^2 - 1)} \text{, where } g = y'(0) \;,$$

Take Laplace inverse for both sides we get $L^{-1}\big(Y(s)\big) = L^{-1}\left(\frac{g s^{\frac{3}{2}-2}}{s^{\frac{3}{2}} + \frac{1}{10}(\lambda^2 - 1)}\right)$

See [17]for inverse Laplace and related to Mittag-leffler we get

So $y(x) = gxE_{\frac{3}{2},2}\left(\frac{1}{10}(1 - \lambda^2)x^{\frac{3}{2}}\right)$ *,where g is a constant and* $g = y'(0)$

From second condition we have $y'(1) - i\lambda y(1) = 0 \rightarrow y'(1) = i\lambda y(1)$

And the solution is $y(x) = y'(0)xE_{\frac{3}{2},2}\left(\frac{1}{10}(1 - \lambda^2)x^{\frac{3}{2}}\right)$

Now $i\lambda y(1) = \; i\lambda y'(0)E_{\frac{3}{2},2}\left(\frac{1}{10}(1 - \lambda^2)\right)$ and

$y'(1) = y'(0)E_{\frac{3}{2},1}\left(\frac{1}{10}(1 - \lambda^2)x^{\frac{3}{2}}\right)$, from properties of Mittag-leffler derivative

So $y'(0)E_{\frac{3}{2},1}\left(\frac{1}{10}(1 - \lambda^2)\right) = \; i\lambda y'(0)E_{\frac{3}{2},2}\left(\frac{1}{10}(1 - \lambda^2)\right)$

$y'(0) \neq 0$ Since if $y'(0) = 0$ the solution is trivial

$$E_{\frac{3}{2},1}\left(\frac{1}{10}(1 - \lambda^2)\right) = \; i\lambda E_{\frac{3}{2},2}\left(\frac{1}{10}(1 - \lambda^2)\right) \rightarrow E_{\frac{3}{2},1}\left(\frac{1}{10}(1 - \lambda^2)\right) - \; i\lambda E_{\frac{3}{2},2}\left(\frac{1}{10}(1 - \lambda^2)\right) = 0$$

By using definition of Mittag-Leffler and open the summation we can find λ

The above fractional Boundary value Problem has solution

$$y(x) = gxE_{\frac{3}{2},2}\left(\frac{1}{10}(1 - \lambda^2)x^{\frac{3}{2}}\right), \quad \lambda \neq 1, -1, -i \;,$$

Example 1.5.2: solve the fractional boundary value problem

$$-{}_{0}^{C}D_{x}^{\frac{5}{3}} y(x) + \frac{1}{4} y(x) = \frac{1}{4}\lambda^2 y(x); \qquad x \in [0,1]$$

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

JOURNAL OF UNIVERSITY OF BABYLON
For Pure and Applied Sciences (JUBPAS)
Vol.30; No.2.| 2022

$y(0) = 0\,, \qquad y'(1) - i\lambda y(1) = 0$

Solution: we have $M = \frac{1}{4}$

By Existence and uniqueness theorem the above fractional boundary value problem has solution if $\frac{1+|\lambda^2|}{|i\lambda-1|} < \frac{40\Gamma\left(\frac{2}{3}\right)}{21}$ because $\frac{a^{\alpha}M(1+|\lambda^2|)(2\alpha-\alpha^2+1)}{(\alpha-1)(|ai\lambda-1|)\Gamma(\alpha+1)} == \frac{1^{\frac{5}{3}}\frac{1}{4}(1+|\lambda^2|)(2.\frac{5}{3}-\frac{25}{9}+1)}{\left(\frac{5}{3}-1\right)(|i\lambda-1|)\Gamma\left(\frac{5}{3}+1\right)} = \frac{\frac{14}{36}}{\frac{2}{3}.\frac{5}{3}\frac{2}{3}\Gamma\left(\frac{2}{3}\right)}\frac{1+|\lambda^2|}{|i\lambda-1|} <$

$\frac{21}{40\Gamma\left(\frac{2}{3}\right)}.\frac{40\Gamma\left(\frac{2}{3}\right)}{21} = 1$

Now $-{}_0^CD_x^{\frac{5}{3}}y(x) + \frac{1}{4}y(x) = \frac{1}{4}\lambda^2 y(x) \rightarrow {}_0^CD_x^{\frac{5}{3}}y(x) = \frac{1}{4}(1-\lambda^2)y(x)$

See reference [10] page 55 said $D^{\alpha}y(t) = hy(t),\ \textit{where: } n-1 < \alpha < n,$

With the Boundary condition $y^{(k)}(0) = b_k,\ b_k \in R,\ k = 0,1,2,\dots,n-1,$

has the solution: $y(t) = \sum_{k=0}^{n-1} b_k t^k E_{\alpha,k+1}(ht^{\alpha})$

We know $h = \frac{1}{4}(1-\lambda^2), \quad 1 < \alpha \le 2, \quad y(0) = b_0 = 0\,, \quad y'(0) = b_1$

So the solution is $y(x) = \sum_{k=0}^{1} \frac{1}{4} b_k x^k E_{\frac{5}{3},k+1}\left(\frac{1}{4}(1-\lambda^2)x^{\frac{5}{3}}\right)$

$y(x) = \sum_{k=0}^{1} b_k x^k E_{\frac{5}{3},k+1}\left(\frac{1}{4}(1-\lambda^2)x^{\frac{5}{3}}\right) = b_0 E_{\frac{5}{3},1}\left(\frac{1}{4}(1-\lambda^2)x^{\frac{5}{3}}\right) + b_1 x^1 E_{\frac{5}{3},2}\left(\frac{1}{4}(1-\lambda^2)x^{\frac{5}{3}}\right) =$

$b_1 x E_{\frac{5}{3},2}\left(\frac{1}{4}(1-\lambda^2)x^{\frac{5}{3}}\right)\ ,\lambda \neq 1,-1,-i$

Example 1.5.3: solve the fractional boundary value problem

$-{}_0^CD_x^{\frac{4}{3}}y(x) + \frac{1}{8}y(x) = \frac{1}{8}\lambda^2 y(x); \qquad x \in [0,1]$

$y(0) = 0\,, \qquad y'(1) - \lambda i y(1) = 0$

Solution: we have $M = \frac{1}{8}$

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

By Existence and uniqueness theorem the above fractional boundary value problem has solution

if $\frac{1+|\lambda^2|}{|i\lambda-1|} < \frac{32\Gamma\left(\frac{1}{3}\right)}{51}$ because

$$\frac{a^{\alpha}M\left(1+|\lambda^2|\right)(2\alpha-\alpha^2+1)}{(\alpha-1)(|ai\lambda-1|)\Gamma(\alpha+1)} = \frac{1^{\frac{4}{3}}.\frac{1}{8}(1+|\lambda^2|)\frac{17}{9}}{\frac{1}{3}.\frac{4}{3}.\frac{1}{3}\Gamma\left(\frac{1}{3}\right)|ai\lambda-1|} = \frac{\frac{17}{72}}{\frac{4}{27}\Gamma\left(\frac{1}{3}\right)}\frac{1+|\lambda^2|}{|i\lambda-1|} < \frac{51}{32\Gamma\left(\frac{1}{3}\right)}\frac{32\Gamma\left(\frac{1}{3}\right)}{51} = 1$$

Now $-{}_0^C D_x^{\frac{4}{3}} y(x) + \frac{1}{8}y(x) = \frac{1}{8}\lambda^2 y(x) \rightarrow {}_0^C D_x^{\frac{4}{3}} y(x) = \frac{1}{8}(1-\lambda^2)y(x)$

Take fractional integral for order $\frac{4}{3}$ for both sides we get

$$I^{\frac{4}{3}}\left\{{}_0^C D_x^{\frac{4}{3}} y(x)\right\} = \frac{1}{8}(1-\lambda^2)I^{\frac{4}{3}}\{y(x)\}$$

By **lemma 1.3** implies that $y(x) + c_0 x + c_1 = \frac{1}{8}(1-\lambda^2)I^{\frac{4}{3}}\{y(x)\}$

Now take Laplace transformation for both sides we get

$$Y(s) + \frac{c_0}{s^2} + \frac{c_1}{s} = \frac{1}{8}(1-\lambda^2)L\{I^{\frac{4}{3}}\{y(x)\}\}$$

By Convolution theorem see [10], [18]and [11]

We have $Y(s) + \frac{c_0}{s^2} + \frac{c_1}{s} = \frac{1}{8}\frac{(1-\lambda^2)Y(s)}{s^{\frac{4}{3}}}$

$$s^{\frac{4}{3}}Y(s) + \frac{1}{8}(\lambda^2-1)Y(s) = -\frac{c_0}{s^{\frac{2}{3}}} - \frac{c_1}{s^{-\frac{1}{3}}}$$

$$Y(s)\left(s^{\frac{4}{3}} + \frac{1}{8}(\lambda^2-1)\right) = -c_0 s^{-\frac{2}{3}} - c_1 s^{\frac{1}{3}}$$

$$Y(s) = -\frac{c_0 s^{-\frac{2}{3}}}{s^{\frac{4}{3}}+\frac{1}{8}(\lambda^2-1)} - \frac{c_1 s^{\frac{1}{3}}}{s^{\frac{4}{3}}+\frac{1}{8}(\lambda^2-1)}$$

Take inverse Laplace for both sides

$$L^{-1}\{Y(s)\} = -c_0 L^{-1}\left\{\frac{s^{\frac{4}{3}-2}}{s^{\frac{4}{3}}+\frac{1}{8}(\lambda^2-1)}\right\} - c_1 L^{-1}\left\{\frac{s^{\frac{4}{3}-1}}{s^{\frac{4}{3}}+\frac{1}{8}(\lambda^2-1)}\right\}$$

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

See [17]for inverse Laplace and related to Mittag-leffler we get

$$y(x) = -c_0 x E_{\frac{4}{3},2}\frac{1}{8}\left((1-\lambda^2)x^{\frac{4}{3}}\right) - c_1 E_{\frac{4}{3},1}\left(\frac{1}{8}(1-\lambda^2)x^{\frac{4}{3}}\right)$$

Example 1.5.4: solve the fractional boundary value problem

$$-{}_0^C D_x^{\frac{4}{3}} y(x) + \frac{1}{9}y(x) = \frac{1}{9}\lambda^2 y(x); \qquad x \in [0,1]$$

$$y(0) = 0\,, \qquad y'(1) - \lambda i y(1) = 0$$

Solution: we have $M = \frac{1}{9}$

By Existence and uniqueness theorem the above fractional boundary value problem has solution if

$\frac{1+|\lambda^2|}{|i\lambda-1|} < \frac{12\Gamma\left(\frac{1}{3}\right)}{17}$ because $\frac{a^{\alpha}M\left(1+|\lambda^2|\right)(2\alpha-\alpha^2+1)}{(\alpha-1)(|ai\lambda-1|)\Gamma(\alpha+1)} = \frac{1^{\frac{4}{3}}.\frac{1}{9}\left(1+|\lambda^2|\right)\frac{17}{9}}{\frac{1}{3}.\frac{4}{3}.\frac{1}{3}\Gamma\left(\frac{1}{3}\right)|ai\lambda-1|} = \frac{\frac{17}{9.9}}{\frac{4}{27}\Gamma\left(\frac{1}{3}\right)}\frac{1+|\lambda^2|}{|i\lambda-1|} < \frac{17}{12\Gamma\left(\frac{1}{3}\right)}\frac{12\Gamma\left(\frac{1}{3}\right)}{17} = 1$

Now $-{}_0^C D_x^{\frac{4}{3}} y(x) + \frac{1}{9}y(x) = \frac{1}{9}\lambda^2 y(x) \rightarrow {}_0^C D_x^{\frac{4}{3}} y(x) = \frac{1}{9}(1-\lambda^2)y(x)$

Take Laplace Transformation for both sides we get

$$L\left\{{}_0^C D_x^{\frac{4}{3}} y(x)\right\} = \left\{s^{\frac{4}{3}}Y(s) - s^{\frac{1}{3}}y(0) - s^{\frac{-2}{3}}y'(0)\right\}$$

Implies that $s^{\frac{4}{3}}Y(s) - s^{\frac{1}{3}}y(0) - s^{\frac{-2}{3}}y'(0) = \frac{1}{9}(1-\lambda^2)Y(s)$

$$s^{\frac{4}{3}}Y(s) - s^{\frac{-2}{3}}y'(0) = \frac{1}{9}(1-\lambda^2)Y(s)$$

$$s^{\frac{4}{3}}Y(s) - \frac{1}{9}(1-\lambda^2)Y(s) = s^{\frac{-2}{3}}y'(0)$$

$$Y(s) = \frac{k}{s^{\frac{2}{3}}\left(s^{\frac{4}{3}} - \frac{1}{9}(1-\lambda^2)\right)} \rightarrow Y(s) = \frac{k}{s^2\left(1-\frac{1}{9}(1-\lambda^2)s^{\frac{-4}{3}}\right)} \;\; ; \;\; k = y'(0)$$

$$Y(s) = \frac{1}{s^2}\frac{k}{1-\frac{\frac{1}{9}(1-\lambda^2)}{s^{\frac{4}{3}}}}$$

By using the geometric series $\frac{1}{1-u} = 1 + u + u^2 + u^3 + \cdots$; $|u| < 1$, We obtain that:

$$Y(s) = \frac{1}{s^2}\frac{k}{1-\frac{\frac{1}{9}(1-\lambda^2)}{s^{\frac{4}{3}}}} = \frac{k}{s^2}\left(1 + \frac{\frac{1}{9}(1-\lambda^2)}{s^{\frac{4}{3}}} + \frac{\frac{1}{81}(1-\lambda^2)^2}{s^{\frac{8}{3}}} - \cdots\right) = \frac{k}{s^2} + \frac{\frac{1}{9}(1-\lambda^2)k}{s^{\frac{10}{3}}} + \frac{\frac{1}{81}(1-\lambda^2)^2 k}{s^{\frac{14}{3}}} - \cdots$$

$$y(x) = \frac{kx}{\Gamma(2)} + \frac{\frac{1}{9}(1-\lambda^2)kx^{\frac{7}{3}}}{\Gamma\left(\frac{10}{3}\right)} + \frac{\frac{1}{81}(1-\lambda^2)^2 kx^{\frac{11}{3}}}{\Gamma\left(\frac{14}{3}\right)} - \cdots \qquad ,; \;\; k = y'(0)$$

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

JOURNAL OF UNIVERSITY OF BABYLON
For Pure and Applied Sciences (JUBPAS)
Vol.30; No.2.| 2022

Example 1.5.5: solve the fractional boundary value problem

$$-{}_0^CD_x^{\frac{3}{2}}y(x) - 0.4x^{\frac{3}{2}}y(x) = \lambda^2 0.4x^{\frac{3}{2}}y(x); \qquad x \in [0,1]\,, \alpha \in (1,2]$$

$$y(0) = 0\,, \qquad y'(1) - i\lambda y(1) = 0.$$

Solution: we have $M = 0.4$

By Existence and uniqueness theorem the above fractional boundary value problem has solution

if $\frac{1+|\lambda^2|}{|i\lambda-1|} < \frac{15\sqrt{\pi}}{28}$ because $\frac{a^{\alpha}M(1+|\lambda^2|)(2\alpha-\alpha^2+1)}{(\alpha-1)(|ai\lambda-1|)\Gamma(\alpha+1)} = \frac{1^{\frac{3}{2}}.(0.4)(\frac{7}{4})}{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\frac{1+|\lambda^2|}{|i\lambda-1|} < \frac{28}{15\sqrt{\pi}}\frac{15\sqrt{\pi}}{28} = 1$

Now $-{}_0^CD_x^{\frac{3}{2}}y(x) - 0.4x^{\frac{3}{2}}y(x) = \lambda^2 0.4x^{\frac{3}{2}}y(x) \rightarrow {}_0^CD_x^{\frac{3}{2}}y(x) + 0.4(1+\lambda^2)x^{\frac{3}{2}}y(x) = 0$

Now by power series method for solving Fractional derivative for fractional order α

See reference [19]said $y(x) = (x - x_0)^{\alpha-1}\sum_{n=0}^{\infty} a_n(x - x_0)^{n\alpha}, (a_n \in R)$

For $x \in (x_0, x_0 + \rho)$ and a_0 is the Boundary condition. its solution for the fractional derivative equation $D^{\alpha}y(x) + p(x)y(x) = 0$,

Now we have ${}_0^CD_x^{\frac{3}{2}}y(x) + 0.4(1+\lambda^2)x^{\frac{3}{2}}y(x) = 0$

Around $x_0 = 0$

The solution is $y(x) = 0.4(1+\lambda^2)(x)^{\frac{3}{2}-1}\sum_{n=1}^{\infty} a_n(0.4(1+\lambda^2)x)^{n\frac{3}{2}}, \qquad (a_n \in R)$

$\rightarrow y(x) = 0.4(1+\lambda^2)(x)^{\frac{1}{2}}\sum_{n=1}^{\infty} a_n(0.4(1+\lambda^2)x)^{n\frac{3}{2}}, \qquad (a_n \in R)$

If we want find the coefficients a_n we can in put $y(x)$ in the above equation and find the coefficients a_n.

Conclusion

In this work, Regge problem of fractional order has been studied. The Banach fixed point theorem (contraction mapping theorem) and Schauder fixed point theorem has been applied to Regge problem of fractional order. The results obtain that the Regge problem in 1.4 and 1.5 has unique solution under a condition. The condition has been obtained with the operator that we defined on the problem.

Acknowledgments: We appreciate the revisionary's helpful remarks and suggestions for improving our manuscript.

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

Conflict of interests.

There are non-conflicts of interest.

References

[1] T. Regge, "Analytic properties of the scattering matrix," *Nuovo Cim.*, vol. 8, no. 5, pp. 671–679, 1958, doi: 10.1007/BF02815247.

[2] G. A. Aigunov, K. H. F. Jwamer, and G. A. Dzhalaeva, "Estimates for the eigenfunctions of the Regge problem," *Math. Notes*, vol. 92, no. 1–2, pp. 132–135, 2012, doi: 10.1134/S0001434612070140.

[3] K. H. F. Jwamer and R. R. Q. Rasul, "Behavior of the eigenvalues and eignfunctions of the regge-type problem," *Symmetry (Basel).*, vol. 13, no. 1, pp. 1–11, 2021, doi: 10.3390/sym13010139.

[4] K. B. Oldham and Jerome Spanier, *The Fractional Calculus Theory and Applications of Differentiation and Integration to Arbitrary Order*. New York: Academic press ,INC, 1974.

[5] J. T. Machado, V. Kiryakova, and F. Mainardi, "Recent history of fractional calculus," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 16, no. 3, pp. 1140–1153, 2011, doi: 10.1016/j.cnsns.2010.05.027.

[6] K. S. Miller and B. Ross, "An Introduction to The Fractional Calculus and Fractional Differential Equations," *John-Wily and Sons*. Wiley-Inter Science, New York, p. 9144, 1993.

[7] I. Podlubny, *Fractional Differential Equations*. San Diego: Elsevier, 1999.

[8] A. Carpinter and F. Mainardi, *Fractals and Fractional Calculus in Continuum Mechanics*. Springer-Verlag Wien GmbH, 1997.

[9] R. Gorenflo and F. Mainardi, "Essentials of Fractional Calculus," *MaPhySto Cent.*, p. 33, 2000.

[10] C. Milici, G. Draganescu, and J.Tenreiro Machado, *Fractional Differential Equations. Introduction to Fractional Differential Equations*, vol. 25. Switzerland: Springer, 2019.

[11] Murray R.Spiegel, *Schaums's -Laplace Transforms*. United states: Mcgraw Hill, 1965.

[12] B. H. E. Ibrahim, Q. Dong, and Z. Fan, "Existence for boundary value problems of two-term Caputo fractional differential equations," *J. Nonlinear Sci. Appl.*, vol. 10, no. 02, pp. 511–520, 2017, doi: 10.22436/jnsa.010.02.16.

[13] Erwin Kreyszig, *Introductory Functional Analysis with Applications*, vol. 46, no. 1. 1989.

[14] R. M. Brooks and K. Schmitt, "The contraction mapping principle and some applications," pp. 1–90, 2009.

[15] M. Fabian, P. Habala, P. Hajek, V. M. Santalucia, J. Pelant, and V. Zizler, "Functional Analysis and Infinite-Dimensional Geometry." Springer, New York, p. 460, 2001.

[16] C. Corduneanu, *Integral Equations and Applications*. New York, 1991.

[17] S. Kazem, "Exact Solution of Some Linear Fractional Differential Equations by Laplace Transform," *Int. J. Nonlinear Sci.*, vol. 16, no. 1, pp. 3–11, 2013.

[18] I. Podlubny, "Geometric and Physical Interpretation of Fractional Integration and Fractional Defferentiation," *Fract. Calc. Appl. Anal.*, vol. 5, no. 4, pp. 199–255, 2002, doi: 10.1007/978-3-642-33911-0_4.

[19] D. Salim, "Solutions of Ordinary Homogenous Fractional Order Differential Equations with Variable Coefficients," 2008.

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com