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# Characteristics of Eigenfunctions and Eigenvalues of a Fourth Order Differential Equations with Spectral Parameter in the Boundary Conditions

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# خصائص الدوال الذاتية والقيم الذاتية للمعادلات التفاضلية من الرتبة الرابعة مع المعلمة الطيفية في الشروط الحدودية

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## ABSTRACT

The characteristics of eigenvalues, and their positions on the real axis are discussed in this article for a fourth order eigenvalue problem with a spectral parameter in the boundary conditions. We have demonstrated that our described problem has a unique solution, as well as, it is shown the characteristics of eigenfunctions and their derivatives in the space $L_p(0,\eta)$ $(1 < p < \infty)$.

Key words:

Eigenvalues, eigenfunctions, spectral parameter, boundary conditions, fourth order.

## الخلاصة

في هذا البحث تمت مناقشة خصائص القيم الذاتية ومواقعها على المحور الحقيقي لمسألة القيمة الحدودية من الرتبة الرابعة مع المعلمة الطيفية في الشروط الحدودية . لقد أثبتنا أن مشكلتنا الموصوفة لها حل وحيد ، و إضافة إلى ذلك ، عرضنا خصائص الدوال الذاتية ومشتقاتها في الفضاء $L_p(0,\eta)$ $(1 < p < \infty)$.

**الكلمات المفتاحية:**

القيم الذاتية، الدوال الذاتية، المعلمة الطيفية، الشروط الحدودية، الرتبة الرابعة.

## INTRODUCTION

The boundary value problems have attracted much attention and are considered one of the important equations since it is used in different wide fields [1]. Boundary value problems are used in applied mathematics, several branches of physics, engineering and chemistry [2]-[3]. The boundary-value problems for ordinary differential operators with spectral parameter in boundary conditions are considered by several researchers for instance [4]-[6].The basis properties of systems for the second order Sturm -Liouville problem of root functions in the space Lp, $1 < p < \infty$ with spectral parameter in one of the boundary condition are studied in [5]-[9]. In [4]-[10]. the basic properties of systems for the fourth order Sturm–Liouville problem with spectral parameter in one of the boundary conditions are investigated, They also study the properties of the second order Sturm–Liouville problem with spectral parameter in both boundary conditions. Aryan found the general solution for the described linear ordinary differential equation of order $n$, he also further identified the boundedness of the eigenfunctions and asymptotic behavior of eigenvalues of the specified 2nd order boundary value problem [11].

Aryan looked for linear independent solutions as well as asymptotic formulas for eigenvalues in both regular and irregular cases for the assigned spectral boundary problem with a spectral parameter in the boundary conditions [12].

In this paper, we mention the features of eigenvalues with identifying their places on the real axis with a spectral parameter in the boundary conditions, in addition the properties of eigenfunctions $z(t)$ and their derivatives in the space $L_p(0,\eta)$ $(1 < p < \infty)$, for a fourth-order eigenvalue problem:

$$[(z''(t))' - (p(t)\, z'(t))]' = \Omega z(t), \qquad 0 < t < \eta \tag{1}$$

$$z\ (0) = 0, \tag{2}$$

$$z''(0) = 0, \tag{3}$$

$$z''\ (\eta) - a\Omega z'(\eta) = 0, \tag{4}$$

$$\big((z'')' - p(t) z'\ \big)_{(\eta)} - b\Omega z(\eta) = 0, \tag{5}$$

where $z(t)$ is eigenfunction, $\Omega$ is a spectral parameter, and $\Omega \in A \equiv (\bigcup_{k=1}^{\infty} A_k) \cup (C \setminus R)$, where $A_k = (\Omega_{k-1}(0), \Omega_k(0)), k \in N$, p is a positive absolutely continuous function on the interval $[0,\eta]$, $\eta$ is finite, a and b are real constant, $a < 0$, $b < 0$ and $F(\Omega) = z''(\eta,\Omega)/z'(\eta,\Omega)$

## 2. Some supplementary facts and theorems

Inserting the boundary condition

$$z'(\eta) \cos\beta + z''(\eta) \sin\beta = 0, \quad \text{where } \beta \in [0,\pi] \tag{6}$$

**Lemma 2.1:** [13,§2, Lemma 2.1] Consider $z(t,\Omega)$ is a nontrivial solution of equation (1) for $\Omega > 0$. If $z, z', z'', Sz$ are zero or positive and at $t = a$ not all equal zero, then they are greater than zero for $t > a$. If $z, -z', z'', -Sz$ are zero or possitive and at $t = a$ not all equal zero, then they are greater than zero for $t < a$.

**Theorem 2.1:** [14,§5, Theorem 5.1]. The eigenvalues of the problem (1), (2),(3) (5), (6) for $\beta \in \left[0, \frac{\pi}{2}\right]$ are simple, real and making a sequence that increase infinitely $\{\Omega_n(\beta)\}_{n=1}^{\infty}$, and in addition,

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$\Omega_n(\beta) > 0, n \in N$. Furthermore, $z_n^{(\beta)}(t), n \in N$ the eigenfunction, correspondes to the eigenvalue $\Omega_n(\beta)$ having exactly $n-1$ simple zeros in the interval of $(0,1)$.

## 3. Features of eigenfunctions and eigenvalues of the given problem

**Theorem 3.1:** There is a nonzero solution $z(t,\Omega)$ of the problem (1), (2), (3), (5) for each fixed $\Omega \in C$ that is unique up to a constant coefficient.

**Proof:** Let $\varphi_k(t,\Omega), k = \overline{1,4}$, be the solutions of equation (1), normalized for $t = 0$ by the Cauchy conditions

$$\varphi_k^{(s-1)}(0,\Omega) = \delta_{ks}, \quad s = \overline{1,3}, S\varphi_k(0,\Omega) = \delta_{k4}, \quad \text{where} \quad Sz = z''' - pz' \tag{7}$$

Where $\delta_{ks}$ is the Kronecker delta.
Seeking the function $z(t,\Omega)$ in the form bellow

$$z(t,\Omega) = \sum_{k=1}^{4} C_k\varphi_k(t,\Omega), \tag{8}$$

where the $C_k, k = \overline{1,4}$, are constant.

From the (7 ) , (8 ) and the boundary condition (2),(3),(5) we get $C_1 = C_3 = 0$

So, we have $z(t,\Omega) = C_2\varphi_2(t,\Omega) + C_4\varphi_4(t,\Omega)$. (9)

Put (9) in boundary condition (5) and let $Sz = z''' - pz'$ we get

$$S(C_2\varphi_2(\eta,\Omega) + C_4\varphi_4(\eta,\Omega)) - b\Omega(C_2\varphi_2(\eta,\Omega) + C_4\varphi_4(\eta,\Omega)) = 0,$$

or

$$C_2(S\varphi_2(\eta,\Omega) - \mathrm{b}\Omega\varphi_2(\eta,\Omega)) + C_4(S\varphi_4(\eta,\Omega) - \mathrm{b}\Omega\varphi_4(\eta,\Omega)) = 0.$$

To complete this theorem's proof it we need to show that

$$|S\varphi_2(\eta,\Omega) - \mathrm{b}\Omega\varphi_2(\eta,\Omega)| + |S\varphi_4(\eta,\Omega) - \mathrm{b}\Omega\varphi_4(\eta,\Omega)| > 0. \tag{10}$$

From lemma ( 2.1 ) it is following that $\varphi_k(\eta,\Omega) > 0, S\varphi_k(\eta,\Omega) > 0, k = \overline{1,4}$, for $\Omega > 0$ so by

$\mathrm{b} < 0$, (10) true for $\Omega > 0$

Let $\Omega \in C \setminus (0,+\infty)$. For such $\Omega$, (10) that is not fulfilled, then we have $\varphi_2(t,\Omega)$ $\varphi_4(t,\Omega)$ solutions of the problem (1), (2), (3), (5). Specifying $v(t,\Omega)$ in the following form:

$v(t,\Omega) = \varphi_4'(\eta,\Omega)\varphi_3(t,\Omega) - \varphi_3'(\eta,\Omega)\varphi_4(t,\Omega).$

Since $v'(\eta,\Omega) = 0$, $v(t,\Omega)$ is an eigenfunction of the problem (1),(2),(3),(5) and (6) for $\beta = 0$ corresponding to the eigenvalue $\Omega \in C \setminus (0,+\infty)$.

From theorem (2.1) having that $\Omega > 0$ that is contradictions of the relation $\Omega \in C \setminus (0,+\infty)$. ■

**Theorem 3.2:** If $\Omega$ is a spectral eigenvalue and $z(t,\Omega)$ be a corresponding eigenfunction of the problem (1)-(5) and $F(\Omega) = z''(\eta,\Omega)/z'(\eta,\Omega)$ then

$$\frac{dF(\Omega)}{d\Omega} = -\frac{1}{z'^2(\eta,\Omega)}\left\{\int_0^\eta z^2(t,\Omega)dt - \mathrm{b}\, z^2(\eta,\Omega)\right\}, \qquad \Omega \in A. \qquad (11)$$

**Proof:** Let $Sz = z''' - pz'$, so equation (1) becomes

$(Sz(t,\Omega))' = \Omega z(t,\Omega), \textit{and by same way we have } (Sz(t,\mu))' = \mu\, z(t,\mu).$

Multiply last two equations by $z(t,\mu)$ and $z(t,\Omega)$ and integrate the resulting equation from 0 to $\eta$, yields

$$\int_0^\eta (Sz(t,\mu))'z(t,\Omega)\,dt - \int_0^\eta (Sz(t,\Omega))'z(t,\mu)\,dt = (\mu-\Omega)\int_0^\eta z(t,\mu)z(t,\Omega)\,dt. \qquad (12)$$

By integrating $\int_0^\eta (Sz(t,\mu))'z(t,\Omega)\,dt$ and $\int_0^\eta (Sz(t,\Omega))'z(t,\mu)\,dt$ by parts and using the stated boundary condition, we obtain

$$\int_0^\eta (Sz(t,\mu))'z(t,\Omega)\,dt - \int_0^\eta (Sz(t,\Omega))'z(t,\mu)\,dt = (\mu-\Omega)[b\, z(\eta,\mu)\, z(\eta,\Omega)] +$$

$$\int_0^\eta [Sz(t,\Omega)z'(t,\mu) - Sz(t,\mu)z'(t,\Omega)]\,dt,$$

$\textit{since } Sz(t,\Omega) = z'''(t,\Omega) - pz'(t,\Omega) \textit{ and } Sz(t,\mu) = z'''(t,\mu) - pz'(t,\mu)$, then

$$\int_0^\eta (Sz(t,\mu))'z(t,\Omega)\,dt - \int_0^\eta (Sz(t,\Omega))'z(t,\mu)\,dt = (\mu-\Omega)[b\, z(\eta,\mu)\, z(\eta,\Omega)] +$$

$$\int_0^\eta [z'''(t,\Omega)z'(t,\mu) - z'''(t,\mu)z'(t,\Omega)]\,dt. \qquad (13)$$

Utilizing the formula of integration by parts for the resulting integration in this equation and using the boundary condition (3), we get

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$$\int_0^{\eta} (Sz(t,\mu))' z(t,\Omega)\,dt - \int_0^{\eta} (Sz(t,\Omega))' z(t,\mu)\,dt = (\mu-\Omega)[b\,z(\eta,\mu)\,z(\eta,\Omega)] +$$

$$z'(\eta,\mu)z''(\eta,\Omega) - z'(\eta,\Omega)z''(\eta,\mu). \qquad (14)$$

Therefore equation (12) reduces to

$$(\mu-\Omega)[b\,z(\eta,\mu)\,z(\eta,\Omega)] + z'(\eta,\mu)z''(\eta,\Omega) - z'(\eta,\Omega)z''(\eta,\mu)$$

$$= (\mu-\Omega)\int_0^{\eta} z(t,\mu)z(t,\Omega)\,dt,$$

or

$$z'(\eta,\mu)z''(\eta,\Omega) - z'(\eta,\Omega)z''(\eta,\mu)$$

$$= (\mu-\Omega)\left(\int_0^{\eta} z(t,\mu)z(t,\Omega)\,dt - b\,z(\eta,\mu)\,z(\eta,\Omega)\right). \qquad (15)$$

Applying the limit as $\mu \to \Omega$ and dividing both sides of equation (15) by $z'(\eta,\Omega)z'(\eta,\mu)$ obtains

$$\frac{dF(\Omega)}{d\Omega} = -\frac{1}{z'^2(\eta,\Omega)}\left\{\int_0^{\eta} z^2(t,\Omega)dt - \mathrm{b}z^2(\eta,\Omega)\right\}.$$

**Theorem 3.3:** In the problem (1)-(5) the eigenvalues are all real.

**Proof:** suppose that $\Omega$ is not real then $\bar{\Omega}$ is a eigenvalue of the stated problem. Since the coefficients $p(t), a, b$ are real, then $z(t,\bar{\Omega}) = \overline{z(t,\Omega)}$.

Settings $\mu = \bar{\Omega}$, in equation (15) we get

$$-\overline{z''(\eta,\Omega)}\,z'(\eta,\Omega) + z''(\eta,\Omega)\overline{z'(\eta,\Omega)} = (\bar{\Omega}-\Omega)\left(\left\{\int_0^{\eta} \overline{z(t,\Omega)}z(t,\Omega)\,dt - b\,\overline{z(\eta,\Omega)}\,z(\eta,\Omega)\right),\right.$$

or

$$-\overline{z''(\eta,\Omega)}\,z'(\eta,\Omega) + z''(\eta,\Omega)\overline{z'(\eta,\Omega)} = (\bar{\Omega}-\Omega)\left\{\int_0^{\eta} |z(t,\Omega)|^2 dt - b|z(\eta,\Omega)|^2\right\}. \qquad (16)$$

By virtue of the specified boundary condition, we gain

$$-a(\bar{\Omega}-\Omega)|z'(\eta,\Omega)|^2 = (\bar{\Omega}-\Omega)\left\{\int_0^{\eta} |z(t,\Omega)|^2 dt - b|z(\eta,\Omega)|^2\right\}.$$



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Because $\bar{\Omega} \neq \Omega$, we have

$$\int_0^\eta |z(t,\Omega)|^2 dt + a|z'(\eta,\Omega)|^2 - b|z(\eta,\Omega)|^2 = 0\,. \tag{17}$$

Furthermore taking both sides of equation (1) and multiplying them by $\overline{z(t,\Omega)}$ and integrating from 0 to l gets

$$\int_0^\eta z^{(4)}(t,\Omega)\overline{z(t,\Omega)}dt - \int_0^\eta \left(P(t)z'(t,\Omega)\right)' \overline{z(t,\Omega)}dt = \Omega \int_0^\eta |z(t,\Omega)|^2 dt. \tag{18}$$

Taking the integration of the resulting equation by parts and taking into consideration the boundary conditions, we gain

$$\int_0^\eta |z''(t,\Omega)|^2 dt + \int_0^\eta p(t)\,|z'(t,\Omega)|^2 dt + \overline{z\,(\eta,\Omega)}\left(z'''(\eta,\Omega) - P(\eta)z'(\eta,\Omega)\right)$$

$$-a\Omega z'(\eta)\overline{z^{'}(\eta,\Omega)} = \Omega \int_0^\eta |z(t,\Omega)|^2 dt.$$

From boundary condition (5) we get

$$\int_0^\eta |z''(t,\Omega)|^2 dt + \int_0^\eta p(t)\,|z'(t,\Omega)|^2 dt + b\Omega z(\eta)\overline{z\,(\eta,\Omega)} - a\Omega|z'(\eta,\Omega)|^2 = \Omega \int_0^\eta |z(t,\Omega)|^2 dt$$

Or

$$\int_0^\eta |z''(t,\Omega)|^2 dt + \int_0^\eta p(t)\,|z'(t,\Omega)|^2 dt = \Omega\left(\int_0^\eta |z(t,\Omega)|^2 + a\Omega|z'(\eta,\Omega)|^2 - b|z(\eta,\Omega)|^2\right). \tag{19}$$

From equations (17) and (19) we get

$$\int_0^\eta |z''(t,\Omega)|^2 dt + \int_0^\eta p(t)\,|z'(t,\Omega)|^2 dt = 0, \tag{20}$$

gives $z\,(t,\Omega) = 0$ which is a contradiction. So the given problem's eigenvalues are real.

**Theorem 3.4:** Problems (1)-(5) have simple eigenvalues that form a countable set with no finite limit point.

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**Proof:** for non-real $\Omega$, the whole function of equation (4) s left-hand side does not equal to zero. As a result, it does not dissolve in the same way. As a result, the zeros make accountable set with infinite limit point.

Let us demonstrate that equation (4) is having only simple roots.

In fact if $\Omega = \tilde{\Omega}$ is a multiple root of (4), then

$$z''(\eta,\tilde{\Omega}) - a\,\tilde{\Omega} z'(\eta,\tilde{\Omega}) = 0, \tag{21}$$

$$\frac{\partial z''(\eta,\tilde{\Omega})}{\partial \Omega} - a\frac{\partial}{\partial \Omega}[\tilde{\Omega}\, y'(\eta,\tilde{\Omega})] = 0,$$

$$\frac{\partial z''(\eta,\bar{\Omega})}{\partial \Omega} - a\, z'(\eta,\tilde{\Omega}) - a\,\tilde{\Omega}\frac{\partial z'(\eta,\bar{\Omega})}{\partial \Omega} = 0. \tag{22}$$

We obtain by multiplying equation (22) by $\frac{1}{\mu-\Omega}$ and going the limit as $\mu \to \Omega$

$$-\frac{\partial z''(\eta,\Omega)}{\partial \Omega} z'(\eta,\Omega) + z''(\eta,\Omega)\frac{\partial z'(\eta,\Omega)}{\partial \Omega} = \int_0^\eta z^2(t,\Omega)dt - \mathrm{b}z^2(\eta,\Omega) \tag{23}$$

After we insert $\Omega = \tilde{\Omega}$ in (23) we get

$$-\frac{\partial z''(\eta,\tilde{\Omega})}{\partial \tilde{\Omega}} z'(\eta,\tilde{\Omega}) + z''(\eta,\tilde{\Omega})\frac{\partial z'(\eta,\tilde{\Omega})}{\partial \tilde{\Omega}} = \int_0^\eta z^2(t,\tilde{\Omega})dt - \mathrm{b}z^2(\eta,\tilde{\Omega}). \tag{24}$$

From equations (21), (22) and (24) we get

$$z'(\eta,\tilde{\Omega})\left(-\mathrm{a}z'(\eta,\tilde{\Omega})\right) = \int_0^\eta z^2(t,\tilde{\Omega})dt - \mathrm{b}z^2(\eta,\tilde{\Omega}),$$

$$\int_0^\eta z^2(t,\tilde{\Omega})dt - \mathrm{b}z^2(\eta,\tilde{\Omega}) + a\, z'^2(\eta,\tilde{\Omega}) = 0\,. \tag{25}$$

Further since $\tilde{\Omega}$ is a real eigenvalue reduce from (19) that

$$\int_0^\eta z''^2(t,\tilde{\Omega})dt + \int_0^\eta p(t)z'^2(t,\tilde{\Omega})dt = \tilde{\Omega}\left(\int_0^\eta z^2(t,\tilde{\Omega})dt - \mathrm{b}z^2(\eta,\tilde{\Omega}) + a\, z'^2(\eta,\tilde{\Omega})\right). \tag{26}$$

From (25) and (26) we acquire



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$$\int_0^\eta z''^2(t,\tilde{\Omega})dt + \int_0^\eta p(t)z'^2(t,\tilde{\Omega})dt = 0. \quad (27)$$

This indicates that $z(t,\tilde{\Omega}) \equiv 0$. The proof of this theorem is finished by contradiction that results.

**Lemma 3.1**: let $F(\Omega) = \frac{1}{M(\eta,\Omega)} = \frac{z''(\eta,\Omega)}{z'(\eta,\Omega)}$, (28)

then the following formula holds

$$\frac{\partial M(\eta,\Omega)}{\partial\Omega} = \frac{1}{z''^2(\eta,\Omega)}\left\{\int_0^\eta z^2(t,\Omega)dt - \mathrm{b}z^2(\eta,\Omega)\right\}. \quad (29)$$

**Proof:** by differentiating (28) with respect to $\Omega$, we obtain

$$F'(\Omega) = \frac{-M'(\eta,\Omega)}{M^2(\eta,\Omega)},$$

$$M'(\eta,\Omega) = -M^2(\eta,\Omega).F'(\Omega). \quad (30)$$

Utilizing equation (28) and equation (11) in equation (30) we deduce

$$M'(\eta,\Omega) = -\left(\frac{z'(\eta,\Omega)}{z''(\eta,\Omega)}\right)^2\left(-\frac{1}{z'^2(\eta,\Omega)}\left(\int_0^\eta z^2(t,\Omega)dt - \mathrm{b}\,z^2(\eta,\Omega)\right)\right)$$

or

$$\frac{\partial M(\eta,\Omega)}{\partial\Omega} = \frac{1}{z''^2(\eta,\Omega)}\left(\int_0^\eta z^2(t,\Omega)dt - \mathrm{b}z^2(\eta,\Omega)\right).$$

**Theorem 3.5:** If $z(t,\Omega)$ is a solution of the stated problem, so we have

$$\int_0^t z^2(s,\Omega)ds = \begin{vmatrix} z''(t,\Omega) & z'(t,\Omega) \\ \frac{\partial}{\partial\Omega}z''(t,\Omega) & \frac{\partial}{\partial\Omega}z'(t,\Omega) \end{vmatrix} + bz^2(t,\Omega), \quad (31)$$

for all $t \in [0,\eta]$.



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**Proof:** dividing both side of equation (15) by $(\mu-\Omega)$ and simplify gives

$$\int_0^t z\,(s,\mu)\,z\,(s,\Omega)ds=\frac{z''(t,\Omega)z'(t,\ \mu)-z''(t,\mu)z'(t,\Omega)}{(\mu-\Omega)}+bz(t,\mu)z(t,\Omega). \quad (32)$$

Using the limit on both sides of (32) as $\mu\to\Omega$, we get

$$\int_0^t z^2(s,\Omega)ds=\lim_{\mu\to\Omega}\left(\frac{z''(t,\Omega)z'(t,\mu)-z''(t,\mu)z'(t,\Omega)}{(\mu-\Omega)}+bz(t,\mu)z(t,\Omega)\right). \quad (33)$$

$let\ m=\mu-\Omega$, hence

$$\int_0^t z^2(s,\Omega)ds=$$

$$z''(t,\Omega)\left(\lim_{m\to 0}\frac{z'(\Omega,\Omega+m)-z'(t,\Omega)}{m}-z'(t,\Omega)\left(\lim_{m\to 0}\frac{z''(\Omega,\Omega+m)-z''(t,\Omega)}{m}+bz^2(t,\Omega)\right.\right.$$

$$\int_0^t z^2(s,\Omega)ds=z''(t,\Omega)\left(\frac{\partial}{\partial t}z'(t,\Omega)\right)-z'(t,\Omega)\frac{\partial}{\partial t}z''(t,\Omega)+bz^2(t,\Omega)$$

or

$$\int_0^t z^2(s,\Omega)ds=\begin{vmatrix} z''(t,\Omega) & z'(t,\Omega)\\ \frac{\partial}{\partial\Omega}z''(t,\Omega) & \frac{\partial}{\partial\Omega}z'(t,\Omega)\end{vmatrix}+bz^2(t,\Omega).$$

**Theorem 3.6:** In each interval $A_k, k=2,3,4,\ldots$, the boundary value problem (1)-(5) can only have one eigenvalue problem.

**Proof:** if $\tilde{\Omega}\in A_{k_0}$ is an eigenvalue of defined problem (1)-(5) for some $k_0\in N\setminus\{1\}$, thus ( 26 ) implies that

$$\int_0^\eta z^2(t,\tilde{\Omega})dt-\mathrm{b}z^2(\eta,\tilde{\Omega})+a\,z'^2(\eta,\tilde{\Omega})>0,$$

by multiplying this inequality by $-\dfrac{1}{z'^2(\eta,\tilde{\Omega})}$, it follows that



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$$-\frac{1}{z'^2(\eta,\tilde{\Omega})}\left(\int_0^{\eta} z^2(t,\tilde{\Omega})dt - \mathrm{b}z^2(\eta,\tilde{\Omega})\right) - a < 0.$$

From theorem (3.2), and above relation we get

$$\frac{dF(\Omega)}{d\Omega} - a < 0,$$

or

$$\frac{d}{d\Omega}(F(\Omega) - a\,\Omega)\Big|_{\Omega=\tilde{\Omega}} < 0.$$

Because $F(\tilde{\Omega}) - a\,\tilde{\Omega} = 0$, As a result of this inequality, the function $F(\Omega) - a\,\Omega$ only takes zero value when the interval $A_{k_0}$ is strictly decreasing. Consequently

$F(\Omega) = \mathrm{a}\Omega$ has a unique solution $\tilde{\Omega}$ in the interval $A_{k_0}$.■

## Conflict of interests.

There are non-conflicts of interest.

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