REVIEW

JOURNAL OF UNIVERSITY OF BABYLON
For Pure and Applied Sciences (JUBPAS)

Vol.30 ; No.2.| 2022

Approximation in Semi-compact and Pre-concave Sets in Metric Spaces

Jawad Khadim Judy[1*]

[1] Ministry of Education-Iraq General Directorate of Babel Education .

***Corresponding author email:** dr.jkjy69@gmail.com

التقريب في المنطقة الشبة المرصوصة والاولية التقعر في الفضاءات المترية

جواد كاظم جودي

[1] المديرية العامة لتربية بابل-وزارة التربية- العراق

Received:	1 /12 /2021	**Accepted:**	5 /6 /2022	**Published:**	30 /6/2022

ABSTRACT

It is known that an approximation in non-compact sets is not guaranteed always same is true for concave sets which do not guarantee get the uniqness of a best approximation element. In this papers we introduce a new method to approximation in a semi-compact and pre-concave set in the metric spaces. This new method accomplished by prove some theorems which used properties of above set that guarantees existences and uniqueness of best approximation element for any element in this set, where these properties that is semi-compactness and pre-concavity alone are considered to be weak topological (metric) properties. Moreover, the other important conclusion is that the semi -compact and pre-concave set in metric space must be become a convex set, more than it must be a strictly convex set and the importance of this conclusion is clear that the convexity of sets is considered a strong properties of sets, also there are others conclusions in the folds of this research.

Key words: convex set, concave set, semi open set, semi closed set, best approximation in normed space, Approximation in metric spaces.

الخلاصة

من المعلوم بأن المجموعة الغير المرصوصة لا توفر أفضل تقريب لعنصر(دالة) ينتمي الى تلك المجموعة كذلك الحال بالنسبة للمنطقة المقعرة حيث أنها لا يمكن أن توفر الوحدانية لدالة (عنصر) أفضل تقريب لدالة معينة(عنصر) تنتمي اليها حيث أن صفة عدم الرص (التراص) وانعدام التحدب من الصفات التبولوجية (المترية) الضعيفة. في هذا البحث سوف أقدم طريقة جديدة في تقريب الدوال في المنطقة شبه المرصوصة (semi-compact set) والاولية التقعر (pre-concave set) تثبت إمكانية الحصول على عنصر أفضل تقريب من المجموعة التي تتميز بالصفتين أعلاه لأي عنصر ينتمي اليها بل أكثر من ذلك امكانية اثبات وحدانية هذا العنصر، عملية ايجاد دالة أفضل تقريب واثبات وحدانيتها تتم من خلال خواص المتتابعات وتقاربها في الفضاء المتري. والاستنتاج المهم الاخر هو أن المجموعة شبة المرصوصة والاولية التقعر بالتأكيد يجب أن تكون مجموعة محدبة (convex set) وهذا من الاستنتاجات المهمة نظرا" لان صفات انعدام التراص والتحدب وكما ذكرنا أعلاه من الصفات التبولوجية (المترية) الضعيفة بينما صفة التحدب من الصفات المترية القوية.

الكلمات المفتاحية: المجموعة المحدبة، المجموعة المقعرة، المجموعة شبة المفتوحة، المجموعة شبة المغلقة، التقريب في الفضاء المعياري، التقريب في الفضاء المتري.

ISSN: 2312-8135 | Print ISSN: 1992-0652

info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

1. Introduction

An approximation in the compact set in terms of providing's the best approximation of any its elements and the convex set considering in terms of providing's the uniqueness of best approximation element for each element that located in this set was studied in several researches and books such as [**1**], [**2**], [**3**] …etc. but the approximation in semi-compact set is not studied in general and for a non-convex set .In our papers we prove an approximation in this set can happen as in theorems 3.1 and 3.3 .

Before we define a new set and in order to, it is necessary to mention some definitions, the value of the best uniform approximation for a function $f \in C_{[a,b]}$ by the set $\boldsymbol{P_n}$ of algebraic polynomials of degree ≤ n is denoted by $E_n(f,p_n)$ or, simply by $E_n(f)$, is define $E_n(f,p_n) = \underbrace{\inf}_{P_n \in \boldsymbol{P_n}} \{\| (f - P_n)_{(x)} \|\}$.The set A is called a semi-open set if $A \subseteq Cl(int(A))$ also A is called a semi-closed if $int(Cl(A)) \subseteq A$ [**4**], [**5**]where $int(A)$ indicted to interior of the set A and $Cl(A)$ indicted to closure of this set ,it is clear that an open set is a semi-open set . The set A is called semi-compact set if every infinite semi-open cover of A has a finite sub cover which cover A where in this paper I will use R^n(real numbers in n-dimensions) with usual topology in R^n i.e. $(R^n.T_{u(R^n)})$,choose this topology to match with use of the convex set in R^n. And not far from our subject the set A is called convex set if the segment which joined between any two points in A is place also in A where the non-convex set is called a concave set, and in the same field the set A is called strictly convex set if the segment which joined between any two points in A is located in $int(A)$ [**6**], [**7**].

Now I will define a new set and I will called it a pre-concave set as follows the set A is called a pre-concave set if the segment which joined between any two points in A is located in $int(Cl(\text{A}))$, And in more detail the segment which joined between any two points in A where A is a pre-convex set be located in $int(CL(\text{A}))$(i.e.$L = \lambda x + (1-\lambda)y \subseteq int\big(Cl(\text{A})\big) \forall x.y \in A, \lambda \in (0.1)$).

It is clear that infinite union of semi open sets is a semi- open set.

Let A_i be a semi-closed set for each $\text{i} = 1,2,\dots,\text{n}$, Now is $\bigcap_{i=1}^{n} \text{A}_i$ also semi-closed set?

Since infinite (also for a finite) union of semi open sets is a semi- open set

then $\bigcup_{i=1}^{n} \text{A}_i \subseteq Cl\big(int(\bigcup_{i=1}^{n} \text{A}_i)\big)$, $i = 1,2,\dots\dots,n$

Then $\big(Cl\big(int(\bigcup_{i=1}^{n} \text{A}_i)\big)\big)^c \subseteq (\bigcup_{i=1}^{n} \text{A}_i)^c$, $i = 1,2,\dots\dots,n$

$int\big(Cl(\bigcap_{i=1}^{n} \text{A}_i^c)\big) \subseteq \bigcap_{i=1}^{n} \text{A}_i^c \; i = 1,2,\dots\dots,n$

Since A_i^c is semi-closed set ,So by the previous step $\bigcap_{i=1}^{n} \text{A}_i^c$ semi-closed set……………..(1.1)

Now, if X is a metric space , $A \subset \text{X}$ and let x belonging in limit point set of A (i.e. $x \in \acute{A}$)
So, for every an open set G and $x \in G$, $G \setminus \{x\} \cap A \neq \phi$.
Here the question is, does this definition still valid if we replace open sets by semi-open sets?

This mean if S be a semi-open set, $S \subset X$ and if $x \in \acute{A}$,then Is $S \setminus \{x\} \cap A \neq \phi$?
It is known that an open set is a semi-open set,
So, suppose that S is a semi-open but not an open set and $x \in S$
Then $S \subseteq Cl(int(S))$ and then $x \in Cl(int(S))$
This mean either $x \in int(S)$ or $x \in (int(S))\grave{}$ limit point of $int(S)$
First part: if $x \in int(S)$
Since $int(S)$ is an open set and $x \in int(S)$,also $x \in \acute{A}$
then $int(S) \setminus \{x\} \cap A \neq \phi$ and by with note $int(S) \subseteq S$
This mean $S \setminus \{x\} \cap A \neq \phi$ which end of part one of proof
Second part: if $x \in (int(S))\grave{}$
And since $x \in \acute{A}$ So, there exists two sequences $< u_n >$ in $int(S)$ and $< a_n >$ in A
Such that $u_n \longrightarrow x$ and $a_n \longrightarrow x$ as $n \longrightarrow \infty$
From our information in analysis, this mean $\lim_{n \longrightarrow \infty} |u_n - a_n| = 0$
So, there exist a subsequence v_n of u_n and a_n such that $v_n \longrightarrow x$
This mean $v_n \subset u_n \subset int(S) \subset S$ and $v_n \subset a_n \subset A$ So, $v_n \subset (S \cap A)$
So, $G \setminus \{x\} \cap (S \cap A) \neq \phi$ where G an open set and $x \in G$ .this mean $S \cap (G \setminus \{x\} \cap A) \neq \phi$
And then $S \setminus \{x\} \cap (G \setminus \{x\} \cap A) \neq \phi$
Since $(G \setminus \{x\} \cap A) \subset A$ then $S \setminus \{x\} \cap A \neq \phi$ which end of the second part of proof
Here our conclusion if $x \in \acute{A}$ and $x \in S$ which S is a semi-open then $S \setminus \{x\} \cap A \neq \phi$……(1.2)

2. Materials and Methods

For semi-compact set A in a metric space X there are two questions regarding, that is, Is the semi-compact set became semi-closed set? As in the case for the compact set. Furthermore, is any infinite set of in a semi-compact set A has a limit point? I will answer these questions in the following theorems:

2.1 <u>Theorem:</u>

Any semi-compact set A in metric space X is a semi-closed set.

<u>Proof:</u>

Let $p \in A$, $q \in A^c$ and W_p and V_q be a semi-open neighborhood of p and q such that $W_p \cap V_q = \phi$ (this by choosing's radius of neighborhood less than $d(p,q) \neq 0$ since $p \neq q$).
Now, $\{W_p\}_{p \in A}$ is a semi-open cover of A
Since A is a semi-compact set
Then there exists finite sets of W_p which cover A (i.e. $A \subset W_{p_1} \cup W_{p_2} \cup \dots \dots \cup W_{p_n}$)
So, $(W_{p_1} \cup W_{p_2} \cup \dots \dots \cup W_{p_n})^c \subset A^c$
Since $V_{q_i} \subset W_{p_i}^c, i = 1,2, \dots \dots, n$ and by (1,1)
Then, $V_q = V_{q_1} \cap V_{q_2} \cap \dots \dots V_{q_n} \subset W_{p_1}^c \cap W_{p_2}^c \cap \dots \dots \cap W_{p_n}^c = (W_{p_1} \cup W_{p_2} \cup \dots \dots \cup W_{p_n})^c \subset A^c$
But V_q is a semi-open neighborhood of $q \in A^c$

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

This means that every $q \in A^c$ has a semi-open set V_q and $q \in V_q$ which fully entirely located in A^c.
So, A^c is infinite union of semi-open sets and then A^c is a semi-open set
And then A is a semi-closed set ■

2.2 Theorem:

Let X be a metric space and $A \subset$ X such that A is a semi-compact set then every infinite subset of A has a limit point.

Proof:

Let M be an infinite set in A and suppose that M has no limit point in A
Then by (1.2) for every $x \in M$ and $V_x \subseteq A$ where $x \in V_x$ semi-open set such that,
$x \in V_x, V_x \setminus \{x\} \cap M = \phi$
So, $V_x \cap M = \{x\}$ but $\bigcup_{x \in M} V_x$ is semi-open cover of A
This mean $A \subseteq \bigcup_{x \in M} V_x$
But, A is a semi-compact set, then there exist a finite numbers of V_x cover A
But this mean M contains a finite number of elements
This result contracts the fact that M is an infinite set
So, M must haves a limit pointe■

3. Results and Discussion

In this part, I will study how the semi-compact set A available the best approximation element for any element $f \in A$ by convergent of series in an infinite set in A in metric spaces where d is the metric distance. The important conclusion here is that the semi-compact set and pre-concave set must be a convex set Furthermore, it become a strictly convex set. The prove of this was provided by properties of semi-closed set and then by all these properties the best approximation element must be unique.

3.1 Theorem:

Let X be a metric space and let $A \subset$ X such that A is a semi-compact set then there exist a best approximation $a^* \in A$ for every $f \in A$.

Proof:

Let $d^* = \underbrace{\inf}_{a \in A}\{d(a, f), f \in A\}$.
Then if a^* exist such that $d^* = d(a^*, f)$,So a^* is a best approximation of f in A
Otherwise since A is an infinite semi-compact set then by theorem 2.2
There exist a sequence $\{a_i\}_{i=1}^{\infty}$ of points in A such that $\lim\limits_{i \to \infty} d(a_i, f) = d^*$
Also, $\{a_i\}_{i=1}^{\infty}$ has limit point say a^*
Then $\forall \in > 0\ \exists k \in N$ such that $d(a_k, f) \le d^* + \frac{\in}{2}$ and $d(a_k, a^*) \le \frac{\in}{2}$

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

So, $d(a^*, f) \leq \mathrm{d}(a_k, a^*) + \mathrm{d}(a_k, f) \leq d^* + \frac{\in}{2} + \frac{\in}{2} = d^* + \in$

Since $\in$ is arbitrary number then $d^* = d(a^*, f)$

So, $d(a^*, f) \leq d(a, f), \forall a \in A$

And then a^* is a best approximation of f in A ■

3.2 Theorem:

Let X be a metric space and let $A \subseteq X$ such that A is a semi-compact and a pre-concave set then A is a convex set and more than A necessary to be strictly convex set.

Proof:

First part of this theorem:

By definition of a pre-concave set and by theorem 2.1 it is concluded:

For each $a, b \in A$ then $L: a\lambda + b(1-\lambda) \subseteq int(cl(A))$

Since A is a semi-compact set

Then A semi-closed set and so, $int(cl(A)) \subseteq A$

So, $L \subseteq A$ and then A is a convex set

For the second part

Since if $A \subseteq B$ for any A, B then $int(A) \subseteq int(B)$ then $int(int(cl(A))) \subseteq int(A)$

So, $int(cl(A))) \subseteq int(A$

And then $L \subseteq int(cl(A))) \subseteq int(A)$

And so, A is strictly convex set■

3.3 Theorem:

Let X be a metric space and let A is a semi-compact and pre-concave set then the best approximation element is unique for every element $f \in A$.

Proof:

By theorem (3.1), theorem (3.2) it is found that:

Since A is a semi-compact set

Then there exist a best approximation element for f in A.

Now, suppose that S_1, S_2 be a best approximation for f in A such that $S_1 \neq S_2$

Let h^* be the error such that $h^* = \underbrace{inf}_{a \in A} \{d(a, f), f \in A\}$

This mean $d(f, S_1) = d(f, S_2) = h^*$

Now, $\frac{1}{2}(d(S_1 + S_2, f)) \leq \frac{1}{2} d(f, S_1) + \frac{1}{2} d(f, S_2)$

Since A is convex set then the form $\frac{1}{2}(S_1 + S_2) \in A$ and then $\frac{1}{2}(S_1 + S_2)$ is also best approximation of f in A

So, $d\left(f, \frac{1}{2}(S_1 + S_2)\right) = h^*$

Now, let $0 < \lambda < 1$ be the largest numbers such that:

The point $S = \frac{1}{2}(S_1 + S_2) + \lambda(f - \frac{1}{2}(S_1 + S_2)) \in A$

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

مجلــة جامعــة بابــل للعلــوم الصــرفة والتطبيقيــة

So, $d(f,S) = (1-\lambda)h^*$ and then $d(f,S) < h^*$ which is a contradiction value of h^*
So, $S_1 = S_2$
And then the best approximation element is unique■

4. Conclusion:

In this papers we proved that an approximation in semi-compact and pre-concave sets in metric spaces is verified in terms of existence and uniqness because we were able to prove that this set is a strictly convex se**t.**

Conflict of interests.

There are non-conflicts of interest.

References

[1] Erwin Kreyszig "Introductory Functional Analysis with Applications", University of Windsor, John Wiley & Sons New York Santa Barbara London Sydney Toronto Copyright© 1978, by John Wiley & Sons. Inc. All rights reserved. Published simultaneously in Canada.

[2] Akram Mahmoud abu Ghazal **"**Best approximation and best co-approximation in normed Space" a thesis submitted in partial fulfilment of the requirement for the degree of master of mathematics, supervised by Associate professor. As'ad y. As'ad, The Islamic University of Gaza, January 26, 2010.

[3] T. D. Narang and Sangeeta "Best Approximation in Metric Spaces"2000 Mathematics Subject Classification. 41A65, 41A50, all content following this page was uploaded by Sangeeta. - on 01 November 2015.

[4] Levine, N., (Semi – open sets and semi – continuity in topological spaces), Ohio State University, Amer Math Monthly 70 (1963), 36-41.

[5] Mashhour, A.S, Abd EL- Monsef, M.E and EL-Deeb, S.N., (On pre-continuous and weak pre continuous mappings), proc. Math. phys . Soc . Egypt .53 (1982), 47 -8.

[6] Valeriu Soltan**, "**Lectures on Convex Sets **"** George University Mason, USA. World Scientific publishing Co. Pte. Ltd,2020.

[7] Lars-_Ake Lindahl," Convexity and Optimization ", c 2016 Lars-_Ake Lindahl, Department of Mathematics, Uppsala University, 2016.

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com