

# Approximation in Real Contra-Continuous Functions Spaces

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## التقريب في فضاء الدوال المستمرة المضادة الحقيقية

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### ABSTRACT

#### Background

It is known that an approximation in general topology did not study an approximation of functions as well as approximate of real contra-continuous functions but it is limited to an approximation of sets in topological spaces in general and simple.

#### Materials and Methods:

In this paper I will study an approximation in real contra-continuous functions space starting from providing a best approximation element of this kind of functions in a compact set and I symbol of this space by  $CO(R)$  where  $R$  is real numbers .

#### Results:

Also in this paper I described contra-continuous function (as continuous functions) in real numbers also, I was able to get an example of this kind of functions in  $R$  (where it very difficult example) and approximate it by Bernstein operator.

#### CONCLUSION:

Here, the important conclusions are that the compact set in real numbers is available best approximation element for any contra-continuous function which is located in it and the other is that the contra-continuous functions must be bounded.

#### Key words:

An approximation of contra-continuous functions, Bernstein operator, Continuous functions, Contra-continuous functions, Compact sets, Closed sets, Open sets.

### الخلاصة

#### مقدمة:

من المعروف بأن دراسة التقريب في الفضاءات التوبولوجية (بصورته البسيطة) لم يتطرق الى موضوع تقريب الدوال فضلاً عن تقريب الدوال المستمرة المضادة بل اقتصر في دراسته على تقريب المجموعات في الفضاءات التوبولوجية بشكل بسيط وعام.

#### طرق العمل:

في هذا البحث سوف أدرس بشيء من التفصيل التقريب في فضاء الدوال المستمرة المضادة في  $R$  (مجموعة الأعداد الحقيقية) مبتدأً بإمكانية تجهيز المجموعة المرصوصة لعنصر أفضل تقريب لدالة حقيقية مستمرة مضادة تنتمي إليها. وسوف أرمز لفضاء هذا النوع من الدوال بالرمز  $CO(R)$ .

#### الاستنتاجات:

من الاستنتاجات المهمة في هذا البحث هي أن الدالة المستمرة المضادة الحقيقية يجب أن تكون مقيدة وبالتالي تمتلك معيار (Norm) أيضاً" المجموعة المرصوصة يجب أن توفر عنصر أفضل تقريب لدالة مستمرة مضادة حقيقية تنتمي إليها أسوة بالدوال المستمرة، كذلك في هذا البحث سوف أعطي مثال خاص لدالة مستمرة مضادة في  $R$  (وهو من الأمثلة الصعبة جداً) ونقره بمؤثر برنشتاين.

#### الكلمات المفتاحية:

تقريب الوال المستمرة المضادة، مؤثر برنشتاين، الدوال المستمرة، الدوال المستمرة المضادة، المجموعات المرصوصة، المجموعات المغلقة، المجموعات المفتوحة.



## INTRODUCTION

An approximation in topological spaces is studied in several researches and books such as [1], [2], [3] ...etc. Those studies were based on an approximation space (or rough set) This is done by studying the interior points of these sets as well as the set's closure, but never not study an approximation of functions as well as contra continuous function in topological spaces.

As I mentioned earlier, what prompted me to write this research is the absence of any example of a real contra-continuous function of course, in the set of real numbers, in addition to the larger issue, which is the approximation of this mysterious type of functions with a real polynomial.

The function  $f: (X, \tau) \rightarrow (Y, \sigma)$  is called contra-continuous function if the invers image of any an open set in  $Y$  (according to  $\sigma$  -topology) is a closed set in  $X$  (according to  $\tau$  -topology) [4],[5]...etc. for example Let  $X = \{1,2,3\}, \tau = \{\{1\}, X, \phi\}$  and  $Y = \{a, b, c\}, \sigma = \{\{a\}, X, \phi\}$  and let  $f: (X, \tau) \rightarrow (Y, \sigma)$  such that  $f(1) = c, f(2) = f(3) = a$ , Then  $f$  is contra - continuous function.

In the real numbers with usual topology (in general) not just an approximation of these functions is not study but there is no any example for contra-continuous function because impossible finding examples for this kind in real numbers( $\mathbb{R}$ ) since in this case some elements in a domain of function be have more than one image in its codomain as I will explain later.

As defined it previously , the function  $f: X \rightarrow Y$  is called contra-continuous function if the invers image of any an open set in  $Y$  is a closed set in  $X$  (where I will used the real numbers  $\mathbb{R}$  with usual topology  $(\mathbb{R}, T_u)$  instead of the sets  $X, Y$  ) and symbolled of the real contra-continuous function space by  $CO(\mathbb{R})$  .

As described (or define) earlier of the continuous function as the following "If  $f: X \rightarrow Y$  and let  $x_0 \in X$  then we say that  $f$  is continuous in  $x_0$  if  $\forall \epsilon > 0 \exists \delta > 0$  such that if  $|x - x_0| < \delta$  then  $|f(x) - f(x_0)| < \epsilon$  [6],[7] ....etc.

So, for the contra-continuous function I will describe (or definition) it as "If  $f: X \rightarrow Y$  and let  $x_0 \in X$  then we say that  $f$  is contra-continuous function in  $x_0$  if  $\forall \epsilon > 0 \exists \delta > 0$  such that if  $|x - x_0| \leq \delta$  then  $|f(x) - f(x_0)| < \epsilon$  .

**Or** "If  $f: X \rightarrow Y$  and let  $x_0 \in X$  then we say that  $f$  is contra-continuous function in  $x_0$  if  $\forall \epsilon > 0 \exists \delta > 0$  such that if  $|x_0| \leq \delta$  then  $|f(x) - f(x_0)| < \epsilon$ .....(1-1) (Note that numbering includes two parts of above results).

This definition (or description) come from definition of a closed set in  $(\mathbb{R}, T_u)$  that since an open set in  $\mathbb{R}$  will be as an open interval then its complement must be a closed set and it will be as a closed intervals or countable sets whether finite or infinite sets with condition that these numbers must be bonded.

So after defined contra-continuous function  $f$  as above then one have the right to ask especially (about studying the possibility of an approximation of this kind of functions) is this function has a norm? before answered this question there is another question for this function that is "is  $f$  bonded ?" and I will study this case with some expansion in theorem 2.3 this by used properties of convergent of sequences in domain of  $f$ .

Now, after proved that  $f$  bonded function ,So sequentially it has a norm and as in case of continuous functions which has two types of norms that is integral norm and supremum norm,



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Here the contra-continuity type cannot be have integral norm because this functions not continuous, So I will use supremum (*sup*) norm (or absolute value norm) when approximate it (i.e.  $\|f\|_{CO[a,b]} = \sup\{|f(x)|: x \in [a, b], f \in CO[a, b]\}$ ).

For an open and a closed set in  $R$  it is clear that an open interval is also an open set and a closed interval must be a closed set (as mentioned above) also clear that every element in an open interval content in an open ball which located in our open interval, Here if a closed set in the form of countable set then every element can be content at least in single set (for example the set of same element “which contains this element” ),So it contained in a closed set, Let's symbolize to this results by the number.....(1-2).

Also for elements in a closed interval, every element can be content in a closed set(disc) which located in our interval except endpoints which only be contained in our closed interval, so every element in a closed set is located in a closed set which continent in our closed set Let's symbolize to this results by the number..... (1-3).

From our information the infinite union of a closed sets not necessary a closed set but if the closed sets in  $(R, \tau_u)$  as form closed interval or countable set then the infinite union of them is a closed set as shown in results (1-2) and (1-3).

We return to the approximation of the contra-continuous functions, initially I will study existences of best approximation element of these functions. It is known that the compact set give this element for any continuous function which content in it, the main question here is that” is this feature available and valid for contra-continuity functions? In this paper I study availability to get a best approximation element in the compact set.

After getting the best approximation element and an example for real contra-continuous one has the right to ask can anyone approximate the contra-continuous function by real polynomial of degree  $n$  ? I talk here of one of the well-known polynomials , such as Bernstein's operator, It is known that Bernstein's polynomials is one of the excellent polynomials that provides the best approximation elements for any continuous function within the closed interval  $[0,1]$  or its expansion, in this paper I will approximate real contra-continuity function  $f$  by Bernstein operator as in theorem 3.3.

## 2. Materials and Methods

For contra-continuous function, there are two questions: is there a relationship between our previous description of it and the possibility of moving the inverse image of an open set to a closed set? Here I will answer this question in the first theorem. Furthermore, does contra-continuous function bounded? Also will answered these questions in the next theorem, before that I will prove convergent of image sequence in contra-continuous function.

**Theorem 2.1.:** If  $f: X \rightarrow Y$  is any function then  $f$  is a contra-continuous function iff the invers image of any an open set in  $Y$  is a closed set in  $X$ .

**Proof:** *Necessary:*

Suppose that  $f$  is a contra-continuous function and let  $V$  be an open set in  $Y$ .

Let  $x_0 \in X$  such that  $f(x_0) \in V$  and let  $y_0 = f(x_0)$ .

Since  $f$  is a contra-continuous function , So by (1-1) we get  $\forall \epsilon > 0 \exists \delta > 0$  such that:

**Either** if  $|x - f^{-1}(y_0)| \leq \delta$  then  $|f(x) - y_0| < \epsilon$  **or** if  $|x_0| \leq \delta$  then  $|f(x) - y_0| < \epsilon$



In both of the above possibilities and since both sets  $X = Y = R$  with usual topology and Since  $V$  is an open set then there exist an open ball  $B$  in  $Y$  its center is  $f(x_0)$  such that  $B \subseteq V$  So, by our describe of contra-continuity of  $f$  (i.e. by 1-1).

There exist a closed set  $\bar{B}$  in  $X$  its center is  $x_0$  such that  $f(\bar{B}) \subseteq B \subseteq V$

So,  $\bar{B} \subseteq f^{-1}(V)$  and then by results (1-2) and (1-3)  $f^{-1}(V)$  is a closed set in  $X$ .

*Sufficiently:*

Suppose that the invers image of any an open set in  $Y$  is a closed set in  $X$

Let  $x_0 \in X$  and let  $\bar{B}$  be an open ball in  $Y$  such that its center is  $f(x_0)$  (i.e.  $f(x_0) \in \bar{B}$ )

Since invers image of any an open set in  $Y$  is a closed set in  $X$ ,

Then  $f^{-1}(\bar{B})$  is a closed in  $X$  and  $x_0 \in f^{-1}(\bar{B})$

By definition of a closed set in metric spaces (1-3), there exist a disc  $B$  in  $X$  its center is  $x_0$ .

Such that  $B \subseteq f^{-1}(\bar{B})$  and then  $f(B) \subseteq \bar{B}$

So, 1-1 is verified and this mean  $f$  is contra-continuous in  $x_0$ . ■

**Theorem 2.2:** Suppose that  $f: X \rightarrow Y$  be any function then  $f$  is contra-continuous function in  $x_0$  iff  $x_n \rightarrow x_0$  (where  $x_n \in X$ ) implies  $f(x_n) \rightarrow f(x_0)$

**Proof:** *Necessary:*

Since  $f$  is contra-continuous function then  $\forall \epsilon > 0 \exists \delta > 0$  such that:

Either if  $|x - x_0| \leq \delta$  then  $|f(x) - f(x_0)| < \epsilon$  or if  $|x_0| \leq \delta$  then  $|f(x) - f(x_0)| < \epsilon$

Now, let  $x_n \in X$  and  $x_n \rightarrow x_0$  then  $|x_n - x_0| \leq \delta$  as  $n \rightarrow \infty$

So, by definition (by 1-1) of contra-continuity for  $f: |f(x_n) - f(x_0)| < \epsilon$ .

This mean that  $f(x_n) \rightarrow f(x_0)$

*Sufficiently:*

Suppose that  $x_n \rightarrow x$  implies  $f(x_n) \rightarrow f(x)$

So, this mean  $|x_n - x| \leq \delta$  implies  $|f(x_n) - f(x)| < \epsilon$  for any  $\delta, \epsilon$ .

This indicates that  $f$  is contra-continuous in the point  $x_n, n \in N$

Now, if  $f$  is not contra-continuous in some points in  $X$

Then  $\exists \epsilon > 0 \forall \delta > 0$  such that:

Either  $|x_n - x| \leq \delta$  but  $|f(x_n) - f(x)| \geq \epsilon$  or  $|x_n| \leq \delta$  but  $|f(x_n) - f(x)| \geq \epsilon$

This mean (in particular), if  $\delta = \frac{1}{n}$ , So there exist  $x_n$  satisfy if  $|x_n - x| \leq \frac{1}{n}$  but  $|f(x_n) - f(x)| \geq \epsilon$  or if  $|x_n| \leq \frac{1}{n}$  but  $|f(x_n) - f(x)| \geq \epsilon$   $n \in N$  as  $n \rightarrow \infty$

Clearly  $x_n \rightarrow x$  but  $f(x_n) \not\rightarrow f(x)$  or  $x_n \rightarrow 0$  but  $f(x_n) \not\rightarrow f(x) \forall x \in X$  with special case  $f(x_n) \not\rightarrow f(0)$

Which is a contradiction, So  $f$  is a contra-continuous function ■

**Theorem 2.3 :** If  $f: [a, b] \rightarrow Y$  is contra-continuous function, then  $f$  must be bounded.

**Proof:** Suppose that  $f$  is not bounded in  $[a, b]$

So,  $\forall \epsilon > 0 \exists n \in N$ , such that if  $x_n \in [a, b]$  then  $|f(x_n)| \geq \epsilon$

So, since  $f$  is a contra-continuous function and by theorem 2.2  $\exists \delta \geq 0$  such that  $|x_n| \geq \delta$

But  $[a, b]$  is a bounded set

And this is a contradiction, So  $f$  must be bounded in  $Y$  ■



Suppose that  $F(x) = (f \cup g)(x) = f(x) \cup g(x)$  then  $F: (R, T_u) \rightarrow (R, T_u)$

It is clear that  $F$  is a function since every  $x \in R$  has one image in codomain of  $F = R$

And it is clear that the rang of  $f$  is only natural numbers  $N$

Now, to prove that  $f(x)$  is contra-continuous function in real numbers with usual topology:

Let  $B$  is an open set in  $R$  (codomain of  $f$ ) such that  $B$  contain only  $x \in N$  then:

$$F^{-1}(B) = (f \cup g)^{-1}(B) = f^{-1}(B) \cup g^{-1}(B) = [x, x + 1) \cup (x, x + 1] = [x, x + 1] = \text{closed set in } (R, T_u)$$

with the same way if the open set  $B$  contine more than one element in codomain of  $F$ .

$$\text{This mean } F^{-1}(B) = (f \cup g)^{-1}\{x_1, x_2, \dots, x_n\} = f^{-1}\{x_1, x_2, \dots, x_n\} \cup g^{-1}\{x_1, x_2, \dots, x_n\} = [x_1, x_n] = \text{closed set in } (R, T_u)$$

And If  $B$  is an open set in  $R$  (codomain of  $F$ ) such that  $B$  contine  $x \notin N$  then

$$F^{-1}(B) = \phi = \text{closed set in } (R, T_u)$$

So,  $f$  is contra-continuous in  $R$ .

In this part we study approximate of above function  $F$  by Bernstein Polynomials where:

$$B_n(f; x) = \sum_{k=0}^n \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}, \text{ for any } f \in C[0,1], x \in [0,1] \text{ and } n \in N. \quad [8]$$

**Theorem 3.3:** Suppose that  $F: [0,1] \rightarrow R$  is contra-continuous function then

$$\lim_{n \rightarrow \infty} (F(x) - B_n(F; x)) = 0 \text{ such that } F \in CO[0,1], x \in [0,1] \text{ and } n \in N.$$

**Proof:** Since  $B_n(F; x) = \sum_{k=0}^n \binom{n}{k} F\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$

And since  $F = \text{natural numbers}$  So,  $F\left(\frac{k}{n}\right) = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{otherwies} \end{cases}$  for all  $k, n \in N$ , (and  $k, n \in [0,1]$ )

Also,  $\binom{n}{k} = 1$  when  $k = n$  because  $k \leq n$

$$\text{So, } B_n(F; x) = \sum_{k=0}^n \binom{n}{k} F\left(\frac{k}{n}\right) x^k (1-x)^{n-k}, x \in N, \text{ for all } n \in N$$

$$= \sum_{k=0}^n \binom{n}{k} (f \cup g)\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$$

$$= \sum_{k=0}^n \binom{n}{k} f\left(\frac{k}{n}\right) \cup g\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$$

Since  $f\left(\frac{k}{n}\right) = g\left(\frac{k}{n}\right) = \begin{cases} 1 & \text{when } k = n \\ 0 & \text{other wise} \end{cases}$  and then  $(1-x)^{n-k} = 1$  also  $\binom{n}{k} = 1$

$$\text{So, } B_n(F; x) = x^n, n \in N$$

Now, since  $x \in [0,1]$  this mean  $x = \frac{a}{b}$  such that  $a \leq b$

$$\text{Then } \lim_{n \rightarrow \infty} x^n = \lim_{n \rightarrow \infty} \left(\frac{a}{b}\right)^n = \lim_{n \rightarrow \infty} \frac{a^n}{b^n} = 0 \text{ So, } B_n(F; x) = 0 \text{ As } n \rightarrow \infty.$$



Also,  $F(x) = (f \cup g)(x) = f(x) \cup g(x) = 0, x \in [0,1]$

So,  $B_n(F; x) = F(x), x \in [0,1]$

If  $a = b$  then  $\lim_{n \rightarrow \infty} \left(\frac{a}{b}\right)^n = \lim_{n \rightarrow \infty} 1 = 1 = B_n(F; x)$

Then  $F(x) = (f \cup g)(x) = 1, x \in (0,1]$  ..... So,  $B_n(F; x) = F(x), x \in (0,1]$

This mean,  $B_n(F; x) = F(x), x \in [0,1]$  As  $n \rightarrow \infty$  ■

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### Conflict of interests.

There are non-conflicts of interest.

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