Article

JOURNAL OF UNIVERSITY OF BABYLON
For Pure and Applied Sciences (JUBPAS)

Vol.31; No.2.| 2023

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

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Some Properties of the Mandelbrot Sets $\mathbb{M}(Q_\alpha)$

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Received: 27 /2 /2023 **Accepted:** 6/6/2023 **Published:** 30/6/2023

Abstract

The aim of this work is to study the concept of the parameter plane, so we are interested in introducing the Mandebrot set,as well as the Mandelbrot set is a fractals ,whereas, if we zoom in on any small piece in the border regions of the shape, we notice that it is similar to the original shape, that is the Mandelbrot set, with special topological specifications, and we introduce of some properties of the Mandelbrot sets for the map in the form $Q_\alpha(z) = \alpha z - \alpha z^2$, where α is a complex constant, and we prove that the conjecture is the Duadi-Hubbard theorem . By studying of the parameter plane that the Mandelbrot set is a component form in one of its details in the dynamical plane it is the Julia set , in other words every small detail of the Mandelbrot set represents the Julia set, so there is more than one characteristic of the Julia set, we will limit ourselves only to presenting the connection of the Julia set for Q_α.

Keywords: Connected set, Filled Julia set , Julia set , simply connected, Mandelbrot set

الخلاصة

الهدف من هذا العمل هو دراسة مفهوم مستوى البارامتر ، لذلك نحن مهتمون بتقديم مجموعة Mandebrot ، كما أن مجموعة Mandelbrot عبارة عن فركتلات ، بينما إذا قمنا بتكبير أي قطعة صغيرة في المناطق الحدودية لـ الشكل ، نلاحظ أنه مشابه للشكل الأصلي ، وهو مجموعة Mandelbrot ، بمواصفات طوبولوجية خاصة ، ونقدم بعض خصائص مجموعات Mandelbrot للخريطة بالصيغة Q_α (z) = αz−αz ^ 2 ، حيث α ثابت معقد ، ونثبت أن التخمين هو نظرية .Duadi−Hubbardمن خلال دراسة مستوى البارامتر الذي تعتبر مجموعة ماندلبروت شكلاً مكونًا في أحد تفاصيله في المستوى الديناميكي ، فإن مجموعة جوليا هي مجموعة جوليا ، بمعنى آخر كل التفاصيل الصغيرة لمجموعة ماندلبروت تمثل مجموعة جوليا ، لذلك هناك أكثر من واحدة. من سمات مجموعة Julia ، سنقتصر فقط على تقديم اتصال مجموعة Julia لـ Q_α.

الكلمات المفتاحية: مجموعة متصلة ، مجموعة جوليا المحشوة ، مجموعة جوليا ، متصلة ببساطة ، مجموعة ماندلبروت

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1. Introduction

Julia (1918) and Fatou (1919) developed the theory of iteration of complex maps. The complex shape of Julia sets was discovered through evolution in computer graphics. The mathematician Benoit Mandelbrot was the first to call the name Mandelbrot set, in the late 1970s brought the set wide acclaim with the help of images on a high–speed computer. By drawing Julia set , hence the Mandelbrot set has become almost fictional on computers, see [3,4,8 and 9] where contain not only much information about Julia set and Mandelbrot set, but also contains exquisite color plates of Julia sets, zooms of the Mandelbrot set and other fractals. The Mandelbrot set is considered by far the most complicated, yet the most fascinating fractal. Through previous studies and research, they paid attention to this aspect, and this is what we see, for example, in [7] Douady and Hubbard proves that the filled Julia set is a connected set. In [6] Devaney has a more detailed discussion of these results. which is one of the important things through which it is concerned to study complex dynamics, where the parameter is one of the complex numbers. Which in turn is one of the topics that have an important relationship in fractal.

2. Preliminaries

We will give some definitions for this work.

Definition (2.1) [2]

A subset G of a metric space X is connected if it cannot be represented as the union of two relatively separate sets, neither of which is empty. Otherwise G is disconnected.

Definition (2.2) [1]

A region is simply connected if its complement with respect to the extended plane is connected.

For example, the open dick $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$ is simply connected

Definition (2.3) [1]

Assume that an analytic map $Q_\alpha: \mathbb{C}_\infty \to \mathbb{C}_\infty$. Define $J(Q_\alpha)$ *as* the closure of all repelling periodic points of Q_α.

Definition (2.4) [1]

For any polynomial $Q_\alpha: \mathbb{C}_\infty \to \mathbb{C}_\infty$ with degree $n \geq 2$. Defined filled Julia set as $\mathbb{K}(Q_\alpha) = \{z \in \mathbb{C}_\infty : \{Q_\alpha^n(z)\}_{n=0}^{\infty} \text{ is bounded}\}$. Also define Escape set $\mathbb{A}(\infty)$ as $= \{z : Q_\alpha^n \to \infty \ as\ n \to \infty\}$.

Remark (2.5) [1]

$J(Q_\alpha) =$ the boundary $\mathbb{K}(Q_\alpha)$.

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Definition (2.6) [3]

The collection of complex number c such that the orbit of 0 for $g_c(z) = z^2 + c$ is bounded is called the "Mandelbrot set", and denoted by $\mathbb{M}$. So

$\mathbb{M} = \{c \in \mathbb{C}: J(g_c) \text{ is a connected set}\}$

By **[5]** $Q_\alpha(z) = \alpha z - \alpha z^2$ is conjugate to $g_c(z) = z^2 + c$ and note that $\alpha = 1 \pm \sqrt{1-4c}$. By above , we will write the definition as

$\mathbb{M} = \{\alpha \in \mathbb{C}: J(Q_\alpha) \text{ is a connected set}\}$.

3. Some Characteristics of the Mandelbrot set

We will study some properties of the Mandelbrot set .

Theorem (3.1) [4]

A polynomial map P has an attracting periodic point , then the basin of attraction of p has a critical point .

Proposition (3.2)

If Q_α has an attracting periodic point , then α is in $\mathbb{M}$.

Proof :

Since $Q_\alpha{'}(z) = \alpha - 2\alpha z$, so Q_α has a critical point 0.5 only. If p is an attracting periodic point of Q_α , then by Theorem (3.1) , so the iterates of 0.5 for Q_α converges to p and its iterates . Hence the iterates of 0.5 are bounded , so α is in $\mathbb{M}$.

Proposition (3.3)

$\mathbb{M}$ is symmetric about the real axis.

Proof:

If $\alpha = x + iy \in \mathbb{M}$, then $\bar{\alpha} = x - iy \in \mathbb{M}$ $\{ |\bar{\lambda}| = |\lambda| \}$.

$$|Q_{\bar{\alpha}}(0.5)| = \left|\frac{1}{4}\bar{\alpha}\right| = \frac{1}{4}|\bar{\alpha}| = \frac{1}{4}|\alpha| = |Q_\alpha(0.5)|$$

$$\left|Q_{\bar{\alpha}}{}^2(0.5)\right| = \frac{1}{4}|(\bar{\alpha})^2| + \frac{1}{16}|(\bar{\alpha})^3| = \frac{1}{4}|(\alpha)^2| + \frac{1}{16}|(\alpha)^3| = \left|Q_\alpha{}^2(0.5)\right|$$

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$|Q_{\bar{\alpha}}{}^n(0.5)| = |Q_\alpha{}^n(0.5)|$, thus $\bar{\alpha} \in \mathbb{M}$. see Figure (1).

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Now, we give the conjecture the Douady – Hubbard Theorem .

Proposition (3.4)

The parameter α is in $\mathbb{M}$ iff $|{Q_\alpha}^n(0.5)| \le 4$ for each $n \ge 1$. $\mathbb{M}$ is compact subset of $\{|\alpha| \le 4\}$. So $\mathbb{C} \backslash \mathbb{M}$ is connected .

Proof :

If $|\alpha| > 4$, and let $= |\alpha| - 1$. If $|z| \ge |\alpha|$

$$|Q_\alpha(z)| = |\alpha z - \alpha z^2| = |\alpha z^2 - \alpha z|$$

$$\ge |z|^2 \left|z - \frac{\alpha}{z}\right|$$

$$\ge |z|^2 \left||z| - \frac{|\alpha|}{|z|}\right| \ge |\alpha|^2 \big||\alpha| - 1\big| = |\alpha|^2 r > |\alpha| r$$

$|Q_\alpha(0.5)| \ge |\alpha| r$ (If $|\alpha| > 4$, and let $r = |\alpha| - 1$. If $|z| \ge |\alpha|$)

$\left|{Q_\alpha}^2(0.5)\right| \ge |\alpha| r^2$

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$|{Q_\alpha}^n(0.5)| \ge |\alpha| r^n$

$\left|{Q_\alpha}^{n+1}(0.5)\right| = |Q_\alpha({Q_\alpha}^n(0.5))| \ge |{Q_\alpha}^n(0.5)| r \ge (|\alpha| r^n) r = |\alpha| r^{n+1}$.

Therefore by Law of Induction , $|{Q_\alpha}^n(0.5)| \ge |\alpha| r^n$ for all n , since $r > 1$, the iterates of 0.5 are unbounded, so that α is not in $\mathbb{M}$. Therefore $\mathbb{M}$ is compact subset of $\{|\alpha| \le 4\}$.

Assume that $|\alpha| \le 4$ and α is not in $\mathbb{M}$, but $|{Q_\alpha}^n(0.5)| = 4 + \beta$ for $\beta > 0$ and $i > 1$. Then $\left|{Q_\alpha}^{i+1}(0.5)\right| = |\alpha| \left|({Q_\alpha}^i(0.5)(1 - {Q_\alpha}^i(0.5)\right|$

$$\ge 4\left|{Q_\alpha}^i(0.5)\right|(\left|{Q_\alpha}^i(0.5)\right| - 1$$

$$= 4(12 + 3\beta + \beta^2)$$

$$\ge 12 + 4\beta + \beta^2$$

$$\ge 12 + 4\beta.$$

By Law of Induction we obtain $\left|{Q_\alpha}^{i+k}(0.5)\right| \ge 12 + 4^k \to \infty$ as $k \to \infty$, thus α is not in $\mathbb{M}$. contradiction with the above part $\mathbb{M}$ is a subset of $\{|\alpha| \le 4\}$. So $\alpha \in \mathbb{M}$.

Fix n, the $\{|\alpha| \le 4\}$ such that if $|{Q_\alpha}^n(0.5)| \le 4$ is a closed set. shows that $\mathbb{M}$ is closed and bounded , thus $\mathbb{M}$ is compact. Let U be any bounded domain in . Then $|{Q_\alpha}^n(0.5)| \le 4$ for all $\alpha \in \partial U$ since $Q_\alpha(0.5)$ is a polynomial in α, from the maximum principle that $|{Q_\alpha}^n(0.5)| \leq 4$ for all $\alpha \in U$. So $U \subset \mathbb{M}$. Therefore $\mathbb{C}\backslash\mathbb{M}$ consists of a single unbounded component.

Proposition (3.5)

The Mandelbrot set $\mathbb{M}$ is connected set .

Proof:

Suppose $D_4 = \{|z| \leq 4\}$ is compact and $P_n(\alpha) = {Q_\alpha}^n(0.5)$ a polynomial. Then $\mathbb{M} = \bigcap_{n=1}^{\infty} P_n^{-1}(D_4)$ is compact subset of D_4. The $\mathbb{C}\backslash\mathbb{M}$ is the union of open connected sets containing ∞, $\mathbb{C}\backslash\mathbb{M} = \bigcup_{n=1}^{\infty} P_n^{-1}\left({D_4}^C\right)$, so $\mathbb{C}\backslash\mathbb{M}$ is connected , that is $\mathbb{M}$ is full and we can show that $\mathbb{C}\backslash\mathbb{M}$ is simply connected , so $\mathbb{M}$ is connected set .

Example (3.6)

The parameter $\alpha = 5$ is not in the Mandelbrot set . We compute the first few terms of the sequence

$Q_5(0.5) = 5.\frac{1}{2} - 5.\frac{1}{4} = \frac{5}{4}$ =1.25 , thus ${Q_5}^2(0.5) = Q_5\left(\frac{5}{4}\right) = 1.5$, and therefore ${Q_5}^3(0.5) = Q_5(1.5) = 4.3$. Since $\left|{Q_5}^3(0.5)\right| = 4.3 > 4$, we see that go to infinty . Hence α is not in the Mandelbrot set .

Example (3.7)

The parameter $\alpha = 1$ is in the Mandelbrot set .

We compute the first few terms of the sequence

$Q_1(0.5) = \frac{1}{2} - \frac{1}{4} = 0.25$, thus ${Q_1}^2(0.5) = Q_1\left(\frac{1}{4}\right) = 0.18$, and therefore

${Q_1}^3(0.5) = Q_1(0.18) = 0.15$. Since $\left|{Q_1}^3(0.5)\right| = 0.15 < 4$.

Hence α is in the Mandelbrot set .

Remark (3.8)

We will see the relationship between the Mandelbrot set and the Julia set for different α as following

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Corollary (3.9) [6]

Let z be the critical point of $Q_\alpha(z) = \alpha z - \alpha z^2$. If $Q_\alpha{}^n(z) \to \infty$, then $J(Q_\alpha)$ is totally disconnected. On the other hand, $Q_\alpha{}^n(z) \to 0$, so $J(Q_\alpha)$ is connected

Proposition (3.10)

If $\alpha \in \mathbb{M}$ then $J(Q_\alpha)$ is connected and $\alpha \notin \mathbb{M}$ then $J(Q_\alpha)$ is totally disconnected .

Proof :

From Proposition (3.4) and Corollary (3.9), thus $\alpha \in \mathbb{M}$ and then $J(Q_\alpha)$ is connected and on the other hand, $J(Q_\alpha)$ is totally disconnected .

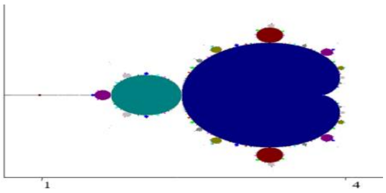

Figure (1) For the value of α

Conclusion

We have $Q_\alpha(z) = \alpha z - \alpha z^2$ and $\mathbb{M}$ is a Mandelbrot set , we introduse the some properties of the Mandelbrot sets of the map $Q_\alpha(z)$ where α is complex constant, and prove that the conjecture the Douady – Hubbard Theorem, also we get as follows:

- $\mathbb{M}$ is symmetric about the real axis.
- If Q_α has an attracting periodic point , then α is in $\mathbb{M}$.
- α is in $\mathbb{M}$ iff $|Q_\alpha{}^n(0.5)| \leq 4$ for each $n \geq 1$. $\mathbb{M}$ is compact subset of $\{|\alpha| \leq 4\}$. So $\mathbb{C} \backslash \mathbb{M}$ is connected.

Acknowledgements:

I would like to thank Iftichar M. T.AL-Shara'a .

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Conflict of interests.

There are non-conflicts of interest.

References

[1] A. G. ,Naoum , , I. M. T. AL-Shara'a and , H.Q.Z. Al Salami, "Some Topological Properties Of Julia Sets of Maps of the form $(\lambda z - \lambda z^2)$", *Journal Of Babylon University* ,Vol. (13) No. 3,2005 .

[2] L. Carleson, and T.W. Gamelin ,*Complex dynamics* ,Springer-Verlag, New York, 1993.

[3] K.J. , Falconer , *Fractal Geometry* , John Wiley & Sons Ltd., England , 1990 .

[4] D., Gulick , *Encounters with chaos* , Mc Graw –Hill ,Inc. U.S.A, 1992 .

[5] A. G. ,Naoum , , I. M. T. AL-Shara'a and , H.Q.Z. Al Salami, "The Geometric Properties Of Julia Sets of Maps of the form $(\lambda z - \lambda z^2)$", *Babylon Journal Of Humanities\ College Of Education*, Vol. (2) No. 3, 2007 .

[6] , R.L , Devaney *An Introduction to Chaotic Dynamical Systems* ,second edition , Addison-Wesley , 1989 .

[7] A. Douady , and J.H., Hubbard , *"On The Dynamics of Polynomial-like mappings"* , Annales Scientifique de l' Ecole Normale Superieur , 18 (1985) pp 287-343 .

[8] A. M. Hubert, G. Jayalalitha and T. Nadu ,, "*Mandelbrot Sets As Complex Polynomial* ", Jour of Adv Research in Dynamical & Control Systems, Vol. 10, 06-Special Issue, 2018.

[9] A. Zireh, "*A Generalized Mandelbrot Set of Polynomials of Type E_d For Bicomplex Numbers*", Georgian Mathematical Journal Volume 15 , No.1, 189–194, 2008.