# Spherical Approximation on Unit Sphere

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## Abstract

In this paper we introduce a Jackson type theorem for functions in $L_p$ spaces on sphere And study on best approximation of functions in $L_p$ spaces defined on unit sphere.

our central problem is to describe the approximation behavior of functions in $L_p$ spaces for $p < 1$ by modulus of smoothness of functions.

**Keywords** : Modulus of smoothness , orthogonal matrices on $R^d$ , approximation of functions in $L_p$ spaces defined on unit sphere .

## الخلاصة

في هذا البحث نقدم نظرية جاكسون للدوال في الفضاءات $L_p$ عندما $0 < p < 1$ على كرات الوحدة و دراسة افضل تقريب للدوال في الفضاءات $L_p$المعرفه على كرات الوحدة مشكلتنا الاساسية هنا هي وصف عملية التقريب لسلوك الدوال في الفضاءات $L_p$, $p < 1$بأستخدام مقاييس معامل النعومة للدالة

**الكلمات المفتاحيه:** مقاييس النعومة , المصفوفات المتعامدة في الفضاء $\mathbf{R}^d$, تقريب الدوال في الفضاءات $L_p$ المعرفة على كرات الوحدة.

We need the following definitions from Ditzian,2008

Definitions 1.1

For $L_p(S^{d-1})$, we denote thespace of functions on the sphere

$(S^{d-1}) = \{x = (x_1, x_2, \dots, x_d): x_1^2 + x_2^2 + \dots + x_d^2 = 1\}$.

Let $f: S^d \to R$,

For functions on $S^{d-1}$ in the function spaces $L_p(S^{d-1})$ , $p < 1$, we define the quasinorm

$$\|f\|_{L_p(S^{d-1})} = \left(\int_{S^d} |f(x_1, x_2, \dots, x_d)|^p \, dx_1 dx_2 \dots dx_d\right)^{\frac{1}{p}}$$

Nawmodulus of smoothness $\omega_r(f,t)_{L_p(S^{d-1})}$ where is recently introduced inDitzian,1999. $\omega_r(f,t)_{L_p(S^{d-1})}$ is given by

$$\omega_r(f,t)_{L_p(S^{d-1})} = \sup\left\{\left\|\Delta_\rho^r f\right\|_{L_p(S^{d-1})} : \rho \in O_t\right\}, t \geqq 0$$

Such that $\Delta_\rho f(x) = f(\rho x) - f(x), \Delta_\rho^r f(x) = \Delta_\rho\left(\Delta_\rho^{r-1} f(x)\right)$

$\Delta_\theta^r f = (TQ - I)^r f$

If $r = 1 \Delta_\theta^1 f = (TQ - I) f$

$TQf(x) = f\big(Q(x)\big), Q \epsilon So(d)$

$O_t = \{\rho \in So(d): \max_{x \in S^{d-1}} [\rho x . x \geq cost\}]$

And $\mathbf{So}(\square)$ is the groupof orthogonal matrices of $d \times d$ real entries with determinant 1

Let $M_0$ be the $d \times d$ (even) matrix given by

$$M\theta = \begin{bmatrix} \begin{matrix} cos\theta sin\theta \\ -sin\theta cos\theta \end{matrix} & & 0 \\ & \ddots & \\ 0 & & \begin{matrix} cos\theta sin\theta \\ -sin\theta cos\theta \end{matrix} \end{bmatrix}$$

Clearly $M_0 = I, (M_\theta)^j = M_{j\theta}$ and $(M_\theta)^{-1} = M_{-\theta}$

$$S_{(\theta,p)}f(x) = \frac{1}{m_\theta}\left(\int_{xy=cos\theta} |f(y)|^p \, d\gamma(y)\right)^{\frac{1}{p}}, \qquad S_\theta I = I,$$

where $d\gamma$ is the measure on the set $\{y \in S^{d-1}: x.y = cos\theta\}$ that it causes by the Lebesgue measure on $S^{\dagger}(d-2)$ ($\{y: x.y = cos\theta\}$ is an isometric map of dilation on $S^{\dagger}(d-2)\}$).
and $m_\theta$ is given by $S_\theta I = I;$
$\tilde{\Delta}$ is the Laplace - Beltrami differential operator given, using the Laplacian operator

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$$

$\tilde{\Delta} f(x) = \Delta F(x)$ for $x \in S^{d-1}$ ,where $F(x) = f\left(\frac{x}{|x|}\right)$

If $f \in L_p(S^{d-1})$ The K- functional of $f$ defined as

$$K_{2\alpha}(f, \tilde{\Delta}, t^{2\alpha})_{L_p(S^{d-1})} \cong \inf\left(\|f - g\|_{L_p(S^{d-1})} + t^{2\alpha} \left\|(-\tilde{\Delta})^{\alpha} g\right\|_{L_p(S^{d-1})} : (-\tilde{\Delta})^{\alpha} g \in L_p(S^{d-1})\right)$$

$$S_\theta f(x) = \frac{1}{m(\theta)} \int_{\{y \in S^{d-1}: x.y = cos\theta\}} f(y) d\gamma(y),$$

$S_\theta I = I, x \in S^{d-1}$

$$m_\ell(u) = \frac{-2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell - j} m(ju)$$

## 2.AuxiliaryResults

In this papper we need the following lemmas.

**Lemma 2.1** Ditzian,2004

For $f \in L_p(S^{d-1}), p < 1$

We have $\int \left( |f(Q^{-1} M_\theta Q_x|^p \, dQ \right)^{1/\square}$

Where $S_{\theta,p}(f) = \frac{1}{m\theta}\left(\int_{xy=cos\theta} |f(y)|^p d\gamma(y)\right)^{1/p}$

**Lemma 2.2**

For $f \in L_p((S)^{d-1}), p < 1$ then

$\|f(\rho x)\|_{L_p(S^{d-1})} = \|f(x)\|_{L_p(S^{d-1})}$, for any $\rho \in So(d)$

**Proof**

Since , ,so that
$\rho \in S$ ,then
$\|f(\rho x)\|_{L_p(S^{d-1})} = \|f\|_{L_p(S^{d-1})}$
and
$\|f(\rho x)\|_{L_p(S^{d-1})} = \|f(x)\|_{L_p(S^{d-1})}$ ■

**Lemma 2 .3**Ditzian,2008

An operation by an element of $so(d)$is an isometry and in most situations under thecondition
$\|f(\rho x) - f(x)\|_{L_p(S^{d-1})} \to 0 as |\rho - I| \to 0$ ,
where $|\rho - \eta|^2 = \max_{x \in S^{d-1}} \left( (\rho x - \eta x).(\rho x - \eta x) \right)$ . (Note that $\max(\rho x.x) \geqq cos$ is equivalent to $|\rho - I| \leqq 2 \left| \frac{\sin t}{2} \right|$)

**Lemma 2.4** Ditzian ,2004
For an integer $\ell$ we have
$$\binom{2\ell}{\ell} + 2\sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell - j} cosj\theta = 4^{\ell} sin^{2l}\frac{\theta}{2} .$$

**Lemma 2.5**Ditzian,2004
$$S_\theta f(x) = \frac{1}{m(\theta)} \int_{\{y \in S^{d-1}:x.y=cos\theta\}} f(y) d\gamma(y), S_\theta I = I,$$
$x \in S^{d-1}$

For $S_{j,\theta} f$ given in $S_{j,\theta} f(x) = \frac{-2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell - j} S_{j\theta} f(x)$
$(S_{j,\theta} f)^\wedge (x) \equiv m_l(2\pi\ell\,|x|) \hat{f}(x)$ and
$$1 - m_l(u) = \frac{2\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \frac{4^l}{\binom{2l}{l}} \int_0^1 \left(\frac{\sin us}{2}\right)^{2\ell} (1 - s^2)^{\frac{d-3}{2}} ds$$
.

**Lemma 2.6**Ditzian ,2004
For $0 < u \leq \pi$ , $0 < C_1 u^{2l} \leq 1 - m_l(u) \leq C_2 u^{2\ell}$ ,
For $u \geq \pi$ , $0 < m_l(u) \leq S_{j,\theta} < 1.$
**Lemma2.7**Ditzian,2004

For
$$\left| \left(\frac{d}{du}\right)^j m_l(u) \right| \leq C_{\ell,j} \left(\frac{1}{1+u}\right)^{\frac{d-1}{2}}$$

**Lemma2.8**

**Proof**

Following the proof of lemma 3.17 step by step in Stein,G.Weiss,1971 we can get the proof ∎

## 3. The Main Results

In this paper we introduce our main results

**Theorem 3.1**

For $f \in L_p(S^{d-1}), p < 1,$ we have

(i) $S_{(\theta,p)} f \in L_p(S^{d-1})$

(ii) $\|S_{(\theta,p)} f\|_{L_p(S^{d-1})} \leq C(p)\|f\|_{L_p(S^{d-1})}$

(iii) $\|S_{(\theta,p)} f - f\|_{L_p(S^{d-1})} \leq C(p)\omega_r(f,\theta)_{L_p(S^{d-1})}$

**Proof of**

(i) Using $\|f(\rho)\|_{L_p(S^{d-1})} = \|f\|_{L_p(S^{d-1})}$

We directly get $S_{(\theta,p)} f \in L_p(S^{d-1})$

(ii) Now we use $\|f(\rho x)\|_{L_p(S^{d-1})} = \|f(x)\|_{L_p(S^{d-1})}$ and hence $\|f(Q^{-1}M_\theta Q_x)\|_{L_p(S^{d-1})} = \|f(x)\|_{L_p(S^{d-1})}$ to obtain

$$\|S_{(\theta,p)} f\|_{L_p(S^{d-1})} = \left\|\int_{so(d)} |f(Q^{-1}M_\theta Q_x)|^{pr} dQ\right\|_{L_p(S^{d-1})}$$

For $Q^{-1}M_\theta Q = \rho$

$$\left(\int_{so(d)} \left| f\left(Q^{-1}M_\theta Q_x\right|^{pr} dQ\right)^{\frac{1}{p}}\right|^{p}\right)^{\frac{1}{\square}})$$

$$= \left(\int_{S^{d-1}} \int_{so(d)} |f(Q^{-1}M_\theta Q_x)|^{pr} dQ\, dx\right)^{\frac{1}{p}}$$

$$\leq C(p)\left(\int_{S^{d-1}} |f(\rho x)|^{pr} dx\right)^{1/p}$$

$$\leq C(p)\|f^r\|_{L_p(S^{d-1})}$$

(iii) The inequality $\|S_\theta f - f\|_{L_p(S^{d-1})} \leq c(p)\omega_r(f,\theta)_{L_p(S^{d-1})}$ follows from to abovewhere

$\Delta_\rho f(x) = f(\rho x) - f(x)$

for $r = 1$ to show that the remainder of

$\|S_\theta f - f\|_{L_p(S^{d-1})} \leq C(p)\omega_r(f,\theta)_{L_p(S^{d-1})}$

we note that

$$\int_{So(d)} f(Q^{-1}M_\theta Q_x) dQ = \int_{So(d)} f(Q^{-1}M_{-\theta} Q_x) dQ$$

$$= \int_{So(d)} f(Q^{-1}M_{-\theta} Q_x^{-1}) dQ$$

and use $\left\|\Delta_\rho^r f\right\|_{L_p(S^{d-1})} = \|(TQ-I)^r\|_{L_p(S^{d-1})}$

$$= \left\|(T-I)^s \sum_{m=0}^{r-s} \binom{r-s}{m} (T-I)^m T^{r-s-m}\right\|_{L_p(S^{d-1})}$$

For $T_\rho f(x) = f(\rho x), \rho = Q^{-1}M_\theta Q$ and $\rho^{-1} = QM_{-\theta}Q^{-1}$ ■

**Theorem 3.2**

If $f \in L_p(S^{d-1}), p \le 1$ and $\|f(\rho x)\|_{L_p(S^{d-1})} = \|f(x)\|_{L_p(S^{d-1})}$ ,for any $\rho \in So(d)$, $\|f(\rho x) - f(x)\|_{L_p(S^{d-1})} \to 0$ as $|\rho - I| \to 0$

where $|\rho-\eta|^2 = \max_{x \in S^{d-1}} \left((\rho x - \eta x).(\rho x - \eta x)\right).$ (Not that $\max(\rho x - x) \ge cos\square$ is equivalent to $|\rho - I| \le 2\left|\frac{\sin t}{2}\right|$ )

then $\left\|C_n^\delta(f,x)\right\|_{L_p(S^{d-1})} \le C(p)\|f\|_{L_p(S^{d-1})}$

**Proof**

In fact $C_n^\delta(f,x) = \int_0^\pi M_n^\delta(\theta) S_\theta(f,x)d\theta$ is known for $f \in L_p(S^{d-1})$

$$\left\|C_n^\delta(f,x)\right\|_{L_p(S^{d-1})} = \left\|\int_0^\pi M_n^\delta(\theta) S_\theta(f,x)d\theta\right\|_{L_p(S^{d-1})}$$

$$\le \left\|\int_0^\pi M_n^\delta(\theta) S_\theta(f,x)d\theta\right\|_{L_1(S^{d-1})}$$

$$\le \int_0^\pi M_n^\delta(\theta)\|S_\theta(f,x)\|_{L_1(S^{d-1})} d\theta$$

$$= \|S_\theta(f,x)\|_{L_1(S^{d-1})}$$

$$= \left(\int_{S^{d-1}} |S_\theta(f,x)|dx\right)$$

$$= \left(\int_{S^{d-1}} |S_\theta|^{1-p+p}\right)^{1-\frac{1}{p}+\frac{1}{p}}$$

$$= \left(\int_{S^{d-1}} |S_\theta|^p |S_\theta|^{1-p}\right)^{\frac{1}{p}} \left(\int_{S^{d-1}} |S_\theta|^{1-p+p}\right)^{1-\frac{1}{p}}$$

$$\le C(p)\left(\int_{S^{d-1}} |S_\theta|^p\right)^{\frac{1}{p}}$$

using $S_{(\theta,p)}f = \frac{1}{m_\theta}\left(\int |f(y)|^p d\gamma(y)\right)^{\frac{1}{p}}$ we get

$\left\|C_n^\delta(f,x)\right\|_{L_p(S^{d-1})} \le C(p)\|f\|_{L_p(S^{d-1})}$ ■

**Theorem 3.3**

Let $f \in L_p(S^{d-1})$ for even $d > 3$ , with $\|f(\rho x)\|_{L_p(S^{d-1})} = \|f\|_{L_p(S^{d-1})}$ for each $\rho \in So(d)$ and $\|f(\rho x) - f(x)\|_{L_p(S^{d-1})} \to 0$ as $|\rho - I| \to 0$ then

$$\left\|C_n^\delta - f\right\|_{L_p(S^{d-1})} \le C(p,\theta)\omega_r(f,\mu)_{L_p(S^{d-1})}.\omega_r(f,\mu)_\infty.\omega_r(f,\mu)_\infty^{\frac{-1}{p}}$$

**Proof**

$$\|C_n^\delta - f\|_{L_p(S^{d-1})} \leq \left\| \int_0^\pi \mu_n^\delta(\theta) \left| [(S]_\theta f - f) d\theta \right| \right\|_{L_1(S^{d-1})}$$

$$\leq \int_0^\pi \mu_n^\delta(\theta) \|S_\theta(f) - f\|_{L_1(S^{d-1})} d\theta$$

Then using the fact that $\int_0^\pi M_n^\delta(\theta) d\theta = 1$ We get

$$\|C_n^\delta - f\|_{L_p(S^{d-1})} \leq \left(\int_{S^{d-1}} |S_\theta(f) - f|^{p-p+1}\, dx\right)^{\frac{1}{p}-\frac{1}{p}+1}$$

$$\leq \left(\int_{S^{d-1}} |S_\theta(f) - f|^p |S_\theta(f) - f|^{p+1}\right)^{\frac{1}{p}} \left([\int_{S^{d-1}} |S_\theta(f) - f|\, d\theta \right)]^{1-\frac{1}{p}}$$

$$\int_{S^{d-1}} |S_\theta(f) - f|^p \, \|S_\theta(f) - f\|_\infty \|S_\theta(f) - f\|_\infty$$

$$\leq C(p,\theta) \|f - s_\theta(f)\|_{Lp(S^{d-1})} \omega_r(f,\mu)_\infty \omega_r(f,\mu)_\infty^{\frac{-1}{p}}$$

$$\leq C(p,\theta) \omega_r(f,\mu)_p \, \omega_r(f,\mu)_\infty \omega_r(f,\mu)_\infty^{\frac{-1}{p}} \blacksquare$$

**Theorem3.4**

If$\in L_p(S^{d-1})$ , then

$$\mathrm{E_n}([f)]_{L_p(S^{d-1})} \leq \mathbf{c(p)} K_{2\propto}(\mathbf{f}, \tilde{\Delta}, n^{-2\propto})_{L_p(S^{d-1})}, \propto > 0$$

**proof**

$$E_n(f) \leq \|f - g\|_{L_p(S^{d-1})}$$

$$\leq c(p) \|f - g\|_{L_p(S^{d-1})} + \mathrm{t}^{2\propto} \left\| \left(-\tilde{\Delta}\right)^\propto \mathbf{g} \right\|_{L_p(S^{d-1})} : \left(-\tilde{\Delta}\right)^\propto \mathbf{g} \in L_p(S^{d-1})$$

$$\leq \mathbf{c(p) K_{2\propto}(f, \tilde{\Delta}, n^{-2\propto})}_{L_p(S^{d-1})} \quad \blacksquare$$

**Theorem 3.5**

If $f \in L_p(S^{d-1})$with even $d > 3$ with

$\|f(\rho)\|_{L_p(S^{d-1})} = \|f(x)\|_{L_p(S^{d-1})}$for any $\rho \in So(d)$

$\|f(\rho) - f(x)\|_{L_p(S^{d-1})} \to 0 \ as|\rho - I| \to 0 \ for\theta \leq \frac{\pi}{2\ell}$

Then
$$\left\| f + \frac{2}{\binom{2\ell}{\ell}} \sum_{j=i}^{\ell} (-1)^j \binom{2\ell}{\ell} S_{j,\theta} f \right\|_{L_p(S^{d-1})} \cong K_{2\ell}(f, \tilde{\Delta}, \theta^{2\ell})_{L_p(S^{d-1})}$$

**Proof**

by definition of the K—functional with

$$\left\| f + \frac{2}{\binom{2\ell}{\ell} \Sigma (-1)^j \binom{2\ell}{\ell} S_{j,\theta} f} \right\|_{L_p(S^{d-1})}$$

$$\|f - S_j \theta\|_{L_p(S^{d-1})} + t\|-\tilde{\Delta} S_\theta f\|_{L_p(S^{d-1})})$$

We realize the operator$\eta_{\propto\theta}(f)$ using the function$\eta(x)$ satisfying $\eta(x) \epsilon C^\infty(R_+)$,

for$0 \leq x \leq 1$, and$\eta(x) = 0$ for$x \geq 2$ the operator$\eta_{\propto\theta}(f)$ is given by

$$\eta_{\propto\theta}(f) = \sum_{k=0}^\infty \eta(\propto \theta k) P_k(f) \quad ,\text{where} \quad f \sim \sum_{k=0}^\infty P_k(f)$$

for

and the definition of the K- functional $K_\ell\left(f,\Delta,t^{2\ell}\right)_{L_p(S^{d-1})}$ we just have to show for all $f\in L_p(R^d)$ andsome fixed $\propto>0$

(as $K_\ell\left(f,\Delta,t^{2\ell}\right)_{L_p(S^{d-1})}\approx K_\ell(f,\Delta,\propto^{-2\ell}t^{2\ell})_{L_p(S^{d-1})}$

displaythat

(i) $\left\|f-S_{j,\theta}f\right\|_{L_p(S^{d-1})}\geq C_1\left\|f-\eta_{\propto\square/t}f\right\|_{L_p(S^{d-1})}$

(ii) $\left\|f-S_{j,\theta}f\right\|_{L_p(S^{d-1})}\geq C_2\,t^{2\ell}\left\|\Delta^\ell\eta_{\propto\square/t}f\right\|_{L_p(S^{d-1})}$

Toprove i- it sufficient to show

$f-S_{j,\theta}f\|_{L_p(S^{d-1})})$

$\leq C_4\left\|f-S_{j,\theta}f\right\|_{L_p(S^{d-1})}$ (1)

since, $as\eta_{1/t}andS_{j,\theta}$ arebounded multiplier operators on $L_p$ ($R^d$),we have

$$\left\|\left(I-\eta_{\propto\square/t}\right)\left(I+S_{j,\theta}+S_{j,\theta}^2+S_{j,\theta}^3+S_{j,\theta}^4\right)\left(f-S_{j,\theta}f\right)\right\|_{L_p(S^{d-1})}\leq C_5\left\|f-S_{j,\theta}f\right\|_{L_p(S^{d-1})}$$

,

where $I$ is the identity operator. To prove(1) wehave to show that

$$\phi(u)=\frac{(1-\eta(u/\propto))m_\ell(u)^5}{1-m_\ell(u)}$$

is abounded multiplier on $L_p\left(R^d\right)or\|D^\vee\phi(u)\|_{L_p(S^{d-1})}\leq\frac{C}{[(1+|u|)|\square]^{d+\propto}},\propto>0$

(At least for $|v|\leq d+1$, but here that restriction does not metter ) while the above iswell known and used manytimes.We show it to below to help the reader.For $\Phi^\vee(\mathbf{x})$ given by $\Phi^\vee(\mathbf{x})=\int_{\mathbf{R}^d}\square\,\Phi(\mathbf{y})\mathbf{e}^{2\pi\square\mathbf{xy}}dy$

Which may be considered as aFourier transform, and prove the lemma2.8

which implies the sufficiency ofshowing that $\|D^\vee\phi(u)\|_{L_p(S^{d-1})}\leq\frac{C}{(1+|u|)^{d+2}}$ for $\propto>0\ and|v|\leq d+1$ .We observation that for $|u|\leq 1,\phi(u)=0$. for$|u|\geq 1$ ,then using Lemma2.7, we recall that the multipliers,and we get that

$$\|D^\vee\phi(u)\|_{L_p(S^{d-1})}\leq C(v)\left(\frac{1}{1+|u|}\right)^{5\left(\frac{d-1}{2}\right)}=C(v)\left(\frac{1}{1+|u|}\right)^{d+\frac{3}{2}d-\frac{5}{2}}$$

and for

The proof of (ii) is step by step of the relation (3.9) of Ditzian,2004■

**Theorem 3.6**

For$f\in L_p(S^{d-1})$ for $d=2n,n=2,3,4,\dots$ with

$\|f(\rho x)\|_{L_p(S^{d-1})} = \|f(x)\|_{L_p(S^{d-1})}$ for any $\rho \in So(d)$
$\|f(\rho x) - f(x)\|_{L_p(S^{d-1})} \to 0$ as $|\rho - I| \to 0$ ,
where $|\rho - \eta|^2 = \max_{x \in S^{d-1}} \left((\rho x - \eta x).(\rho x - \eta x)\right)$ (Note that $\max(\rho x.x) \geq cos\square$ is equivalent to $|\rho - I| \leq 2\left|sin\frac{t}{2}\right|$) then

$E_n(f)_{L_p(S^{d-1})} \leq c(p)\omega^2\left(f,\frac{1}{n}\right)_{L_p(S^{d-1})}$

**Proof**

We have $E(n)_{L_p(S^{d-1})} \leq C(p)K_{2\alpha}(f,\tilde{\Delta},n^{-2\alpha})_{L_p(S^{d-1})}$ by

$$\left\|f + \frac{2}{\binom{2\ell}{\ell}}\sum_{j=1}^{\ell}(-1)^j\binom{2\ell}{\ell j}S_{j\theta}f\right\|_{L_p(S^{d-1})} \approx K_{2\ell}(f,\tilde{\Delta},\theta^{2\ell})_{L_p(S^{d-1})}$$

We have

$$E_n(f)_{L_p(S^{d-1})} \leq C(p)K_{2\alpha}(f,\tilde{\Delta},n^{-2\alpha})_{L_p(S^{d-1})} \leq C(p)\left\|f + \frac{2}{\binom{2\ell}{\ell}\sum(-1)^j\binom{2\ell}{\ell-j}S_{j\theta}f}\right\|_{L_p(S^{d-1})}$$

And by $\|S_\theta f - f\|_{L_p(S^{d-1})} \leq C(p)\omega^2(f,\theta)$
we get
$E_n(f)_{L_p(S^{d-1})} \leq C(p)\omega^2(f,\theta)_{L_p(S^{d-1})}$ ■

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