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JOURNAL OF UNIVERSITY OF BABYLON
For Pure and Applied Sciences (JUBPAS)
Special Issue 2: Proceedings of Al-Muthanna University's First Conference/Directing Scientific Research Towards Achieving Sustainable

Vol, 33; S.2 | 2025

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ISSN: 2312-8135 | Print ISSN: 1992-0652

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Applications of Zalcman Conjecture for Analytic Functions Linked to Cosine Function

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تطبيقات على حدسية زالكمان للدوال التحليلية المرتبطة بدالة الجيب تمام

قاسم علي شاكر

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Published: **31/8/2025**

ABSTRACT

In this paper, we investigate applications of the Zalcman conjecture for analytic functions associated with the cosine function. We introduce and analyze a new class of normalized analytic functions within the open unit disk, leveraging subordination principles. The coefficient estimates for this class are derived, and explicit upper bounds for the first five coefficients are established. Furthermore, we prove the validity of the Zalcman conjecture for this function class by obtaining an upper bound for the Zalcman functional. Our results contribute to the broader study of analytic function theory and extend existing findings on function classes linked to trigonometric functions.

Key words: Analytic function, subordination, Cosine function, Zalcman conjecture.

1. INTRODUCTION

The study of analytic functions and their subclasses has been a fundamental area in complex analysis, particularly in the geometric function theory. One important conjecture in this field is the Zalcman conjecture, which provides bounds on coefficients of analytic and univalent functions.

This paper extends previous results on classes of analytic functions associated with trigonometric functions, specifically focusing on the cosine function. The importance of this research lies in its potential applications in mathematical physics, engineering, and applied sciences, where cosine-based functions frequently model periodic phenomena.

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We define a new subclass of normalized analytic functions associated with the cosine function and establish coefficient estimates and subordination results. Furthermore, we validate the Zalcman conjecture for this class by obtaining an upper bound for the relevant functional.

This work expands upon recent developments in subordination theory and function coefficient estimates, offering a refined approach that strengthens existing mathematical frameworks.

Define $\mathcal{A}$ as the class of all normalized analytic functions f within the open unit disk $\mathbb{U} = \{z: z \in \mathbb{C}, |z| < 1\}$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (z \in \mathbb{U}). \tag{1.1}$$

In the next, an analytic function f is subordinate to another g if there exists analytic function $\omega: \mathbb{U} \longrightarrow \mathbb{U}$ with $\omega(0) = 0$ satisfying $f(z) = g(\omega(z))$ $(z \in \mathbb{U})$, and written $f \prec g$.

Should the function g be univalent in $\mathbb{U}$, the following equivalence holds:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Ma and Minda [1] defined a class of starlike and convex functions by using the method of subordination and studied classes $S^*(\phi)$ and $C^*(\phi)$ which is defined by

$$S^*(\phi) = \left\{f \in H: \frac{zf'(z)}{f(z)} \prec \phi(z), z \in \mathbb{U}\right\},$$

and

$$C^*(\phi) = \left\{f \in H: 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z), z \in \mathbb{U}\right\}.$$

Definition (1.1). Let us define a new family $Q_\rho(cos)$, with $0 \le \rho \le 1$ connected with the cosine function, as follows:

$$Q_\rho(cos) = \left\{f \in \mathcal{A}: \frac{\rho z^2 f''(z) + zf'(z)}{\rho z f'(z) + (1-\rho) f(z)} \prec cos\, z = \Psi(z)\right\} \tag{1.2}$$

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Remark (1.1).

(i) If we take $\rho = 0$ in Definition (1.1), the class $\mathcal{Q}_\rho(cos)$ reduces to the class S^*_{cos} which was studied recently by Bano and Raza (see[2]).

(ii) If we take $\rho = 1$ in Definition (1.1), the class $\mathcal{Q}_\rho(cos)$ reduces to the class S^c_{cos} which was studied recently by Marimuthu et al (see[3]).

The cosine function exhibits different behaviors along the real and imaginary axes:

- Along the real axis $(y = 0)$, it behaves like the standard cosine function.
- Along the imaginary axis $(x = 0)$, it exhibits exponential growth and decay.

Here are the 3D plots of the cosine function in the complex plane:

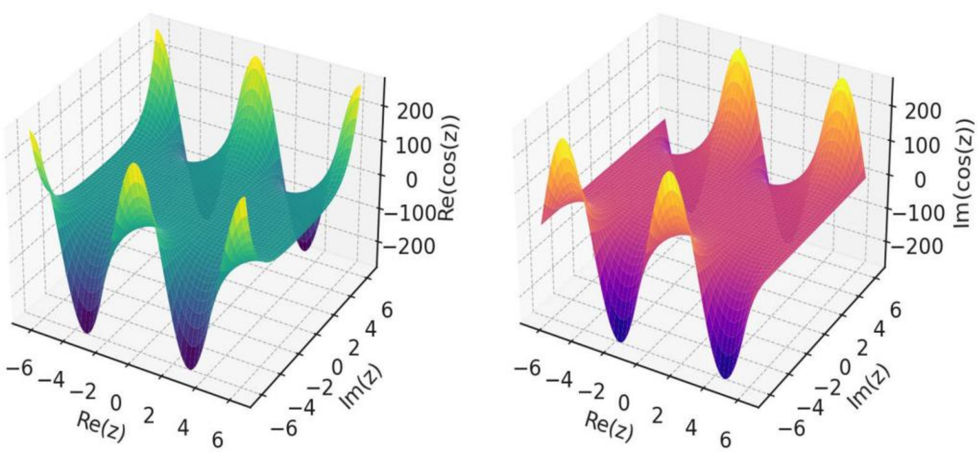

Figure 1. (a) Real part of cos(x + iy) ; (b) Imaginary part of cos(x + iy)

- The left plot illustrates the real component of $cos\ z$.
- The right plot illustrates the imaginary component of $cos\ z$.

Lemma (1.1). Consider $p(z) = 1 + t_1 z + t_2 z^2 + \cdots \in \mathcal{P}$, while $\mathcal{P}$ denotes the collection of all functions p that are analytic in $\mathbb{U}$ as well as fulfill $Re\{p(z)\} > 0$ for $z \in \mathbb{U}$. Subsequently

$$|t_n| \leq 2, \text{ for } n \geq 1 \tag{1.3}$$

$$|t_{i+j} - \mu t_i t_j| \leq 2\, max\{1;\ |1 - 2\mu|\} \tag{1.4}$$

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and for any number $\eta \in \mathbb{C}$, we have

$$\left|t_2 - \eta t_1^2\right| \leq 2\, max\{1;\ |2\eta - 1|\} \tag{1.5}$$

Lemma (1.2) [4]. Consider $p(\mathfrak{z}) = 1 + t_1\mathfrak{z} + t_2\mathfrak{z}^2 + \cdots \in \mathcal{P}$, while $\mathcal{P}$ denotes the collection of all functions p that are analytic in $\mathbb{U}$ and satisfy $Re\{p(\mathfrak{z})\} > 0$ for $\mathfrak{z} \in \mathbb{U}$. Then

$$\left|\alpha t_1^3 - \beta t_1 t_2 + \gamma t_3\right| \leq 2|\alpha| + 2|\beta - 2\alpha| + 2|\alpha - \beta + \gamma| \tag{1.6}$$

2. LITERATURE REVIEW AND RELATED WORK

Several studies have explored function classes associated with trigonometric functions. [1] provided a unified treatment of special univalent function classes through subordination principles. More recently, [2] studied starlike functions linked to the cosine function, while [3] examined coefficient estimates for related function classes. Additionally, [4] investigated the third Hankel determinant for sine-related function subsets.

However, the Zalcman conjecture in the context of cosine-associated analytic functions remains underexplored. Prior research on this conjecture has primarily focused on starlike and convex function classes, with contributions from [5,6,7]. Our work builds upon these findings by explicitly addressing the Zalcman functional in a new class of functions associated with the cosine function.

3. COEFFICIENTS ESTIMATES FOR THE CLASS $\mathcal{Q}_\lambda(cos)$

This section examines the coefficients of the functions within the class $\mathcal{Q}_\rho(cos)$ along with determining the upper bounds for the initial five coefficients.

Theorem (3.1). Assuming $f \in \mathcal{Q}_\rho(cos)$ conforms to the structure (1.1), then

$$|a_2| = 0,$$

$$|a_3| \leq \frac{2}{(1 + 2\rho)},$$

$$|a_4| \leq \frac{1}{3(1 + 3\rho)},$$

$$|a_5| \leq \frac{167}{96(1 + 4\rho)}.$$

Proof. if f belongs to $\mathcal{Q}_\rho(cos)$, then there exists a function u that is analytic in $\mathbb{U}$ and has the requirements $u(0) = 0$ with $|u(\mathfrak{z})| < 1$ for every $\mathfrak{z} \in \mathbb{U}$, in which case

$$\frac{\rho \mathfrak{z}^2 f''(\mathfrak{z}) + \mathfrak{z} f'(\mathfrak{z})}{\rho \mathfrak{z} f'(\mathfrak{z}) + (1-\rho) f(\mathfrak{z})} = \Psi\big(u(\mathfrak{z})\big) + \cos u(\mathfrak{z}), \qquad (\mathfrak{z} \in \mathbb{U}). \tag{3.1}$$

Since f is of the form (1.1), it follows that

$$\frac{\rho \mathfrak{z}^2 f''(\mathfrak{z}) + \mathfrak{z} f'(\mathfrak{z})}{\rho \mathfrak{z} f'(\mathfrak{z}) + (1-\rho) f(\mathfrak{z})} = 1 + (1+\rho) a_2 \mathfrak{z} + [2(1+2\rho)a_3 - (1+\rho)^2 a_2^2]\mathfrak{z}^2$$

$$+[3(1+3\rho)a_4 - 3(1+\rho)(1+2\rho)a_2 a_3 + (1+\rho)^3 a_2^3]\mathfrak{z}^3$$

$$+[4(1+4\rho)a_5 - 4(1+\rho)(1+3\rho)a_2 a_4 - 2(1+2\rho)^2 a_3^2$$

$$+2(1+2\rho)(1+\rho)^2 a_3 a_2^2 - (1+\rho)^4 a_2^4]\mathfrak{z}^4. \tag{3.2}$$

Assume that

$$p(\mathfrak{z}) = \frac{1+u(\mathfrak{z})}{1-u(\mathfrak{z})} = 1 + t_1 \mathfrak{z} + t_2 \mathfrak{z}^2 + t_3 \mathfrak{z}^3 + \cdots,$$

we obtain that $p \in \mathcal{P}$ and

$$u(\mathfrak{z}) = \frac{p(\mathfrak{z}) - 1}{p(\mathfrak{z}) + 1} = \frac{t_1 \mathfrak{z} + t_2 \mathfrak{z}^2 + \cdots}{2 + t_1 \mathfrak{z} + t_2 \mathfrak{z}^2 + \cdots}.$$

On the other side,

$$\cos u(\mathfrak{z}) = 1 - \frac{t_1^2}{8}\mathfrak{z}^2 + \left(-\frac{t_1 t_2}{4} + \frac{t_1^3}{8}\right)\mathfrak{z}^3 + \left(-\frac{35 t_1^4}{384} - \frac{t_2^2}{8} + \frac{3 t_2 t_1^2}{8} - \frac{t_1 t_3}{4}\right)\mathfrak{z}^4 + \cdots. \tag{3.3}$$

By equating the appropriate coefficients of (3.2) and (3.3), we derived

$$a_2 = 0, \tag{3.4}$$

$$a_3 = -\frac{t_1^2}{2(1+2\rho)}, \tag{3.5}$$

$$a_4 = -\frac{t_1}{12(1+3\rho)}\left(t_2 - \frac{t_1^2}{2}\right), \tag{3.6}$$

$$a_5 = -\frac{35 t_1^4}{1536(1+4\rho)} - \frac{t_1}{4(1+4\rho)}\left(\frac{t_3}{4} - \frac{3 t_1 t_2}{8} + \frac{3 t_1^3}{2}\right) - \frac{t_2^2}{32(1+4\rho)}, \tag{3.7}$$

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By take the absolute value of (3.5), we have

$$|a_3| = \frac{|t_1|^2}{2(1+2\rho)},$$

and from (1.3), we get

$$|a_3| \leq \frac{2}{(1+2\rho)}.$$

Applying (1.5) to (3.6), we get

$$|a_4| \leq \frac{1}{3(1+3\rho)}.$$

And,

$$|a_5| = \left| -\frac{35t_1^4}{1536(1+4\rho)} - \frac{t_1}{4(1+4\rho)}\left(\frac{t_3}{4} - \frac{3t_1t_2}{8} + \frac{3t_1^3}{2}\right) - \frac{t_2^2}{32(1+4\rho)} \right|,$$

By applying the triangle inequality we get

$$|a_5| \leq \frac{35|t_1|^4}{1536(1+4\rho)} + \frac{|t_1|}{4(1+4\rho)}\left|\frac{t_3}{4} - \frac{3t_1t_2}{8} + \frac{3t_1^3}{2}\right| + \frac{|t_2|^2}{32(1+4\rho)}.$$

Now, from (1.3) and Lemma (1.2) with $\alpha = \frac{1}{2}, \beta = \frac{3}{8}$ and $\gamma = \frac{1}{4}$, we get

$$|a_5| \leq \frac{167}{96(1+4\rho)}.$$

Consequently, this completes the proof.

By assigning $\rho = 0$ in Theorem (3.1), we obtain the subsequent corollary:

Corollary (3.1). If $f \in \mathcal{Q}_0(cos) := S^*_{cos}$ has the form (1.1), then

$|a_2| = 0,$

$|a_3| \leq 2,$

$|a_4| \leq \frac{1}{3},$

$|a_5| \leq \frac{167}{96}.$

By putting $\rho = 1$ in Theorem (3.1), we obtain the subsequent corollary:

Corollary (3.2). Consider $f \in \mathcal{Q}_1(cos) := S^c_{cos}$. Then

$|a_2| = 0,$

$|a_3| \leq \frac{2}{3},$

$|a_4| \leq \frac{1}{12},$

$|a_5| \leq \frac{167}{480}.$

4. THE ZALCMAN CONJECTURE ESTIMATE FOR CLASS $\mathcal{Q}_\lambda$(cos)

In 1960, Zalcman conjectured that for functions $f \in S$ of the form (1.1), the coefficients fulfill the inequality:

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2, n \geq 2.$$

Recently, the Zalcman functional has garnered considerable attention from scholars (see, for instance, [5–10]). Moreover, beyond its study in the context of starlike and convex functions, this functional can also be effectively linked with the class of bi-univalent functions, opening new directions for establishing coefficient estimates and determinant bounds in this setting (see [11–14]).

We provide an upper bound for the Zalcman functional within the class $\mathcal{Q}_\rho(cos)$ for $n = 3$, therefore demonstrating the validity of the Zalcman conjecture in this instance.

Theorem (4.1). Assuming $f \in \mathcal{Q}_\rho(cos)$, then

$$|a_n^2 - a_{2n-1}| \leq \frac{668\rho^2 + 2204\rho + 551}{96(1+4\rho)(1+2\rho)^2}. \quad (4.1)$$

Proof. From (3.5) and (3.7), we get

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Vol, 33; S.2 | 2025

$$a_3^2 - a_5 = \frac{t_1^4}{4(1+2\rho)^2} + \frac{35t_1^4}{1536(1+4\rho)} + \frac{t_1}{4(1+4\rho)}\left(\frac{t_3}{4} - \frac{3t_1t_2}{8} + \frac{3t_1^3}{2}\right) + \frac{t_2^2}{32(1+4\rho)},$$

By take the absolute value of the above relation and from the triangle inequality, we have

$$\left|a_3^2 - a_5\right| = \frac{t_1^4}{4(1+2\rho)^2} + \frac{35|t_1|^4}{1536(1+4\rho)} + \frac{|t_1|}{4(1+4\rho)}\left|\frac{t_3}{4} - \frac{3t_1t_2}{8} + \frac{3t_1^3}{2}\right| + \frac{|t_2|^2}{32(1+4\rho)},$$

By using (1.3) of Lemma (1.1) and (1.6) of Lemma (1.2), we obtain (4.1).

Corollary (4.1). If $f \in Q_0(cos) := S_{cos}^*$ has the form (1.1), then

$$|a_n^2 - a_{2n-1}| \le \frac{551}{96}.$$

Corollary (4.2). If $f \in Q_1(cos) := S_{cos}^c$ has the form (1.1), then

$$|a_n^2 - a_{2n-1}| \le \frac{3423}{4320}.$$

CONCLUSIONS

This paper examines the applications of the Zalcman conjecture for analytic functions related to the cosine function. We presented and examined a novel category of normalized analytic functions in the open unit disc, employing the principle of subordination. By doing thorough coefficient analysis, definitive upper bounds for the initial five coefficients were derived, offering significant insights into the framework of this family of functions.

Furthermore, we established an upper bound for the Zalcman functional, confirming the validity of the Zalcman conjecture within the defined function class. These findings contribute to the broader field of analytic function theory by extending known results on function classes linked to trigonometric functions.

Future research could explore the generalization of these results to more complex function classes, particularly those associated with other transcendental functions. Additionally, investigating connections between the Zalcman conjecture and other extremal problems in geometric function theory may lead to further theoretical advancements.

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Conflict of interest.
There are non-conflicts of interest.

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