ARTICLE
JOURNAL OF UNIVERSITY OF BABYLON
For Pure and Applied Sciences (JUBPAS)
Vol.33; No.3.| 2025

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

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A Fractional Cubic Spline Approach for Solving Models of Dynamical System

Karwan S. Mohammed[1] , Faraidun K. Hamasalh[2]

[1]Department of Mathematics, College of Education, University of Sulaimani, Kurdistan Region
[1]karwansalim8787@gmil.com

[2]Bakrajo Technical Institute, Sulaimani Polytechnic University, Kurdistan Region, IRAQ
[2]faraidun.hamasalh@univsul.edu.iq

طريقة سبلاين المكعبة الكسرية لحل نماذج النظام الديناميكي

كاروان سليم محمد[1]، فريدون قادر حمەصالح[2]

[1]قسم الرياضيات، كلية التربية، جامعة السليمانية، إقليم كردستان، العراق
2معهد بكرجو التقني، جامعة السليمانية التقنية، إقليم كردستان، العراق

Accepted: /20257/17 **Published:** 30 /9/2025

ABSTRACT

This article presents a novel numerical approach for solving dynamic system models using fractional cubic splines. The proposed method leverages the smoothness and flexibility of fractional cubic splines to approximate solutions of differential equations. By constructing fractional cubic polynomials, we develop a numerical scheme that ensures high accuracy and stability in capturing the complex behaviors inherent in dynamic systems. The effectiveness of the method is demonstrated through numerical experiments on benchmark problems, showcasing its superiority over traditional spline-based techniques in terms of convergence and computational efficiency. The results highlight the potential of fractional cubic splines as a robust tool for the numerical analysis of dynamic systems.

MSC 2010 Classifications: 26A33, 11S82, 65Dxx, 97Nxx.

Keywords: Fractional derivatives, dynamic systems, convergence analysis, error estimation.

1. INTRODUCTION AND PRELIMINARIES

Systems of differential equations are prevalent in many disciplines, including biology, engineering, and physics, since they represent intricate dynamic systems for which conventional analytical solutions are frequently challenging to obtain. As a result, numerical techniques have emerged as crucial resources for estimating solutions to these problems. These dynamic systems also include well-known models of disease propagation, which are widely studied to understand and predict complex behaviors in real-world scenarios. The primary numerical techniques are the topic of this study fractional spline [1-3] .

Fractional splines have been effectively used to solve differential equations, which are often challenging due to their non-local properties. The Caputo fractional derivative is commonly applied in these contexts to ensure accurate solutions [4, 5] . Numerical methods based on fractional cubic splines have been developed to address both linear and nonlinear fractional equations, showcasing their versatility and effectiveness [5, 6] . Numerical Experiments Empirical tests validate the effectiveness of these methods, indicating their potential for widespread application in engineering and physics [7].

The aim of the study is to derive a new law from coefficients that strengthens the understanding of differential equations, especially in the context of dynamical systems [8, 9], we propose modifications to the existing fractional spline interpolation method to improve its computational efficiency and accuracy. Our approach introduces an optimized selection of interpolation nodes and fractional derivative approximations [10, 11] to enhance stability while reducing computational cost. Additionally, we refine the interpolation formulation to ensure better performance for a broader class of fractional differential equations. Numerical experiments of the model describe typhoid fever as such, and its parts are identified. Six human compartments and one pathogen compartment make up the aforementioned model, are conducted to validate the effectiveness of our method, demonstrating superior accuracy compared to existing techniques [12].

The various definitions of the fractional derivative that we employed in our work will be covered in this section. Various techniques are employed to define. The Riemann-Liouville and Caputo derivatives are the most widely used fractional derivatives.

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1.1 Riemann-Liouville Fractional Derivative:

The Riemann-Liouville fractional derivative [13, 14] of order α (where α > 0) is defined as:

$$D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha-1} f(t)\, dt,$$

1.2 Caputo Fractional Derivative:[13, 14]

The Caputo fractional derivative of order α is given by:

$$^{C}D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)\, dt.$$

1.3 Fractional Taylor Series:

The fractional Taylor series [15] extends the classical Taylor series to fractional orders, allowing function approximations in terms of fractional derivatives. The expansion of a function f(x) around a point x_0 is given by:

$$f(x) = \sum_{n=0}^{\infty} \frac{D^{\alpha} f(x_0)}{\Gamma(n\alpha+1)} (x-x_0)^{n\alpha}.$$

1.4 Maximum Norm [16]

The set of all bounded functions $f: x \to R$ is denoted by (x) . Given its numerous good qualities, including being algebra over R and a super space for a number of other well-known spaces, this set is deserving of the name space. The uniform norm of (x) , often known as the sup-norm, is defined as follows:

$$\| f \|_{\infty} = \sup_{x \in \Omega} |f(x)|.$$

2. MODEL FORMULATION AND ANALYSIS

The research focuses on the mathematical formulation of a fractional polynomial, specifically

$$s(x) = a + b(x-x_i) + c(x-x_i)^2 + d(x-x_i)^{\frac{7}{3}} + e(x-x_i)^3 \qquad (2.1)$$

This is designed to satisfy specific conditions at given points, and the polynomial will serve as a foundation for analyzing differential equation systems within dynamic models. The study aims to derive a new law from the coefficients a, b, c, d, and e that will enhance the understanding of fractional differential equations, particularly in the context of dynamic systems [17-19]. The coefficients will be analyzed to establish a new law that can be applied to differential equations. This approach is supported by existing literature on differential equations, which emphasizes the importance of variational methods in finding solutions. While the focus is on fractional polynomials, it is essential to consider the limitations of such models, as they may not capture all dynamics present in more complex systems. The interplay between fractional and classical derivatives remains an area of ongoing research, suggesting that further exploration is necessary to fully understand their implications in dynamic modeling [20-22].

2.1 Polynomial Structure and Conditions:

Fractional spline interpolation is a method that approximates a function by considering both function values and derivative values at given points. Given a set of distinct points, along with function values and derivative values, the spline interpolation polynomial is constructed to satisfy the following conditions:

$$\begin{cases} S(x_i) = y_i & S(x_{i+1}) = y_{i+1} \\ S'(x_i) = {y'}_i & S''(x_i) = {y''}_i \\ & S^{\left(\frac{7}{3}\right)}(x_i) = y^{\left(\frac{7}{3}\right)}{}_i \end{cases} \tag{2.2}$$

In this work, we develop a numerical method based on the fractional spline interpolation formula to solve some models of dynamic system as in [12]. The approach builds on the classical spline interpolation method but extends it to accommodate fractional calculus, particularly fractional derivatives, to improve the accuracy of the numerical solution. The method involves several steps, including interpolation node selection, approximation of fractional derivatives, and the construction of the corresponding spline interpolating polynomial. The following steps outline the methodology in detail.

Theorem 2.1: let $s_i(x)$ be the fractional spline interpolating polynomial defined in equation (2.1), can be shown that the spline function is unique.

Proof:

Now find the first and second fractional derivatives for s(x)

$$\begin{cases} s'(x) = b + 2c(x - x_i) + \frac{7}{3}d(x - x_i)^{\frac{4}{3}} + 3e(x - x_i)^2 \\ s''(x) = 2c + \frac{28}{9}d(x - x_i)^{\frac{1}{3}} + 6e(x - x_i) \\ s^{\frac{7}{3}}(x) = \frac{28}{27}\Gamma\left(\frac{1}{3}\right)d + \frac{9}{\Gamma\left(\frac{2}{3}\right)}e(x - x_i)^{\frac{2}{3}} \end{cases} \tag{2.3}$$

Now from polynomial structure and condition in equation (2.2), $s(x_i) = y_i$, and equation (2.1), we get:

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$$s(x_i) = a \,,\, y_i = a \tag{2.4}$$

Also from $S'(x_i) = y'_i$ in equation (2.4) and equation (2.3), we obtain

$$s'(x_i) = b, \tag{2.5}$$

Also from $S''(x_i) = y''_i$, and equation (2.4), we get:

$$c_i = \frac{y''_i}{2} \tag{2.6}$$

Now from fractional derivative $s^{\frac{7}{3}}(x)$ in equation (2.4) and $S^{(\frac{7}{3})}(x_i) = y^{(\frac{7}{3})}{}_i$ in equation(2.2), we get :

$$s^{\frac{7}{3}}(x) = \frac{28}{27}\Gamma\left(\frac{1}{3}\right) d + \frac{9}{\Gamma\left(\frac{2}{3}\right)} e(x_i - x_i)^{\frac{2}{3}} \text{, and } \frac{27\, y^{(\frac{7}{3})}{}_i}{28\,\Gamma\left(\frac{1}{3}\right)} = d. \tag{2.7}$$

The interval $[a, b]$ is divided into n equal subintervals using the grid points $x_i = a + ih$; $i = 0,1, \ldots\ldots, n$ where$h = (b - a)/n$, in order to create the spline approximation to (2.1). Examine the following limitation. Si of S for every $[x_i, x_{i+1}]$ subinterval, $= 0,1, \ldots\ldots, n$. now from $S(x_{i+1}) = y_{i+1}$ and equation (2.1) we get :

$$S(x_{i+1}) = a + b(x_{i+1} - x_i) + c(x_{i+1} - x_i)^2 + d(x_{i+1} - x_i)^{\frac{7}{3}} + e(x_{i+1} - x_i)^3$$

Using the conditions of equation (2.3), we obtain

$$e = \frac{y_{i+1} - y_i}{h^3} - \frac{y'_i}{h^2} - \frac{y''_i}{2h} - \frac{27\, y^{(\frac{7}{3})}{}_i}{28\,\Gamma\left(\frac{1}{3}\right) h^{\frac{2}{3}}} \tag{2.8}$$

The coefficients in equation (2.1) have the following form:

$$\begin{cases} a_i = y_i \,, \\ b_i = y'_i \\ c_i = \frac{y''_i}{2} \\ d_i = \frac{27\, y^{(\frac{7}{3})}{}_i}{28\,\Gamma\left(\frac{1}{3}\right)} \\ e_i = \frac{y_{i+1} - y_i}{h^3} - \frac{y'_i}{h^2} - \frac{y''_i}{2h} - \frac{27\, y^{(\frac{7}{3})}{}_i}{28\,\Gamma\left(\frac{1}{3}\right) h^{\frac{2}{3}}} \end{cases}$$

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Since $s_i(x)$ are linearly independent and their span contains all possible spline interpolates, they form a basis [15] for the spline interpolation space. Thus, the space of spline interpolating polynomial which is the unique function satisfying all conditions.

3. CONTINUITY CONDITION OF SPLINE FUNCTION:

The following consistency relations are obtained by applying the first derivative continuities at the knots, i.e. $s_i^{(m)}(x_i) = s^{(m)}{}_{i-1}(x_i)$, where m=1,

$$s_i{}'(x_i) = b_i + 2c_i(x_i - x_i) + \frac{7}{3}d_i(x_i - x_i)^{\frac{4}{3}} + 3e_i(x_i - x_i)^2$$

$$s_i{}'(x_i) = b_i,\ s_i{}'(x_i) = y'{}_i \qquad (3.1)$$

Also

$$s'{}_{i-1}(x_i) = b_{i-1} + 2c_{i-1}(x_i - x_{i-1}) + \frac{7}{3}d_{i-1}(x_i - x_{i-1})^{\frac{4}{3}} + 3e_{i-1}(x_i - x_{i-1})^2$$

$$s'{}_{i-1}(x_i) = y'{}_{i-1} + y''{}_i h + \frac{9\, y^{\left(\frac{7}{3}\right)}{}_i}{4\,\Gamma\left(\frac{1}{3}\right)} h^{\frac{4}{3}} + 3\left(\frac{y_i - y_{i-1}}{h^3} - \frac{y'{}_i}{h^2} - \frac{y''{}_i}{2h} - \frac{27\, y^{\left(\frac{7}{3}\right)}{}_{i-1}}{28\,\Gamma\left(\frac{1}{3}\right) h^{\frac{2}{3}}}\right) h^2$$

$$s'{}_{i-1}(x_i) = \frac{3(y_i - y_{i-1})}{h} - 2y'{}_{i-1} - \frac{h}{2}y''{}_{i-1} - \frac{9\, y^{\left(\frac{7}{3}\right)}{}_{i-1}\, h^{\frac{4}{3}}}{14\,\Gamma\left(\frac{1}{3}\right)} \qquad (3.2)$$

Now by continuity condition and from equations (3.1) and (3.2), we get

$$s_i{}'(x_i) = s'{}_{i-1}(x_i)$$

$$y_i = y_{i-1} + \frac{h}{3}\left(y'{}_i + 2y'{}_{i-1}\right) + \frac{h^2}{2}y''{}_{i-1} + \frac{3\, y^{\left(\frac{7}{3}\right)}{}_{i-1}\, h^{\frac{7}{3}}}{14\,\Gamma\left(\frac{1}{3}\right)} \qquad (3.3)$$

This rule is derived from the continuity conditions of the spline function used in the approximation of dynamic system models. It provides a high-accuracy estimation for y_i by incorporating the function's derivative, second derivative, and fractional derivative of order $\frac{7}{3}$. Such a formulation allows for more precise numerical solutions of differential systems, especially in capturing complex dynamic behavior [13, 23-25].

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4. CONVERGENCE ANALYSIS

This section covers the topic of fractional cubic spline convergence in addition to some significant theorems and lemmas.

Theorem 4.1: Let $s(x)$ be a fractional cubic spline interpolate of a sufficiently smooth function y(x) over an interval [a, b] using nodes$\{x_i\}_{i=0}^{n}$. Then the interpolation error $e(x) = s(x) - y(x)$ satisfies $\left|s\,(x) - y\,_i(x)\right| \le \frac{27}{28\,\Gamma\left(\frac{1}{3}\right)}\ h^{\frac{7}{3}}\,\omega(h)$, where $(h) = \left|y^{\left(\frac{7}{3}\right)}(\beta) - y^{\left(\frac{7}{3}\right)}(\alpha)\right|$.

Proof: Now first find $s_i(x)$ from equation (2.1), we have

$$s(x) = a + b(x - x_i) + c(x - x_i)^2 + d(x - x_i)^{\frac{7}{3}} + e(x - x_i)^3$$

since $x = x_{i+1}$, we get

$$s(x_{i+1}) = a + b(x_{i+1} - x_i) + c(x_{i+1} - x_i)^2 + d(x_{i+1} - x_i)^{\frac{7}{3}} + e(x_{i+1} - x_i)^3$$

Now $h = x_{i+1} - x_i$

$$s(x_{i+1}) = a + bh + ch^2 + dh^{\frac{7}{3}} + eh^3 \qquad (3.4)$$

Putting the value of the coefficients of the equation (3.1), we get

$$s(x_{i+1}) = y_i + y'_i h + \frac{y''_i}{2}h^2 + \frac{27\,y^{\left(\frac{7}{3}\right)}{}_i}{28\,\Gamma\left(\frac{1}{3}\right)}h^{\frac{7}{3}} + h^3\left(\frac{y_{i+1}-y_i}{h^3} - \frac{y'_i}{h^2} - \frac{y''_i}{2h} - \frac{27\,y^{\left(\frac{7}{3}\right)}{}_i}{28\,\Gamma\left(\frac{1}{3}\right)h^{\frac{2}{3}}}\right) \qquad (3.5)$$

Now from (3.2) and $\left|s\,(x_{i+1}) - y\,_i(x)\right|$, we get

$$\left|s\,(x_{i+1}) - y\,_i(x)\right| = \left|\frac{27\,y^{\left(\frac{7}{3}\right)}{}_i}{28\,\Gamma\left(\frac{1}{3}\right)}h^{\frac{7}{3}} - \frac{27\,y^{\left(\frac{7}{3}\right)}{}_i}{28\,\Gamma\left(\frac{1}{3}\right)}\ h^{\frac{7}{3}}\right| \le \frac{27}{28\,\Gamma\left(\frac{1}{3}\right)}\ h^{\frac{7}{3}}\left|y^{\left(\frac{7}{3}\right)}(\beta) - y^{\left(\frac{7}{3}\right)}(\alpha)\right|$$

$$\le \frac{27}{28\,\Gamma\left(\frac{1}{3}\right)}\ h^{\frac{7}{3}}\,\omega(h),\ \text{where}\ \omega(h) = \left|y^{\left(\frac{7}{3}\right)}(\beta) - y^{\left(\frac{7}{3}\right)}(\alpha)\right|.$$

Lemma 4.1: Let $s(x)$ be fractional cubic spline and $x = x_{i+1}$, then error estimation

$$\left|s''(x) - y''_i(x)\right| \le \left|hy_i{}^{(3)}(\varepsilon) - w(x)\right|,\ \text{where}\ (x) = \frac{39\,y^{\left(\frac{7}{3}\right)}{}_i\,h^{\frac{7}{3}}}{14\,\Gamma\left(\frac{1}{3}\right)}\,.$$

Proof: Now first find $s''_i(x)$ from equation (3.3), we get

$$s''(x) = 2c + \frac{28}{9}d(x - x_i)^{\frac{1}{3}} + 6e(x - x_i)$$

Put the coefficients c, d , e and $x = x_{i+1}$ in $s''_i(x)$

$$s''(x_{i+1}) = 2\frac{y''_i}{2} + \frac{28}{9} * \frac{27\,y^{\left(\frac{7}{3}\right)}{}_i}{28\,\Gamma\left(\frac{1}{3}\right)}(x_{i+1} - x_i)^{\frac{1}{3}} + 6 * \left(\frac{y_{i+1}-y_i}{h^3} - \frac{y'_i}{h^2} - \frac{y''_i}{2h} - \frac{27\,y^{\left(\frac{7}{3}\right)}{}_i}{28\,\Gamma\left(\frac{1}{3}\right)h^{\frac{2}{3}}}\right)(x_{i+1} - x_i)$$

$$s''(x_{i+1}) = -2y''_i - \frac{6}{h}y'_i - \frac{6}{h^2}y_i - \frac{39\, y^{\left(\frac{7}{3}\right)}_i\, h^{\frac{7}{3}}}{14\, \Gamma\left(\frac{1}{3}\right)} + \frac{6}{h^2}y_{i+1} \tag{3.6}$$

By expanding y_i using the Taylor series, we get:

$$y_{i+1} = y_i + hy'_i + \frac{h^2}{2!}y''_i + \frac{h^3}{3!}y_i^{(3)}(\varepsilon) \tag{3.7}$$

Now from (3.6) and (3.7), we get

$$s''(x_{i+1}) = -2y''_i - \frac{6}{h}y'_i - \frac{6}{h^2}y_i - \frac{39\, y^{\left(\frac{7}{3}\right)}_i\, h^{\frac{7}{3}}}{14\, \Gamma\left(\frac{1}{3}\right)} + \frac{6}{h^2}y_i + \frac{6}{h}y'_i + 3y''_i + hy_i^{(3)}(\varepsilon)\,)$$

$$s''(x_{i+1}) = y''_i - \frac{39\, y^{\left(\frac{7}{3}\right)}_i\, h^{\frac{7}{3}}}{14\, \Gamma\left(\frac{1}{3}\right)} + hy_i^{(3)}(\varepsilon)\,) \tag{3.8}$$

from (3.8) and $\left|s''(x) - y''_i(x)\right|$, we get

$$\left|s''(x_{i+1}) - y''_i(x)\right| = \left|y''_i - \frac{39\, y^{\left(\frac{7}{3}\right)}_i\, h^{\frac{7}{3}}}{14\, \Gamma\left(\frac{1}{3}\right)} + hy_i^{(3)}(\varepsilon)\ - y''_i(x)\right|,$$

$$\left|s''(x_{i+1}) - y''_i(x)\right| = |y''_i(x) - \frac{39\, y^{\left(\frac{7}{3}\right)}_i\, h^{\frac{7}{3}}}{14\, \Gamma\left(\frac{1}{3}\right)} + hy_i^{(3)}(\varepsilon)\ - y''_i(x)|\,,$$

$$= |hy_i^{(3)}(\varepsilon) - \frac{39\, y^{\left(\frac{7}{3}\right)}_i\, h^{\frac{7}{3}}}{14\, \Gamma\left(\frac{1}{3}\right)}\ | \le |hy_i^{(3)}(\varepsilon) - \frac{39\, y^{\left(\frac{7}{3}\right)}_i\, h^{\frac{7}{3}}}{14\, \Gamma\left(\frac{1}{3}\right)}\ |\ = |hy_i^{(3)}(\varepsilon) - w(x)\ |, \text{ where } w(x) = \frac{39\, y^{\left(\frac{7}{3}\right)}_i\, h^{\frac{7}{3}}}{14\, \Gamma\left(\frac{1}{3}\right)}.$$

5. NUMERICAL RESULT

This approach has unique benefits and works well for a range of numerical analytic issues. The particular characteristics of the differential equation to be solved and a high enough degree of accuracy determine which approach is best. We present the simulation of this system, which computes all the results using MATLAB. We analyze the dynamic system using a functional square table such as [12] that uses each of the parameters of the number table in the system. The analysis of the system is represented in numbers compared to the detailed treatment, which is derived from [12], each of which is compared with figure and tables.

$$\frac{dx}{dt} = -\frac{bxy}{x+y}$$

$$\frac{dy}{dt} = \frac{bxy}{x+y} - cy$$

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$$\frac{dz}{dt} = cy$$

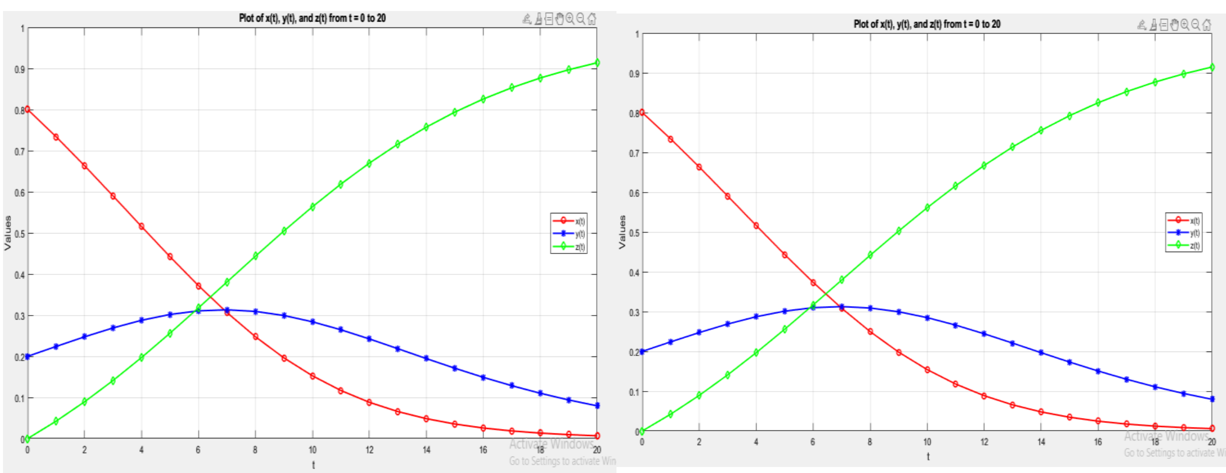
Example 1: The model (5) will be examined with parameters $b = 0.8$, $c = 0.2$ the initial condition $x_0 = 0.8$, $y_0 = 0.2$, $z_0 = 0$, where h=1 and t=(0 , 20), the exact solutions are:

$$x(t) = x_0\left(1+\frac{y_0}{x_0}\right)^{\frac{b}{b-c}}\left(1+\frac{y_0}{x_0}e^{(b-c)(t-t_0)}\right)^{-\frac{b}{b-c}}$$

$$y(t) = y_0\left(1+\frac{y_0}{x_0}\right)^{\frac{b}{b-c}}\left(1+\frac{y_0}{x_0}e^{(b-c)(t-t_0)}\right)^{-\frac{b}{b-c}}\ e^{(b-c)(t-t_0)}$$

$$x(t) = (x_0+y_0+z_0) - (x_0+y_0)^{\frac{b}{b-c}}\left(x_0+y_0e^{(b-c)(t-t_0)}\right)^{-\frac{b}{b-c}}$$

In Figure (1a), we present the approximate solution, while in Figure 1b, we provide the exact solution. Additionally, Table (5.1) displays the approximate solution obtained using the continuity condition of the spline function based on equation number (3.3), alongside which the exact solution, derived from [12] is shown. Furthermore, in Table (5.2), we present the approximate local truncation error calculated between the exact solution and the approximate solution.



Figure(1a) and Figure(1b) Approximate Solutions of the Exact Solutions of the dynamic system.

Figure(1a) Our suggested spline-based construction Method was used to produce approximate solutions of the dynamical system equations for the SLR model, which were then confirmed by MATLAB simulations. Step size h = 1 and parameter values b = 0.8 and c = 0.2 were used to compute the

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results. The graphic shows how well the system's dynamic behaviour over the given interval is captured by the numerical method. In Figure(1b) Exact Solutions of the dynamic system

equation where h=1 and b=0.8 , c=0.2 The image is drawn using MATLAB due to the exact solution found for the system in [12].

Table (5.1) exact solution Approximate Solutions and of the dynamic system equation (5)

t	x(t) approximate	x(t) exact	y(t) approximate	y(t) exact	z(t) approximate	z(t) exact
0	0.80000000	0.80000000	0.20000000	0.20000000	0.00000000	0.00000000
1	0.73359967	0.73359372	0.22400031	0.22400335	0.04240002	0.04240293
2	0.66300639	0.66312693	0.24741565	0.24731728	0.08957796	0.08955579
3	0.58963439	0.59002154	0.26904020	0.26877234	0.14132542	0.14120612
4	0.51524930	0.51603102	0.28757409	0.28711204	0.19717677	0.19685694
5	0.44186311	0.44311205	0.30173951	0.30112586	0.25639738	0.25576209
6	0.37154795	0.37324497	0.31045008	0.30980423	0.31800221	0.31695080
7	0.30620930	0.30823173	0.31298252	0.31248533	0.38080842	0.37928295
8	0.24737119	0.24951099	0.30910621	0.30895901	0.44352251	0.44153000
9	0.19601882	0.19802879	0.29913147	0.29950110	0.50484963	0.50247011
10	0.15253671	0.15418940	0.28385744	0.28482854	0.56360545	0.56098206
11	0.11675359	0.11788909	0.26443943	0.26598767	0.61880659	0.61612324
12	0.08806297	0.08861484	0.24220546	0.24420426	0.66973165	0.66718090
13	0.06558452	0.06557761	0.21847706	0.22072994	0.71593850	0.71369245
14	0.04832192	0.04784877	0.19442848	0.19671403	0.75724921	0.75543720
15	0.03528664	0.03447604	0.17100519	0.17311743	0.79370825	0.79240654
16	0.02557888	0.02456685	0.14889093	0.15067173	0.82553028	0.82476142
17	0.01843069	0.01733763	0.12852604	0.12987663	0.85304335	0.85278574
18	0.01321440	0.01213433	0.11014242	0.11102374	0.87664326	0.87684193
19	0.00943548	0.00843240	0.09381110	0.09423451	0.89675349	0.89733309
20	0.00671468	0.00582455	0.07949095	0.07950238	0.91379445	0.91467307

The approximate solutions we have been able to find by using construction (3.3) and the exact solution of the dynamical system equation (5) we have been able to find by exact which are given in [12] where h=1 and b=0.8 , c=0.2 .

Table (5.2) relative error estimated relative local truncation error between exact and approximate solution.

t	x(t)	y(t)	z(t)
0	0	0	0
1	5.95×10^{-6}	3.04×10^{-6}	2.91×10^{-6}
2	1.2054×10^{-4}	9.837×10^{-5}	2.217×10^{-5}
3	3.8715×10^{-4}	2.6786×10^{-4}	1.1930×10^{-4}
4	7.8172×10^{-4}	4.6205×10^{-4}	3.1983×10^{-4}
5	1.24894×10^{-3}	6.1365×10^{-4}	6.3529×10^{-4}
6	1.69702×10^{-3}	6.4585×10^{-4}	1.05141×10^{-3}
7	2.02243×10^{-3}	4.9719×10^{-4}	1.52547×10^{-3}
8	2.13980×10^{-3}	1.4720×10^{-4}	1.99251×10^{-3}
9	2.00997×10^{-3}	3.6963×10^{-4}	2.37952×10^{-3}
10	1.65269×10^{-3}	9.7110×10^{-4}	2.62339×10^{-3}
11	1.13550×10^{-3}	1.54824×10^{-3}	2.68335×10^{-3}
12	5.5187×10^{-4}	1.99880×10^{-3}	2.55075×10^{-3}
13	6.91×10^{-6}	2.25288×10^{-3}	2.24605×10^{-3}
14	4.7315×10^{-4}	2.28555×10^{-3}	1.81201×10^{-3}
15	8.1060×10^{-4}	2.11224×10^{-3}	1.30171×10^{-3}
16	1.01203×10^{-3}	1.78080×10^{-3}	7.6886×10^{-4}
17	1.09306×10^{-3}	1.35059×10^{-3}	2.5761×10^{-4}
18	1.08007×10^{-3}	8.8132×10^{-4}	1.9867×10^{-4}
19	1.00308×10^{-3}	4.2341×10^{-4}	5.7960×10^{-4}
20	8.9013×10^{-3}	1.143×10^{-5}	8.7862×10^{-4}

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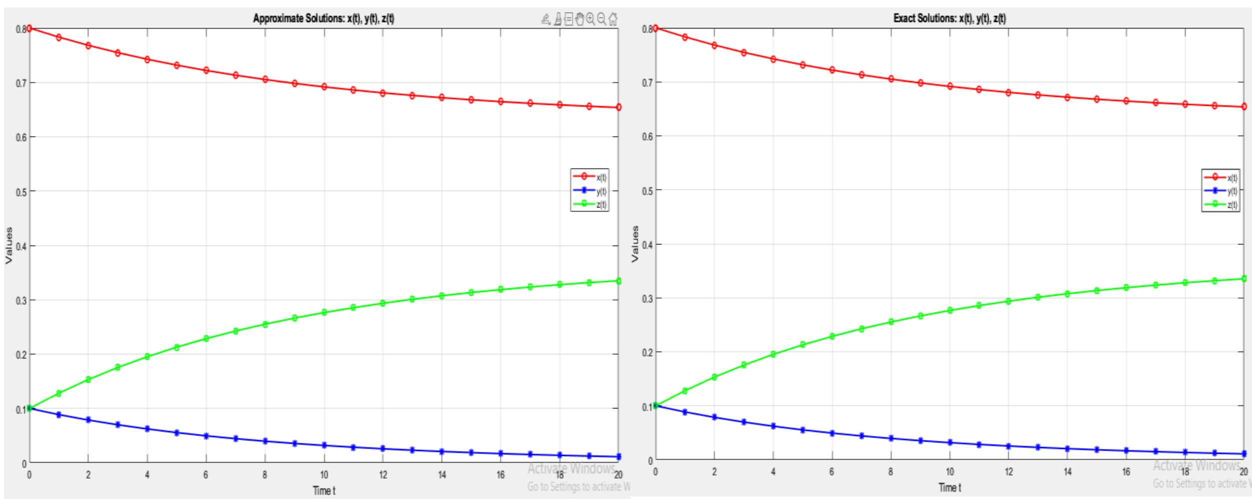
Example 2: Model (5) will be investigated using Parameters $b < c$ where $b = 0.2$, $c = 0.3$ the initial condition $x_0 = 0.8$, $y_0 = 0.1$, $z_0 = 0.1$ where h=1 and t=(0 , 20)

In Figure(2a) , the approximate solution is presented, while Figure (2b) displays the complete solution. Additionally, the approximate solution is included in Table (5.3), and Table (5.4) illustrates the approximate truncation error between the exact and approximate solutions.



Figure(1a) and Figure(1b) are approximate solutions of the exact solutions of the dynamic system.

Figure(2a) Our suggested spline-based construction Method was used to produce approximate solutions of the dynamical system equations for the SLR model, which were then confirmed by

MATLAB simulations. Step size h = 1 and parameter values b = 0.2 and c = 0.3 were used to compute the results. The graphic shows how well the system's dynamic behaviour over the given interval is captured by the numerical method. In Figure(2b) Exact Solutions of the dynamic system equation where h=1 and b=0.2 , c=0.3 The image is drawn using MATLAB due to the exact solution found for the system in [12]

Table (5.3) exact solution Approximate Solutions and of the dynamic system equation (5)

t	x(t) approximate	x(t) exact	y(t) approximate	y(t) exact	z(t) approximate	z(t) exact
0	0.80000000	0.80000000	0.10000000	0.10000000	0.10000000	0.10000000
1	0.78317203	0.78317165	0.08858098	0.08858038	0.12824699	0.12824797
2	0.76811578	0.76809888	0.07862683	0.07860827	0.15325739	0.15329284
3	0.75462373	0.75458670	0.06991647	0.06987645	0.17545981	0.17553686
4	0.74252090	0.74246370	0.06227233	0.06221104	0.19520677	0.19532526
5	0.73165462	0.73157896	0.05554631	0.05546563	0.21279908	0.21295540
6	0.72189083	0.72179931	0.04961396	0.04951648	0.22849522	0.22868420
7	0.71311146	0.71300703	0.04437005	0.04425860	0.24251850	0.24273437
8	0.70521220	0.70509788	0.03972526	0.03960261	0.25506255	0.25529951
9	0.69810069	0.69797939	0.03560341	0.03547216	0.26629591	0.26654845
10	0.69169499	0.69156945	0.03193930	0.03180177	0.27636572	0.27662878
11	0.68592231	0.68579497	0.02867691	0.02853516	0.28540079	0.28566987
12	0.68071780	0.68059087	0.02576793	0.02562375	0.29351427	0.29378538
13	0.67602371	0.67589908	0.02317059	0.02302550	0.30080571	0.30107542
14	0.67178844	0.67166777	0.02084863	0.02070390	0.30736293	0.30762833
15	0.66796589	0.66785056	0.01877050	0.01862720	0.31326361	0.31352224
16	0.66451481	0.66440599	0.01690865	0.01676766	0.31857654	0.31882635
17	0.66139827	0.66129689	0.01523899	0.01510101	0.32336274	0.32360211
18	0.65858313	0.65848994	0.01374037	0.01360596	0.32767651	0.32790411
19	0.65603966	0.65595525	0.01239419	0.01226379	0.33156615	0.33178096
20	0.65374119	0.65366598	0.01118408	0.01105801	0.33507474	0.33527601

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The approximate solutions we have been able to find by using construction (3.3) and the exact solution of the dynamical system equation (5) we have been able to find by exact which are given in [12] where h=1 and b=0.2 , c=0.3.

Table (5.4) exact solution Approximate Solutions and of the dynamic system equation (5)

t	x(t)	y(t)	z(t)
0	0	0	0
1	3.8×10^{-7}	6.0000×10^{-7}	9.8000×10^{-7}
2	1.690×10^{-5}	1.8560×10^{-5}	3.5450×10^{-5}
3	3.703×10^{-5}	4.0020×10^{-5}	7.7050×10^{-5}
4	5.720×10^{-5}	6.1290×10^{-5}	1.1849×10^{-4}
5	7.566×10^{-5}	8.0680×10^{-5}	1.5632×10^{-4}
6	9.152×10^{-5}	9.7480×10^{-5}	1.8898×10^{-4}
7	1.0443×10^{-4}	1.1145×10^{-4}	2.1587×10^{-4}
8	1.1432×10^{-4}	1.2265×10^{-4}	2.3696×10^{-4}
9	1.2130×10^{-4}	1.3125×10^{-4}	2.5254×10^{-4}
10	1.2554×10^{-4}	1.3753×10^{-4}	2.6306×10^{-4}
11	1.2734×10^{-4}	1.4175×10^{-4}	2.6908×10^{-4}
12	1.2693×10^{-4}	1.4418×10^{-4}	2.7111×10^{-4}
13	1.2463×10^{-4}	1.4509×10^{-4}	2.6971×10^{-4}
14	1.2067×10^{-4}	1.4473×10^{-4}	2.6540×10^{-4}
15	1.1533×10^{-4}	1.4330×10^{-4}	2.5863×10^{-4}
16	1.0882×10^{-4}	1.4099×10^{-4}	2.4981×10^{-4}
17	1.0138×10^{-4}	1.3798×10^{-4}	2.3937×10^{-4}
18	9.319×10^{-5}	1.3441×10^{-4}	2.2760×10^{-4}
19	8.441×10^{-5}	1.3040×10^{-4}	2.1481×10^{-4}
20	7.521×10^{-5}	1.2607×10^{-4}	2.0127×10^{-4}

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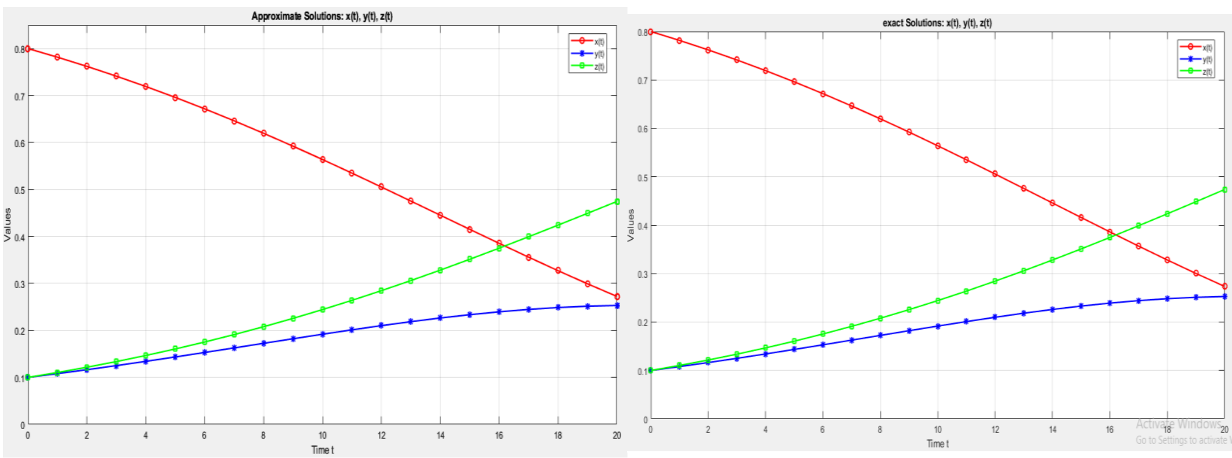
Vol.33; No.3.| 2025

Example 3: Model (5) will be investigated using parameters $b > c$ where $b = 0.2$, $c = 0.1$ the initial condition $x_0 = 0.8$, $y_0 = 0.1$, $z_0 = 0.1$ where h=1 and t=(0 , 20)

In Figure(3a) , the approximate solution is presented, while Figure (3b) displays the complete solution. Additionally, the approximate solution is included in Table (5.5), and Table(5.6) illustrates the approximate truncation error between the exact and approximate solutions.



Figure(3a) and Figure(3b) are approximate and exact solutions of the dynamic system.

Figure(3a) Our suggested spline-based construction Method was used to produce approximate solutions of the dynamical system equations for the SLR model, which were then confirmed by MATLAB simulations. Step size h = 1 and parameter values b = 0.2 and c = 0.1 were used to compute the results. The graphic shows how well the system's dynamic behaviour over the given interval is captured by the numerical method. In Figure(3b) Exact Solutions of the dynamic system equation where h=1 and b=0.2 , c=0.1 The image is drawn using MATLAB due to the exact solution found for the system in [12].

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Table (5.5) exact solution Approximate Solutions and of the dynamic system equation (5)

t	x(t) approximate	x(t) exact	y(t) approximate	y(t) exact	z(t) approximate	z(t) exact
0	0.80000000	0.80000000	0.10000000	0.10000000	0.10000000	0.10000000
1	0.78162579	0.78162565	0.10797862	0.10797874	0.11039558	0.11039561
2	0.76204682	0.76204571	0.11634782	0.11634559	0.12160537	0.12160870
3	0.74125058	0.74125033	0.12508047	0.12507291	0.13366895	0.13367676
4	0.71923844	0.71924291	0.13413985	0.13412304	0.14662171	0.14663405
5	0.69602695	0.69604206	0.14347829	0.14344742	0.16049476	0.16051052
6	0.67164963	0.67168344	0.15303644	0.15298588	0.17531392	0.17533068
7	0.64615874	0.64622132	0.16274273	0.16266624	0.19109853	0.19111244
8	0.61962660	0.61972986	0.17251319	0.17240427	0.20786021	0.20786587
9	0.59214660	0.59230390	0.18225178	0.18210407	0.22560162	0.22559203
10	0.56383358	0.56405917	0.19185114	0.19165897	0.24431528	0.24428186
11	0.53482355	0.53513177	0.20119402	0.20095309	0.26398243	0.26391514
12	0.50527252	0.50567698	0.21015534	0.20986334	0.28457214	0.28445969
13	0.47535457	0.47586706	0.21860487	0.21826218	0.30604056	0.30587076
14	0.44525895	0.44588839	0.22641053	0.22602083	0.32833052	0.32809078
15	0.41518624	0.41593770	0.23344223	0.23301293	0.35137153	0.35104937
16	0.38534383	0.38621764	0.23957610	0.23911856	0.37508007	0.37466381
17	0.35594073	0.35693186	0.24469883	0.24422828	0.39936043	0.39883987
18	0.32718202	0.32827979	0.24871201	0.24824713	0.42410597	0.42347308
19	0.29926315	0.30045138	0.25153607	0.25109828	0.44920078	0.44845035
20	0.27236453	0.27362202	0.25311372	0.25272605	0.47452175	0.47365193

The approximate solutions we have been able to find by using construction (3.3) and the exact solution of the dynamical system equation (5) we have been able to find by exact which are given in [12] where h=1 and b=0.2 , c=0.1.

Table (5.6) exact solution Approximate Solutions and of the dynamic system equation (5)

t	x(t)	y(t)	z(t)
0	0	0	0
1	1.4000×10^{-7}	1.2000×10^{-7}	3.0000×10^{-8}
2	1.1100×10^{-6}	2.2300×10^{-6}	3.3300×10^{-6}
3	2.5000×10^{-7}	7.5600×10^{-6}	7.8100×10^{-6}
4	4.4700×10^{-6}	1.6810×10^{-5}	1.2340×10^{-5}
5	1.5110×10^{-5}	3.0870×10^{-5}	1.5760×10^{-5}
6	3.3810×10^{-5}	5.0560×10^{-5}	1.6760×10^{-5}
7	6.2580×10^{-5}	7.6490×10^{-5}	1.3910×10^{-5}
8	1.0326×10^{-4}	1.0892×10^{-4}	5.6600×10^{-6}
9	1.5730×10^{-4}	1.4771×10^{-4}	9.5900×10^{-6}
10	2.2559×10^{-4}	1.9217×10^{-4}	3.3420×10^{-5}
11	3.0822×10^{-4}	2.4093×10^{-4}	6.7290×10^{-5}
12	4.0446×10^{-4}	2.9200×10^{-4}	1.1245×10^{-4}
13	5.1249×10^{-4}	3.4269×10^{-4}	1.6980×10^{-4}
14	6.2944×10^{-4}	3.8970×10^{-4}	2.3974×10^{-4}
15	7.5146×10^{-4}	4.2930×10^{-4}	3.2216×10^{-4}
16	8.7381×10^{-4}	4.5754×10^{-4}	4.1626×10^{-4}
17	9.9113×10^{-4}	4.7055×10^{-4}	5.2056×10^{-4}
18	1.0978×10^{-3}	4.6488×10^{-4}	6.3289×10^{-4}
19	1.1882×10^{-3}	4.3779×10^{-4}	7.5043×10^{-4}
20	1.2575×10^{-3}	3.8767×10^{-4}	8.6982×10^{-4}

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6. CONCLUSION

The fractional cubic spline method offers an effective and accurate tool for solving systems of modeling in dynamic systems. By combining the smooth approximation properties of cubic splines with the nonlocal nature of fractional calculus, this method provides a powerful framework for capturing complex dynamic behaviors, including memory and hereditary effects. Compared to classical numerical methods, the fractional cubic spline approach delivers better flexibility, higher precision, and greater stability. As dynamic systems in engineering, physics, and biology continue to grow in complexity, fractional cubic splines present a promising tool for advancing both theoretical understanding and practical applications. Future research could explore optimal fractional-order selection, stability under varying conditions, and real-time implementation for control systems, further solidifying their role in modern computational science.

Conflict of interests.

There are non-conflicts of interest.

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