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A Midpoint Quadratic Approach for Solving Numerically Multi-Order Fractional Integro-Differential Equation

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ABSTRACT

Background:

This article presents a method for finding numerical solutions to Fredholm integro-differential equations (FIFDEs) with multi-fractional orders of one or less, using a useful algorithm.

Materials and Methods:

A finite difference approximation to Caputo's derivative using collocation points is used to build the midpoint method for the quadrature rule, which forms the basis of the approach.

Results:

Our method simplifies the evaluation of treatments by transforming the FIFDEs into algebraic equations with operational matrices. After calculating the Caputo derivative at a specific point using the finite difference method, we use the quadrature method, which includes the midpoint rule, to create a finite difference formula for our fractional equation.

Conclusions:

Additionally, numerical examples are provided to demonstrate the validity and use of the approach as well as comparisons with earlier findings. The results are expressed using a program created in MATLAB.

Key words: Fractional calculus, Caputo-fractional derivative, integral-differential equation, midpoint rules, forward difference approximation.

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INTRODUCTION

The Fredholm integro-differential equations (FIFDEs) of multi-fractional orders with variable coefficients that lie in the interval (0,1] in the Caputo sense are the class of equations investigated in this paper. Their general form is as follows:

$$ {}_a^C D_t^{\alpha_n} u(t) + \sum_{i=1}^{n-1} \mathcal{P}_i(t)\, {}_a^C D_t^{\alpha_{n-i}} u(t) + \mathcal{P}_n(t) u(t) $$

$$ = f(t) + \lambda \int_a^b \sum_{j=0}^{m} \mathcal{K}_j(t,s)\, {}_a^C D_s^{\beta_{m-j}} u(s) ds\,, \qquad a \le t \le b \qquad (1) $$

subject to the boundary condition:

$$ g_{11} u(a) + h_{11} u(b) = C_1 \qquad (2) $$

where g_{11}, h_{11} and $C_1 \in \mathbb{R}$. The fractional orders are $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n \le 1$, and $0 = \beta_0 < \beta_1 < \cdots < \beta_m \le 1$ such that $\mu = \max\{\lceil \alpha_n \rceil, \lceil \beta_m \rceil\} = 1$. The unknown function to be found in equation (1) is u. Additionally, the functions $f, \mathcal{P}_i \in C([a,b],\mathbb{R})$, $\mathcal{K}_j \in C(\mathrm{E},\mathbb{R})$, and the $\mathrm{E} = \{(t,s): a \le t < s \le b\}$ represented the known continuous function for all $i = 1,2,\ldots,n$ and $j = 0,1,\ldots,m$. A scalar parameter is the λ. Since $\rho = \{\alpha_{n-i}, \beta_{m-j}\} \in (0,1]$ for all $i = 0,1,\ldots,n$ and $j = 0,1,\ldots,m, n, m \in \mathbb{Z}^+ \cup \{0\}$, the ${}_a^C D_t^{\rho}$ represents the ρ-Caputo fractional differential operator of the real-valued function $u(t)$ on the closed bounded interval $[a,b]$.

The purpose of this research is to solve FIFDEs (1) with boundary conditions (2) using the quadrature rule: the midpoint approach in matrix form. This numerical method for estimating a solution at a point can be used to approximate the value of an unknown function at that given point. The quadrature techniques are the basis of every numerical method for finding solutions of integral parts in functional equations. Abdullo, Samandar and Bobomurod [1] used quadrature methods to solve the first kind Abel integral equation; Al-Rawi [2] used quadrature methods to solve the first kind of integral equations of the convolution type; Al-Nasir [3] applied it to solve Volterra integral equations of the second kind; and Jafar and Mahdi [4] used the modified trapezoid quadrature method for solving Fredholm integral equations of the second kind, although Rahbar and Hashemizadeh [5]. While Emamzadeh and Kajani [6] used the quadrature technique for the second kind of nonlinear Fredholm integral equation. Moreover, Samuel and Robert [7] applied it to solve singular Volterra types. Furthermore, Saadati with Shakeri [8] and Al-Jawary [9] are solving linear integro-differential equations and applying quadrature techniques. Also, Shazad and Shokhan [10] used it to numerically treat the solution of the most general linear Volterra integro-fractional differential equations.

The structure of the paper is as |The necessary definitions and fundamental introduction to fractional calculus are provided in Section 2. Section 3 provides a fundamental review of the formulation of Quadrature-Midpoint techniques. Numerical techniques are derived for FIFDEs in

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detail in Section 4. Additionally, this section's algorithm explains the scheme's primary phases. The numerical results are shown in Section 5, and Section 6 offers the conclusions.

The purpose of this study is to use quadrature techniques for Caputo derivative terms that depend on collocation points and convert to an algebraic system using the finite difference approximation. Additionally, it aims to evaluate the multi-order linear FIFDEs' approximate solution.

2. FUNDAMENTAL DEFINITIONS OF FRACTIONAL DERIVATIVES:

This section outlines the fundamental definitions, features, and attributes of fractional derivatives. In addition, a number of basic ideas and lemmas that are used in this study were described.

Definition 1 ([11]). If there is a real number $\rho > \eta$ such that $u(t) = (t-a)^{\rho} u_0(t)$, with $u_0 \in C[a,b]$, then a real function u, defined on $[a,b]$, belongs to the space $C_\eta[a,b], \eta \in \mathbb{R}$. It is also said to belong to the space $C_\eta^m[a,b]$ if and only if $u^{(m)} \in C_\eta[a,b], m \in \mathbb{Z}^+$.

Definition 2 ([12,13]). For a function $u \in C_\eta[a,b], \eta \geq -1$, the left-sided Riemann-Liouville fractional integral of order $\rho > 0$ is defined as

$${}_aJ_t^{\rho} u(t) = \frac{1}{\Gamma(\rho)} \int_a^t (t-\xi)^{\rho-1} u(\xi) d\xi, \qquad \rho \in \mathbb{R}^+, \qquad a \leq t \leq b$$

Here, $\Gamma(.)$ denotes the gamma function, and for $\rho = 0$, we have the Riemann-Liouville identity operator, ${}_aJ_t^0 u(t) = u(t)$.

Definition 3 ([13,14]). The operator ${}_a^RD_t^{\rho} u(t)$, for a function $u \in C_{-1}^{\lceil\rho\rceil}[a,b]$ of order $\rho \geq 0$ and $t > a$, defined as

$${}_a^RD_t^{\rho} u(t) = D_t^{\lceil\rho\rceil} \, {}_aJ_t^{\lceil\rho\rceil-\rho} u(t)$$

is called the Riemann-Liouville fractional derivative of order ρ. Where $\lceil . \rceil$denotes the ceiling function and for $\rho = 0$, we have the Riemann-Liouville identity derivative operator, ${}_a^RD_t^0 u(t) = u(t)$.

Definition 4 ([13,14]). The operator ${}_a^CD_t^{\rho} u(t)$, for a function $u \in C_{-1}^{\lceil\rho\rceil}[a,b]$ of order $\rho \geq 0$ and $t > a$, defined as

$${}_a^CD_t^{\rho} u(t) = {}_aJ_t^{\lceil\rho\rceil-\rho} D_t^{\lceil\rho\rceil} u(t) = \frac{1}{\Gamma(\lceil\rho\rceil-\rho)} \int_a^t (t-\xi)^{\lceil\rho\rceil-\rho-1} \frac{d^{\lceil\rho\rceil} u(\xi)}{d\xi^{\lceil\rho\rceil}} d\xi$$

is called the Caputo fractional differential operator of order ρ. In Caputo manner derivative for $\rho = 0$, we have the Caputo identity derivative operator, ${}_a^CD_t^0 u(t) = u(t)$.

The following properties hold:

- ${}_aJ_t^{\rho_1} \, {}_aJ_t^{\rho_2} u(t) = {}_aJ_t^{\rho_1+\rho_2} u(t) = {}_aJ_t^{\rho_2} \, {}_aJ_t^{\rho_1} u(t)$ for all $\rho_1, \rho_2 \geq 0$.

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- ${}_a^RD_t^\rho \mathcal{A} = \mathcal{A}\frac{(t-a)^{-\rho}}{\Gamma(1-\rho)}$ and ${}_a^CD_t^\rho \mathcal{A} = 0$; $\mathcal{A}$ is any constant;($\rho \geq 0, \rho \notin \mathbb{N}$).
- ${}_a^CD_t^\rho \; {}_aJ_t^\rho u(t) = u(t)$, for $\lceil\rho\rceil - 1 < \rho \leq \lceil\rho\rceil,\; a \leq t \leq b$.
- ${}_aJ_t^\rho \, {}_a^CD_t^\rho u(t) = u(t) - \sum_{k=0}^{\lceil\rho\rceil-1} \frac{u^k(t=a)}{k!}(t-a)^k$, for $\lceil\rho\rceil - 1 < \rho \leq \lceil\rho\rceil$.
- ${}_a^CD_t^\rho u(t) = {}_a^RD_t^\rho \left[u(t) - T_{\lceil\rho\rceil-1}[u;a]\right]$, and $T_{\lceil\rho\rceil-1}[u;a]$ denotes the Taylor polynomial of degree $\lceil\rho\rceil - 1$ for the function u, centered at a.

Lemma 1 ([15]) If we have $t > a$ and for $u(t) = (t-a)^\gamma$, for $\gamma > -1$ and $\rho \in \mathbb{R}^+$ then the following statement hold:

$${}_aJ_t^\rho u(t) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\rho+1)}(t-a)^{\gamma+\rho}.$$

Lemma 2 ([15]) The function $u(t) = (x-a)^\gamma$, for $\gamma \geq 0$, has a Caputo derivative of order $\rho \geq 0$, which is formed as: For $\gamma \in \{0,1,2,\cdots,\lceil\rho\rceil - 1\}$: ${}_a^CD_t^\rho u(t) = 0$ and for $\gamma \in \mathbb{N}$ and $\gamma \geq \lceil\rho\rceil$ or $\gamma \notin \mathbb{N}$ and $\gamma > \lceil\rho\rceil - 1$:

$${}_a^CD_t^\rho u(t) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\rho+1)}(t-a)^{\gamma-\rho}.$$

Lemma 3 ([16]) For fractional order $0 < \rho \leq 1$ at specified points $t = t_{r+1}; r = 0,1,\ldots,N-1$ and $h = (b-a)/N$, the Caputo derivative finite difference approximation is created

$${}_a^CD_t^\rho \, u(t_{r+1}) = \frac{h^{-\rho}}{\Gamma(2-\rho)}\sum_{\ell=0}^{r}[u(t_{r-\ell+1}) - u(t_{r-\ell})]\, b_\ell^\rho \tag{3}$$

where $b_\ell^\rho = (\ell+1)^{1-\rho} - \ell^{1-\rho}$.

Lemma 4 ([17,18]) Let's say $\rho \geq 0$, $\rho \notin \mathbb{N}$. Additionally, assume that $u \in C_{-1}^{\lceil\rho\rceil}[a,b]$. Then $[{}_a^CD_t^\rho u(t)]_{t=t_0} = 0$, that is $\lim_{t\to a}[{}_a^CD_t^\rho u(t)] = 0$, and the Caputo fractional derivative ${}_a^CD_t^\rho u(t)$ is continuous on $[a,b]$.

3. QUADRATURE-MIDPOINT RULE ([19,20,21]):

The weighted sum of a finite number of integrand function sample values is known as the quadrature rule. Consider the real-valued function $g(t)$, which is defined on the restricted interval $a \leq t \leq b$. By using $\sum_{j=1}^{N} w_j g(t_j) + R[g]$, we want to calculate the value of the integral $\int_a^b g(t)dt$. With $R[g]$ as the remainder and the quadrature rule $\{w_j, t_j\}_{j=1}^N$ may be found in tabular form, where the integration nodes are represented by the real numbers t_j and the quadrature weights, or constants, are represented by w_j, [17,18]. The Midpoint Method is the simplest case of (unweighted) quadrature formula which is based on the uses node sets $\left\{\frac{t_{\ell-1}+t_\ell}{2}\right\}$, the midpoint of subinterval $[t_{\ell-1}, t_\ell]$ to interpolate g on it using a constant polynomial$\big(P_\ell(t) =$

$g\left(\frac{t_{\ell-1}+t_\ell}{2}\right)$ for all $\ell = 1,2,\dots,N\Big)$. The corresponding estimate of the definite integral over $[a,b]$, with N-subintervals $[t_{\ell-1}, t_\ell]$ and equal spaced $h = (b-a)/N$, $N \geq 1$, is given by:

$$\int_a^b g(t)dt = h\sum_{\ell=1}^{N} g\left(t_{\ell-1/2}\right) \tag{4}$$

where $t_{\ell-\frac{1}{2}} = \frac{t_{\ell-1}+t_\ell}{2}$ for the global error $-\frac{(b-a)h^2}{24}g''(\theta), a < \theta < b$ and $g\left(t_{\ell-1/2}\right)$ can be found from the quadratic Newton-Gregory Forward-difference formula on subintervals $[t_{\ell-1}, t_\ell], \forall \ell = 1,2,\dots,N$:

$$P_2(t) = g_{\ell-1} + \binom{S}{1}\Delta g_{\ell-1} + \binom{S}{2}\Delta^2 g_{\ell-1}$$

where $t_s = t_{\ell-1} + sh$ and $\Delta^n g_{\ell-1} = \Delta^{n-1}g_\ell - \Delta^n g_{\ell-1}, n \geq 1$ and hence,

$$t_{\ell-\frac{1}{2}} = t_0 + \left(\ell - \frac{1}{2}\right)h = t_0 + \ell h - \frac{1}{2}h = t_0 + (\ell-1)h + \frac{1}{2}h = t_{\ell-1} + \frac{1}{2}h$$

Thus putting $s = \frac{1}{2}$ so the interpolate $g\left(t_{\ell-\frac{1}{2}}\right)$ by $P_2(t = t_s)$ as:

$$\begin{aligned} g\left(t_{\ell-1/2}\right) &\cong P_2\left(t_{\ell-1} + \frac{1}{2}h\right) = g_{\ell-1} + \binom{S}{1}\Delta g_{\ell-1} + \binom{S}{2}\Delta^2 g_{\ell-1} \\ &= \frac{3}{8}g_{\ell-1} + \frac{3}{4}g_\ell - \frac{1}{8}g_{\ell+1} \end{aligned} \tag{5}$$

which called quadrature interpolation formula.

Now, by applying the quadratic midpoint idea (equations 5 into 4) at each first $(N-1)$ sub-intervals $[t_0, t_1], [t_1, t_2], \dots, [t_{N-2}, t_{N-1}]$ and apply the trapezoidal rule for the last sub-interval $[t_{N-1}, t_N]$,we obtain the compose formula:

$$\begin{aligned} \int_a^b g(t)\,dt &= \left[\int_{t_0}^{t_1} + \int_{t_1}^{t_2} + \dots + \int_{t_{N-2}}^{t_{N-1}} + \int_{t_{N-1}}^{t_N}\right] g(t)dt \\ &= h\sum_{\ell=1}^{N-1}\left[\frac{3}{8}g_{\ell-1} + \frac{3}{4}g_\ell - \frac{1}{8}g_{\ell+1}\right] + \frac{h}{2}[g_{N-1} + g_N] \end{aligned} \tag{6}$$

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4. A NUMERICAL TECHNIQUE UTILIZING THE QUADRATURE-MIDPOINT RULE:

This section presents a new approach that uses quadrature methods with the aid of the finite difference approximation to treat multi-fractional orders of FIDEs with variable coefficients. Recall equation (1) for $0 < \max_{i,j}\{\alpha_i, \beta_j\} \leq 1$ with strictly decreasing for α_i and β_j for all $i = 0,1, \dots, n; j = 0,1, \dots, m$. Thus, for obtaining an approximation of the solution $u(t)$ in a given set of $(N+1)$-equally spaced grid points $t_r = t_0 + rh, (r = 0,1, \dots, N)$ with $t_0 + Nh = b$, consists in approximating the linear fredholm IFDEs (1) in the discretized equations:

$$\left[\sum_{i=0}^{n-1} \mathcal{P}_i(t)\, {}_a^C D_t^{\alpha_{n-i}} u(t) + \mathcal{P}_n(t) u(t)\right]_{t=t_r} = f(t_r) + \lambda \sum_{j=0}^{m-1} \int_a^b \mathcal{K}_j(t_r, s)\, {}_a^C D_s^{\beta_{m-j}} u(s) ds + \lambda \int_a^b \mathcal{K}_m(t_r, s) u(s) ds \quad (7)$$

This leads to a system of $N+1$ linear algebraic equations in $N+1$ unknowns $\tilde{u}(t_r) = \tilde{u}_r$, which approximate $u(t_r)$. Here, the Fredholm integral part in (7) is approximated by the open Newton-Cotes formula (Midpoint rule) and the fractional differential parts are approximated by using forward differences. By applying rule (6) to calculate the integral part in equation (7) at each point $t = t_r$ $(r = 0,1, \dots, N)$. First, at the point $t = t_0$, we have:

$$\begin{aligned}\mathcal{P}_{n,0}\tilde{u}_0 = f_0 + \lambda h \sum_{j=0}^{m-1} \Bigg\{ \sum_{d=1}^{N-1} \mathcal{K}_{0,d-1/2}^{j} \Bigg[\frac{3}{8} \left[{}_a^C D_s^{\beta_{m-j}} u(s)\right]_{s=s_{d-1}} + \frac{3}{4} \left[{}_a^C D_s^{\beta_{m-j}} u(s)\right]_{s=s_d} \\ - \frac{1}{8} \left[{}_a^C D_s^{\beta_{m-j}} u(s)\right]_{s=s_{d+1}} \Bigg] \Bigg\} \\ + \frac{\lambda h}{2} \sum_{j=0}^{m-1} \left\{ \mathcal{K}_{0,N-1}^{j} \left[{}_a^C D_s^{\beta_{m-j}} u(s)\right]_{s=s_{N-1}} + \mathcal{K}_{0,N}^{j} \left[{}_a^C D_s^{\beta_{m-j}} u(s)\right]_{s=s_N} \right\} \\ + \lambda h \sum_{d=1}^{N-1} \left\{ \mathcal{K}_{0,d-1/2}^{m} \left[\frac{3}{8} \tilde{u}_{d-1} + \frac{3}{4} \tilde{u}_d - \frac{1}{8} \tilde{u}_{d+1} \right] \right\} \\ + \frac{\lambda h}{2} \left[\mathcal{K}_{0,N-1}^{m} \tilde{u}_{N-1} + \mathcal{K}_{0,N}^{m} \tilde{u}_N \right] \quad (8)\end{aligned}$$

Take into account the lemmas 3 and 4, since $\left[{}_a^C D_s^{\beta_{m-j}} u(s)\right]_{s=s_0} = 0$, then the results formed as follows:

$$\mathcal{P}_{n,0}\tilde{u}_0 = f_0 + \lambda h \sum_{j=0}^{m-1}\left\{\sum_{d=2}^{N-1}\mathcal{K}_{0,d-1/2}^{j}\left[\frac{3}{8}A_m^{\beta}(j)\sum_{\ell=0}^{d-2}(\tilde{u}_{d-\ell-1}-\tilde{u}_{d-\ell-2})b_\ell^{\beta_{m-j}}\right]\right\}$$
$$+\lambda h \sum_{j=0}^{m-1}\left\{\sum_{d=1}^{N-1}\mathcal{K}_{0,d-1/2}^{j}\left[\frac{3}{4}A_m^{\beta}(j)\sum_{\ell=0}^{d-1}(\tilde{u}_{d-\ell}-\tilde{u}_{d-\ell-1})b_\ell^{\beta_{m-j}}\right.\right.$$
$$\left.\left.-\frac{1}{8}A_m^{\beta}(j)\sum_{\ell=0}^{d}(\tilde{u}_{d-\ell+1}-\tilde{u}_{d-\ell})b_\ell^{\beta_{m-j}}\right]\right\}$$
$$+\frac{\lambda h}{2}\sum_{j=0}^{m-1}\left\{\mathcal{K}_{0,N-1}^{j}A_m^{\beta}(j)\sum_{\ell=0}^{N-2}(\tilde{u}_{N-\ell-1}-\tilde{u}_{N-\ell-2})b_\ell^{\beta_{m-j}}\right.$$
$$\left.+\mathcal{K}_{0,N}^{j}A_m^{\beta}(j)\sum_{\ell=0}^{N-1}(\tilde{u}_{N-\ell}-\tilde{u}_{N-\ell-1})b_\ell^{\beta_{m-j}}\right\}$$
$$+\lambda h\sum_{d=1}^{N-1}\left\{\mathcal{K}_{0,d-1/2}^{m}\left[\frac{3}{8}\tilde{u}_{d-1}+\frac{3}{4}\tilde{u}_d-\frac{1}{8}\tilde{u}_{d+1}\right]\right\}+\frac{\lambda h}{2}\left[\mathcal{K}_{0,N-1}^{m}\tilde{u}_{N-1}\right.$$
$$\left.+\mathcal{K}_{0,N}^{m}\tilde{u}_N\right] \qquad (9)$$

Next, evaluate equation (7) at points $t = t_{\bar{r}+1}$ ($\bar{r} = 0,1,\dots,N-1$; $\bar{r} = r-1$) by the same procedure as before using the quadratic midpoint formula (6) for each integral part, we obtain:

$$\sum_{i=0}^{n-1}\mathcal{P}_{i,\bar{r}+1}\left[\,{}_a^C D_t^{\alpha_{n-i}}u(t)\,\right]_{t=t_{\bar{r}+1}}+\mathcal{P}_{n,\bar{r}+1}\tilde{u}_{\bar{r}+1}$$
$$= f_{\bar{r}+1}$$
$$+\lambda\sum_{j=0}^{m-1}\left\{h\sum_{d=1}^{N-1}\mathcal{K}_j\left(t_{\bar{r}+1},s_{d-1/2}\right)\left[{}_a^C D_s^{\beta_{m-j}}u(s)\right]_{s=s_{d-1/2}}\right.$$
$$\left.+\frac{h}{2}\left[\mathcal{K}_j(t_{\bar{r}+1},t_{N-1})\left[{}_a^C D_s^{\beta_{m-j}}u(s)\right]_{s=s_{N-1}}+\mathcal{K}_j(t_{\bar{r}+1},t_N)\left[{}_a^C D_s^{\beta_{m-j}}u(s)\right]_{s=s_N}\right]\right\}$$
$$+\lambda\left\{h\sum_{d=1}^{N-1}\mathcal{K}_m\left(t_{\bar{r}+1},s_{d-1/2}\right)\tilde{u}\left(s_{d-1/2}\right)\right.$$
$$\left.+\frac{h}{2}\left[\mathcal{K}_m(t_{\bar{r}+1},t_{N-1})\tilde{u}(s_{N-1})+\mathcal{K}_m(t_{\bar{r}+1},t_N)\tilde{u}(s_N)\right]\right\} \qquad (10)$$

apply lemma 3 and 4 then we conclude:

$$\sum_{i=0}^{n-1}\mathcal{P}_{i,\bar{r}+1}\,A_n^{\alpha}(i)\sum_{\ell=0}^{\bar{r}}[\tilde{u}_{\bar{r}-\ell+1}-\tilde{u}_{\bar{r}-\ell}]b_\ell^{\alpha_{n-i}}+\mathcal{P}_{n,\bar{r}+1}\tilde{u}_{\bar{r}+1}=f_{\bar{r}+1}$$

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$$
\begin{aligned}
&+\frac{3\lambda h}{8}\sum_{j=0}^{m-1}A_m^{\beta}(j)\left\{\sum_{d=2}^{N-1}\mathcal{K}_{\bar{r}+1,d-1/2}^{j}\left[\sum_{\ell=0}^{d-2}(\tilde{u}_{d-\ell-1}-\tilde{u}_{d-\ell-2})b_\ell^{\beta_{m-j}}\right]\right\}\\
&+\lambda h\sum_{j=0}^{m-1}A_m^{\beta}(j)\left\{\sum_{d=1}^{N-1}\mathcal{K}_{\bar{r}+1,d-1/2}^{j}\left[\frac{3}{4}\sum_{\ell=0}^{d-1}(\tilde{u}_{d-\ell}-\tilde{u}_{d-\ell-1})b_\ell^{\beta_{m-j}}\right.\right.\\
&\left.\left.-\frac{1}{8}\sum_{\ell=0}^{d}(\tilde{u}_{d-\ell+1}-\tilde{u}_{d-\ell})b_\ell^{\beta_{m-j}}\right]\right\}\\
&+\frac{\lambda h}{2}\sum_{j=0}^{m-1}A_m^{\beta}(j)\left\{\mathcal{K}_{\bar{r}+1,N-1}^{j}\sum_{\ell=0}^{N-2}(\tilde{u}_{N-\ell-1}-\tilde{u}_{N-\ell-2})b_\ell^{\beta_{m-j}}\right.\\
&\left.+\mathcal{K}_{\bar{r}+1,N}^{j}\sum_{\ell=0}^{N-1}(\tilde{u}_{N-\ell}-\tilde{u}_{N-\ell-1})b_\ell^{\beta_{m-j}}\right\}\\
&+\lambda h\sum_{d=1}^{N-1}\left\{\mathcal{K}_{\bar{r}+1,d-1/2}^{m}\left[\frac{3}{8}\tilde{u}_{d-1}+\frac{3}{4}\tilde{u}_d-\frac{1}{8}\tilde{u}_{d+1}\right]\right\}+\frac{\lambda h}{2}\left[\mathcal{K}_{\bar{r}+1,N-1}^{m}\tilde{u}_{N-1}\right.\\
&\left.+\mathcal{K}_{\bar{r}+1,N}^{m}\tilde{u}_N\right] \qquad (11)
\end{aligned}
$$

where $A_\ell^{\sigma}(k)$ for fractional orders $\sigma=\alpha$ or β and $\ell=n$ or m respectively for all $k=\overline{0:\ell}$ $(\ell\in\mathbb{Z}^+)$ are defined as:

$$
A_\ell^{\sigma}(k)=\frac{h^{-\sigma_{\ell-j}}}{\Gamma(2-\sigma_{\ell-j})} \qquad (12)
$$

and $\mathcal{K}_{rp}^{j}=\mathcal{K}_j(t_r,s_p)$ all kernels' values for each $r,p=\overline{0:N}$ and $j=\overline{0:m}$.

After some simple manipulation of linear algebraic equations (9) and (11) we construct a linear system of equations that can be written in matrix form:

$$
[L-\lambda hI]\tilde{U}=F \qquad \ldots(13)
$$

where $L=[L_{k\ell}]_{N+1\times N+1}$ is a lower triangular matrix and define each element $L_{k\ell}$ for all $k,\ell=0,1,\ldots,N$ as:

$$
\left.\begin{aligned}
&L_{k,\ell}=0 && \text{for all } k<l\\
&L_{k,k}=\mathcal{H}_n^{\alpha}(k) && \text{for each } k=\overline{0:N}\\
&L_{k,0}=-\sum_{i=0}^{n-1}\mathcal{P}_{i,k}A_n^{\alpha}(i)b_{k-1}^{\alpha_{n-i}} && \text{for all } k=\overline{1:N}\\
&L_{k,\ell}=\sum_{i=0}^{n-1}\mathcal{P}_{i,k}A_n^{\alpha}(i)C_{k-\ell}^{\alpha_{n-i}} && \text{such that all } k>l\\
&\text{for each } k=2,3,\ldots,N \text{ and with } \ell=1,2,\ldots,k-1
\end{aligned}\right\} \qquad (14)
$$

while

$$\mathcal{H}_n^{\alpha}(r) = \begin{cases} \mathcal{P}_{n,0} & \text{if } r = 0 \\ \mathcal{P}_{n,r} + \sum_{i=0}^{n-1} \mathcal{P}_{i,r} A_n^{\alpha}(i) & \text{o.w.} \end{cases} \quad (15)$$

and the coefficients b_ℓ^σ and C_ℓ^σ $(\ell = \overline{0{:}N})$ for any real number $\sigma \in (0,1]$, $(\sigma = \alpha \text{ or } \beta)$ defined as:

$$\left.\begin{aligned} b_\ell^\sigma &= (1+\ell)^{1-\sigma} - \ell^{1-\sigma} \quad ; \quad b_0^\sigma = 1 \\ C_\ell^\sigma &= b_\ell^\sigma - b_{\ell-1}^\sigma \quad ; \quad C_0^\sigma = 1 \quad \text{and assume } b_{-i}^\sigma = 0, \forall i \end{aligned}\right\} \quad (16)$$

Moreover, the matrix $I = [I_{s\ell}]_{N+1\times N+1}$ is a square matrix of dimension $(N+1)$ and each element are defined for all $s, \ell = 0,1,\dots,N$, $N \geq 2$:

$$I_{s,0} = \frac{3}{8}\mathcal{K}_{s,{}^1/_2}^m - \sum_{j=0}^{m-1} A_m^\beta(j) \left\{ \sum_{d=1}^{N-1} \left[\mathcal{K}_{s,d-{}^1/_2}^j \left(\frac{3}{8} b_{d-2}^{\beta_{m-j}} + \frac{3}{4} b_{d-1}^{\beta_{m-j}} - \frac{1}{8} b_d^{\beta_{m-j}} \right) \right] + \frac{1}{2}\left(\mathcal{K}_{s,N-1}^j b_{N-2}^{\beta_{m-j}} + \mathcal{K}_{s,N}^j b_{N-1}^{\beta_{m-j}} \right) \right\}$$

$$I_{s,\ell} = \left[\frac{3}{8}\mathcal{K}_{s,\ell+{}^1/_2}^m + \frac{3}{4}\mathcal{K}_{s,\ell-{}^1/_2}^m - \frac{1}{8} w_\ell \mathcal{K}_{s,\ell-{}^3/_2}^m \right] + \sum_{j=0}^{m-1} A_m^\beta(j) \left\{ \sum_{d=1}^{N-1} \mathcal{K}_{s,d-{}^1/_2}^j \left[\frac{3}{8} C_{d-\ell-1}^{\beta_{m-j}} + \frac{3}{4} C_{d-\ell}^{\beta_{m-j}} - \frac{1}{8} C_{d-\ell+1}^{\beta_{m-j}} \right] + \frac{1}{2}\left[\mathcal{K}_{s,N-1}^j C_{N-\ell-1}^{\beta_{m-j}} + \mathcal{K}_{s,N}^j C_{N-\ell}^{\beta_{m-j}} \right] \right\} \qquad \ell = \overline{1{:}N-2}$$

$$I_{s,N-1} = \left[\frac{3}{4}\mathcal{K}_{s,N-{}^3/_2}^m - \frac{1}{8} w_{N-1} \mathcal{K}_{s,N-{}^5/_2}^m + \frac{1}{2}\mathcal{K}_{s,N-1}^m \right] + \sum_{j=0}^{m-1} A_m^\beta(j) \left\{ \sum_{d=N-1}^{N} \mathcal{K}_{s,d-{}^3/_2}^j \left[\frac{3}{4} C_{d-N}^{\beta_{m-j}} - \frac{1}{8} w_{d-2}^* C_{d-N+1}^{\beta_{m-j}} \right] + \frac{1}{2}\left[\mathcal{K}_{s,N-1}^j C_0^{\beta_{m-j}} + \mathcal{K}_{s,N}^j C_1^{\beta_{m-j}} \right] \right\}$$

$$I_{s,N} = \sum_{j=0}^{m} A_m^\beta(j) \left[\frac{-1}{8}\mathcal{K}_{s,N-{}^3/_2}^j + \frac{1}{2}\mathcal{K}_{s,N}^j \right]$$

(17)

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with

$$w_\ell = \begin{cases} 0 & if\ \ell = 1 \\ 1 & o.w \end{cases}$$

and

$$w_\ell^* = \begin{cases} 0 & if\ \ell < 0 \\ 1 & o.w \end{cases}$$

Furthermore,

$F = [f_0 \quad f_1 \quad \cdots \quad f_N]^T$ and $\widetilde{U} = [\tilde{u}_0 \quad \tilde{u}_1 \quad \cdots \quad \tilde{u}_N]^T$

Here, $f_i = f(t_i)$ and $\tilde{u}_i$ $(i = \overline{0{:}N})$ are the approximate values of $u_i = u(t_i)$.

Finally, in this technique, a boundary condition of equation (1) is added as a new row in the system (13) can be formed in matrix form, this gives:

$$B\overline{U} = C \tag{18}$$

where

$$B = [g_{11} \quad 0 \quad \cdots \quad 0 \quad h_{11}]_{N+1}\ , \qquad \widetilde{U} = [\tilde{u}_0 \quad \tilde{u}_1 \quad \cdots \quad \tilde{u}_N]^T$$

$$\text{and } C = [C_1]$$

from using the boundary condition equation in matrix form (18) and obtaineing a new matrix by adding (18) to (13) for a difference value of $N \geq 2$, yields:

$$D\widetilde{U} = E \tag{19}$$

where

$$D = \begin{bmatrix} L - \lambda hI \\ \dots \dots \dots \\ B \end{bmatrix}_{(N+\mu+1)\times(N+1)} \quad \text{and} \quad E = \begin{bmatrix} F \\ \dots \\ C \end{bmatrix}_{(N+\mu+1)\times 1}$$

To determine the approximate column vector $\widetilde{U}$'s, store the matrix D and compute $D^T D$ and $D^T E$ then use LU-factorization procedure to solve $[D^T D]\widetilde{U} = [D^T E]$. The approximate solution for all $\tilde{u}_i$ at each point t_i $(i = \overline{0{:}N})$ is obtained for fractional order linear FIDE's (1).

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5. THE ALGORITHM (AFIFM)

The following steps can be used to describe the approximate solution for linear IFDEs of Fredholm type with variable coefficients using the open Newton-Cotes formula (quadrature midpoint rule) and finite difference approximation:

Step 1:

a. Input $N \in \mathbb{Z}^+$, take $h = (b-a)/N$ and $t_r = a + rh$.
b. Input the coefficients of boundary conditions g_{11}, h_{11} and C_1.

Step 2: To compute $A_\ell^\sigma(k)$ for each $k = 0,1,\dots,\ell, (\in \mathbb{Z}^+)$ and for all $\sigma = \alpha$ or β and $\ell = n$ or m, respectively, applied equation (12).

Step 3: Using equation (15) and step 2 for all fractional orders $\alpha_n > \alpha_{n-1} > \cdots > \alpha_1 > \alpha_0 = 0$ to evaluate $\mathcal{H}_n^\alpha(r), r \in \mathbb{Z}^+$.

Step 4: For all $\ell = 0,1,\dots,N$ find the constant coefficients (b_ℓ^σ and C_ℓ^σ) for fractional orders $\sigma = \alpha$ and β respectively using equation (16).

Step 5: For all $k, \ell = 0,1,\dots,N$ evaluate each element $L_{k,\ell}$ using formulas in equation (14) with steps (2,3 and 4). Finally, construct the lower triangular matrix $L = [L_{k\ell}]_{N+1\times N+1}$.

Step 6: Evaluate the values of kernels at each given point, $\mathcal{K}_{s\ell}^j = \mathcal{K}_j(t_s, t_\ell)$ for all $j = 0,1,\dots,m$ and $s, \ell = 0,1,\dots,N$.

Step 7: For all $s, \ell = 0,1,\dots,N$ calculate each elements $I_{s\ell}$ using formulas in equation (17) calling steps (4 and 6). Finally, construct the square matrix $I = [I_{s\ell}]_{N+1\times N+1}$.

Step 8: Compute all elements of column vector F at points t_r by $f_r = f(t_r), t_r = a + rh$ $(r = 0,1,\dots,N)$.

Step 9: Putting boundary conditions g_{11}, h_{11} and C_1 into matrices B and C to form (18).

Step 10: Construct the matrices D and E that are represented in the system (19).

Step 11: Apply the numerical method for the system, which is obtained in step 10 after multiplying both sides by D^T, to compute the column-approximate values $\widetilde{U}$ of exact solution U.

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6. NUMERICAL PERFORMANCE:

The numerical section uses the L_2 error norm to verify the correctness and efficacy of the proposed schemes. The suggested algorithm AFIFM produces numerical results that are compared, and MATLAB is used to generate both the numerical and graphical results.

Example 1. ([22]) Consider a multi-fractional order IDE of the Fredholm type with variable coefficients:

$$
{}_0^C D_t^{0.7} u(t) + \sinh(t) u(t) = \frac{6}{\Gamma(2.3)} t^{1.3} + \sinh(t)(3t^2+2) - \frac{6e^t}{4.2\Gamma(2.2)} - \frac{6}{3.5\Gamma(2.5)} t^2 + \frac{6}{\Gamma(3.5)} - 5e^{t+1} + 8e^t + \int_0^1 [(s^2 e^t)\,{}_0^C D_s^{0.8} u(s) + (st^2 - 1)\,{}_0^C D_s^{0.5} u(s) + (e^{s+t}) u(s)] ds
$$

subjected to the boundary conditions: $u(0) + u(1) = 7$. It has the exact solution is $u(t) = 3t^2 + 2$. This example obtained the numerical computation at every point by applying the suggested strategy which is a linear IFDE of Fredholm type with variable coefficients for fractional order lies in (0,1): Take $N = 10$ and $t_r = t_0 + rh$, $(r = \overline{0:N})$. Since $(n, m) = (1,2)$ and the fractional orders are$\alpha_1 = 0.7$, $\alpha_0 = 0$ and $\beta_2 = 0.8$, $\beta_1 = 0.5$, $\beta_0 = 0$ with boundary coefficients $g_{11} = h_{11} = 1$ and $C_1 = 7$. We obtained the numerical results by using the suggested approach to the mentioned situation. Set the MATLAB application Main N-CQudratureMid, and use equation (12) to get the following results:

$$
A_1^\alpha(0) = 5.5844412044 \quad A_1^\alpha(1) = 1
$$
$$
A_2^\beta(0) = 6.8719105251 \quad A_2^\beta(1) = 3.5682482323 \quad A_2^\beta(2) = 1
$$

Table 1 contains all values of $\mathcal{H}_2^\alpha(r)$ for each $t_r = 0(0.1)1$ for $r = \overline{1:10}$ with $\mathcal{H}_2^\alpha(0) = 0$ using equation (15). Also, the various values of b_ℓ^σ and C_ℓ^σ for fractions $\sigma = \alpha$ and β for all $\ell = 0,1, \dots ,10$ are presented in tables 2 and 3 respectively, by applying equation (16).

Table 1. Include value of $\mathcal{H}_2^\alpha(r)$ for each $t_r (r = 0,1, \dots ,10)$ with $\mathcal{H}_2^\alpha(0) = 0$

t_r	0.1	0.2	0.3	0.4	0.5
$\mathcal{H}_2^\alpha(r)$	5.6846079545	5.7857772070	5.8889614979	5.9951935302	6.1055365099
t_r	0.6	0.7	0.8	0.9	1.0
$\mathcal{H}_2^\alpha(r)$	6.2210947866	6.3430249063	6.4725471866	6.6109579301	6.7596423981

Table 2. Contain all values of b_ℓ^σ for fractions $\sigma = \alpha$ and β for all $\ell = 0,1,\ldots,10$

orders	α-fractional		β-fractional		
ℓ	$b_\ell^{\alpha_0}$	$b_\ell^{\alpha_1}$	$b_\ell^{\beta_0}$	$b_\ell^{\beta_1}$	$b_\ell^{\beta_2}$
1	1	1	1	1	1
2	1	0.2311444133	1	0.4142135623	0.1486983549
3	1	0.1592447569	1	0.3178372451	0.0970325846
4	1	0.1253273961	1	0.2679491924	0.0737769711
5	1	0.1049400301	1	0.2360679774	0.0602217506
6	1	0.0911132627	1	0.2134217652	0.0512394196
7	1	0.0810201031	1	0.1962615682	0.0448040804
8	1	0.0732760205	1	0.1826758136	0.0399434049
9	1	0.0671160618	1	0.1715728752	0.0361290074
10	1	0.0620802700	1	0.1622776601	0.0330476185

Table 3. Contain all values of C_ℓ^σ for fractions $\sigma = \alpha$ and β for all $\ell = 0,1,\ldots,10$

orders	α-fractional		β-fractional		
ℓ	$C_\ell^{\alpha_0}$	$C_\ell^{\alpha_1}$	$C_\ell^{\beta_0}$	$C_\ell^{\beta_1}$	$C_\ell^{\beta_2}$
1	1	1	1	1	1
2	0	−0.7688555866	0	−0.5857864376	−0.8513016450
3	0	−0.0718996563	0	−0.0963763171	−0.0516657703
4	0	−0.0339173607	0	−0.0498880527	−0.0232556134
5	0	−0.0203873660	0	−0.0318812149	−0.0135552204
6	0	−0.0138267674	0	−0.0226462122	−0.0089823310
7	0	−0.0100931596	0	−0.0171601970	−0.0064353391
8	0	−0.0077440825	0	−0.0135857545	−0.0048606755
9	0	−0.0061599586	0	−0.0111029384	−0.0038143975
10	0	−0.0050357918	0	−0.0092952150	−0.0030813888

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The matrices L and I in the finite differences quadratic midpoint methods are formed as in equations (14 for L-matrix) and (17 for I-matrix), running program to obtain:

$$
\boldsymbol{L} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-5.5844 & 5.6846 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1.2908 & -4.2936 & 5.7857 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.8892 & -0.4015 & -4.2936 & 5.8889 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.6998 & -0.1894 & -0.4015 & -4.2936 & 5.9951 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.5860 & -0.1138 & -0.1894 & -0.4015 & -4.2936 & 6.1055 & 0 & 0 & 0 & 0 & 0 \\
-0.5088 & -0.0772 & -0.1138 & -0.1894 & -0.4015 & -4.2936 & 6.2210 & 0 & 0 & 0 & 0 \\
-0.4524 & -0.0563 & -0.0772 & -0.1138 & -0.1894 & -0.4015 & -4.2936 & 6.3430 & 0 & 0 & 0 \\
-0.4092 & -0.0432 & -0.0563 & -0.0772 & -0.1138 & -0.1894 & -0.4015 & -4.2936 & 6.4725 & 0 & 0 \\
-0.3748 & -0.0343 & -0.0432 & -0.0563 & -0.0772 & -0.1138 & -0.1894 & -0.4015 & -4.2936 & 6.6109 & 0 \\
-0.3466 & -0.0281 & -0.0343 & -0.0432 & -0.0563 & -0.0772 & -0.1138 & -0.1894 & -0.4015 & -4.2936 & 6.7596
\end{bmatrix}_{11\times 11}
$$

$$
\boldsymbol{I_{Mid}} = \begin{bmatrix}
10.7183 & -0.1575 & 0.1100 & 0.0740 & 0.0624 & 0.0797 & 0.1329 & 0.2359 & 0.2625 & 3.1164 & 2.5439 \\
10.6012 & -0.0696 & 0.1842 & 0.1500 & 0.1442 & 0.1720 & 0.2424 & 0.3726 & 0.4109 & 3.8111 & 2.9662 \\
10.3944 & 0.0180 & 0.2590 & 0.2290 & 0.2318 & 0.2741 & 0.3668 & 0.5315 & 0.5871 & 4.6305 & 3.4596 \\
10.0972 & 0.1066 & 0.3353 & 0.3117 & 0.3262 & 0.3869 & 0.5074 & 0.7141 & 0.7927 & 5.5819 & 4.0285 \\
9.7086 & 0.1972 & 0.4140 & 0.3991 & 0.4283 & 0.5116 & 0.6655 & 0.9220 & 1.0293 & 6.6736 & 4.6780 \\
9.2276 & 0.2910 & 0.4963 & 0.4924 & 0.5395 & 0.6495 & 0.8425 & 1.1570 & 1.2989 & 7.9145 & 5.4135 \\
8.6530 & 0.3895 & 0.5833 & 0.5927 & 0.6608 & 0.8019 & 1.0400 & 1.4211 & 1.6037 & 9.3145 & 6.2412 \\
7.9836 & 0.4941 & 0.6762 & 0.7012 & 0.7936 & 0.9703 & 1.2597 & 1.7164 & 1.9459 & 10.8848 & 7.1677 \\
7.2179 & 0.6066 & 0.7765 & 0.8194 & 0.9395 & 1.1565 & 1.5038 & 2.0454 & 2.3282 & 12.6374 & 8.2007 \\
6.3546 & 0.7288 & 0.8857 & 0.9490 & 1.1001 & 1.3622 & 1.7742 & 2.4107 & 2.7534 & 14.5859 & 9.3481 \\
5.3918 & 0.8627 & 1.0057 & 1.0916 & 1.2773 & 1.5896 & 2.0735 & 2.8154 & 3.2247 & 16.7450 & 10.6193
\end{bmatrix}_{11\times 11}
$$

From the boundary condition equation, the matrix form is computed as:

$$[B; C] = [1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ 1; 7]$$

Substituting the above matrices for the fundamental equation, the augmented matrix is obtained based on a condition that is:

$$
[\boldsymbol{D}; \boldsymbol{E}]_{\boldsymbol{Mid}} = \left[\begin{array}{ccccccccccc|c}
-1.0718 & 0.0157 & -0.0110 & -0.0074 & -0.0062 & -0.0079 & -0.0132 & -0.0235 & -0.0262 & -0.3116 & -0.2543 & ; -5.0825 \\
-6.6445 & 5.6915 & -0.0184 & -0.0150 & -0.0144 & -0.0172 & -0.0242 & -0.0372 & -0.0410 & -0.3811 & -0.2966 & ; -5.3588 \\
-2.3302 & -4.2954 & 5.7598 & -0.0229 & -0.0231 & -0.0274 & -0.0366 & -0.0531 & -0.0587 & -0.4630 & -0.3459 & ; -5.5977 \\
-1.8990 & -0.4121 & -4.3271 & 5.8577 & -0.0326 & -0.0386 & -0.0507 & -0.0714 & -0.0792 & -0.5581 & -0.4028 & ; -5.8421 \\
-1.6707 & -0.2091 & -0.4429 & -4.3335 & 5.9523 & -0.0511 & -0.0665 & -0.0922 & -0.1029 & -0.6673 & -0.4678 & ; -6.0952 \\
-1.5087 & -0.1429 & -0.2390 & -0.4507 & -4.3475 & 6.0405 & -0.0842 & -0.1157 & -0.1298 & -0.7914 & -0.5413 & ; -6.3517 \\
-1.3741 & -0.1161 & -0.1721 & -0.2486 & -0.4675 & -4.3738 & 6.1170 & -0.1421 & -0.1603 & -0.9314 & -0.6241 & ; -6.6014 \\
-1.2508 & -0.1057 & -0.1448 & -0.1839 & -0.2687 & -0.4985 & -4.4196 & 6.1713 & -0.1945 & -1.0884 & -0.7167 & ; -6.8303 \\
-1.1310 & -0.1039 & -0.1340 & -0.1591 & -0.2078 & -0.3050 & -0.5519 & -4.4981 & 6.2397 & -1.2637 & -0.8200 & ; -7.0203 \\
-1.0102 & -0.1072 & -0.1318 & -0.1512 & -0.1872 & -0.2500 & -0.3668 & -0.6425 & -4.5689 & 5.1523 & -0.9348 & ; -7.1490 \\
-0.8858 & -0.1144 & -0.1349 & -0.1524 & -0.1840 & -0.2361 & -0.3212 & -0.4709 & -0.7239 & -5.9681 & 5.6977 & ; -7.1889 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & ; 7
\end{array}\right]
$$

Solving the system above by$[D^T D; D^T E]$, the approximate solutions $\tilde{u}(t)$ are obtained. Table 4 shows a comparison between the exact solution and numerical solutions $\tilde{u}(t) = u_N(t)$ for $N = 10$ of the finite difference Quadratic Midpoint Method depending on the least square error and running

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time. Also, the results, least square error, and the required time for running the programs for different values, i.e., different step size h, are shown in Table 5.

Figure 1: displays a comparison between the approximate solutions and the exact solution with a step size of $h = 0.1$ and $N = 10$ for example 1. The absolute errors are also depicted in Figure 2.

Table 4. Numerical results for different values of t and comparison between methods

t_r	Exact Solution	Approximate Solution		
		Trapezoidal ([19])	Simpson ([19])	Proposed Method: Q-Midpoint
0	2	1.9247264228	1.9695561787	1.9779087104
0.1	2.03	1.9709652084	2.0125419097	2.0203666961
0.2	2.12	2.0763138666	2.1131947559	2.1201273765
0.3	2.27	2.2416391970	2.2722603624	2.2780889009
0.4	2.48	2.4671462581	2.4902016222	2.4945824415
0.5	2.75	2.7528411826	2.7670192724	2.7697732560
0.6	3.08	3.0986472401	3.1029149354	3.1037712876
0.7	3.47	3.5044460741	3.4978462512	3.4966686664
0.8	3.92	3.9700966058	3.9519418411	3.9485538829
0.9	4.43	4.4954462045	4.4652026508	4.4595144645
1	5	5.0803238017	5.0376880152	5.0296387137
$L.S.E.$		2.681636×10^{-02}	6.657162×10^{-03}	5.091074×10^{-03}
$R.Time/Sec$		0.773492	6.73552	0.872449

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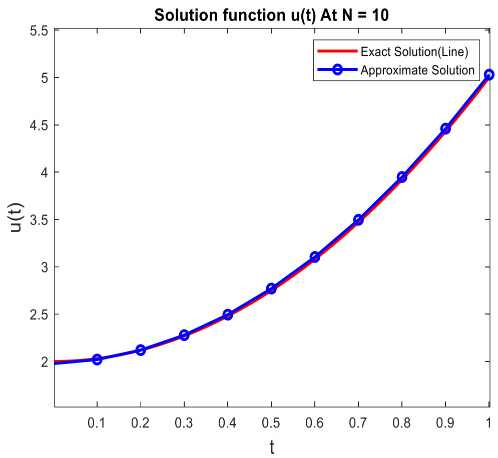

Figure 1. For example, 1 with a step size of h=0.1, the approximate solutions are shown by bullets, while the precise solution is shown by a solid line.

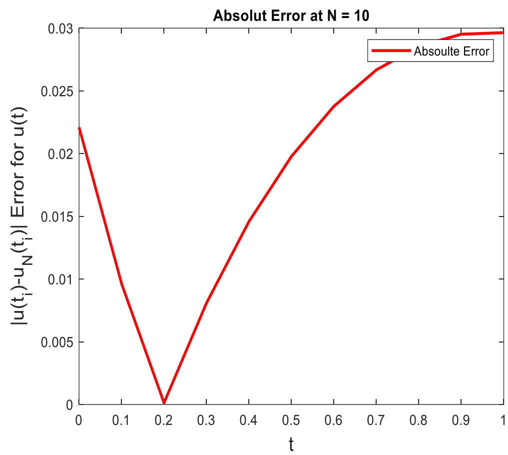

Figure 2. Absolute error plot function $|u(t) - u_N(t)|$ for $N = 10, h = 0.1$ for example 1.

Table 5. Shows the running times and least square errors for the suggested quadrature Midpoint technique and results in [19], with varying step size h values.

h / Methods	0.1		0.02		0.01	
	$L.S.E.$	$R.Time\ /Sec$	$L.S.E.$	$R.Time\ /Sec$	$L.S.E.$	$R.Time\ /Sec$
Trap. M.	2.681636×10^{-02}	0.773492	5.058864×10^{-04}	5.540379	1.246178×10^{-04}	19.32022
Simp. M.	6.657162×10^{-03}	6.73552	3.614007×10^{-04}	544.96858	1.111846×10^{-04}	4290.0595
Q-Mid. M.	5.091074×10^{-03}	0.872449	3.373301×10^{-04}	7.208868	1.100893×10^{-04}	27.078209

Example 2 ([22]) Consider a linear IFDE containing various fractional orders on the closed bounded interval [0,1] with variable coefficients:

$$
\begin{aligned}
{}_0^C D_t^{2\alpha} u(t) - t^2\, {}_0^C D_t^{\alpha} u(t) + \sin(t) u(t) \\
&= \frac{12}{\Gamma(4-2\alpha)} t^{3-2\alpha} - \frac{6}{\Gamma(3-2\alpha)} t^{2-2\alpha} - \frac{12}{\Gamma(4-\alpha)} t^{5-\alpha} + \frac{6}{\Gamma(3-\alpha)} t^{4-\alpha} \\
&+ (2t^3 - 3t^2 + 1)\sin(t) - \left[\frac{24}{(5-\beta)\Gamma(4-\beta)} - \frac{12}{(4-\beta)\Gamma(3-\beta)}\right] t^2 \\
&- \frac{\lambda}{2}\left(\frac{3}{10} t^2 + 1\right) + \lambda \int_0^1 \left[2st^2\, {}_0^C D_s^{\beta} u(s) + (1 + st^2) u(s)\right] ds
\end{aligned}
$$

if $0 < \alpha \leq 0.5$ and $0 < \beta \leq 1$ with boundary condition $u(0) + u(1) = 1$ and $\lambda \in \mathbb{R}$. while the exact solution is $u(t) = 2t^3 - 3t^2 + 1$.

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Putting $\alpha = 0.2$ and $\beta = 0.5$ with $\lambda = 1$. Take $N = 10$ and $t_r = 0{:}\,0.1{:}\,1$ for $r = 0,1,2 \dots N$. Here $n = 2, m = 1$ and applying the AFIFM-algorithm, we obtain:

$$A_2^{\alpha}(0) = 2.8112403816 \;\;,\;\; A_2^{\alpha}(1) = 1.7016542931 \;\;,\;\; A_2^{\alpha}(2) = 1$$

$$A_1^{\beta}(0) = 3.5682482323 \;\;,\;\; A_1^{\beta}(1) = 1$$

Table 6 contains all values of $\mathcal{H}_2^{\alpha}(r)$ for each $t_r = 0(0.1)1$ for $r = \overline{1{:}\,10}$ with $\mathcal{H}_2^{\alpha}(0) = 0$. Moreover, Table 7 shows a comparison between the exact solution and numerical solutions of the finite difference Quadratic Midpoint Method depending on the least square error and running time. Also, the results, least square error, and the required time for running the programs for different values, i.e., different step sizes h, are presented in Table 8. Also, Figure 3 displays a comparison between the approximate solutions and the exact solution with a step size of h and $N = 10$ for example 2. The absolute errors are also depicted in Figure 4.

Table 6. Contain all values of $\mathcal{H}_2^{\alpha}(r)$ for each $t_r(r = \overline{1{:}\,10})$ with $\mathcal{H}_2^{\alpha}(0) = 0$

t_r	0.1	0.2	0.3	0.4	0.5
$\mathcal{H}_2^{\alpha}(r)$	2.8940572553	2.9418435406	2.9536117018	2.9283940370	2.8652523469
t_r	0.6	0.7	0.8	0.9	1.0
$\mathcal{H}_2^{\alpha}(r)$	2.7632873094	2.6216474652	2.4395377248	2.2162273137	1.9510570732

Table 7. Numerical results for different values of t and comparison between methods

t_r	Exact Solution	Approximate Solution		
		Trapezoidal ([19])	Simpson ([19])	Proposed Method: Q-Midpoint
0	1	1.0128515271	1.0135428804	1.0118444092
0.1	0.972	0.9776809241	0.9784876143	0.9764714336
0.2	0.896	0.8986620799	0.8994831238	0.8974204538
0.3	0.784	0.7851674849	0.7859527299	0.7839696749
0.4	0.648	0.6483087225	0.6490279319	0.6472104466
0.5	0.5	0.4996302203	0.5002444799	0.4986831141
0.6	0.352	0.3508188633	0.3513007042	0.3500782391
0.7	0.216	0.2135916114	0.2138882783	0.2131234868
0.8	0.104	0.0996167036	0.0996853063	0.0995072617
0.9	0.028	0.0204121447	0.0201559050	0.0207673005
1	0	−0.0128012553	−0.0135214467	−0.0117031689
$L.S.E.$		4.539727×10^{-04}	5.104847×10^{-04}	3.860882×10^{-04}
$R.Time/Sec$		1.189265	4.275868	1.242617

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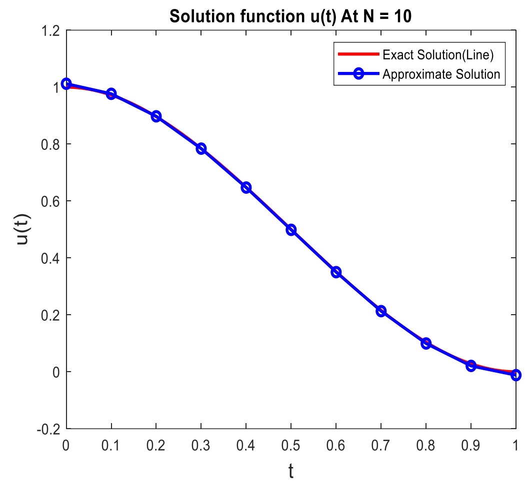

Figure 3. For example, 2 with a step size of h=0.1, the approximate solutions are shown by bullets, while the precise solution is shown by a solid line.

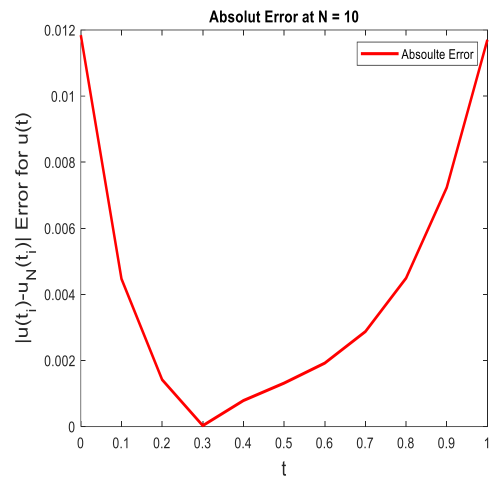

Figure 4. Absolute error plot function $|u(t) - u_N(t)|$ for $N = 10, h = 0.1$ for example 2.

Table 8. Shows the running times and least square errors for the suggested quadrature Midpoint technique and results in [19], with varying step size h values.

h / Methods	0.1		0.02		0.01	
	$L.S.E.$	$R.Time/Sec$	$L.S.E.$	$R.Time/Sec$	$L.S.E.$	$R.Time/Sec$
Trap. M.	4.539727×10^{-04}	1.176075	1.202408×10^{-05}	14.596982	3.006153×10^{-06}	54.893073
Simp. M.	5.104847×10^{-04}	4.238554	1.375432×10^{-05}	301.97256	3.344898×10^{-06}	2261.5876
Q-Mid. M.	3.860882×10^{-04}	1.247561	1.388952×10^{-05}	15.872927	3.450703×10^{-06}	60.766833

Example 3. Consider a multi-fractional order linear IFDE with variable coefficients on the closed bounded interval $[a, b]; a, b \in \mathbb{R}$:

$$
\begin{aligned}
{}_a^C D_t^{\alpha_1} u(t) + t^2 u(t) \\
&= t^2 - t^2 e^{t-a} \\
&- \lim_{M\to\infty} \sum_{k=0}^{M} \left[\frac{(t-a)^{k-\alpha_1+1}}{\Gamma(k-\alpha_1+2)} - \frac{\lambda \sin(t)\,(b-a)^{k-\beta_2+2}}{\Gamma(k-\beta_2+3)} - \frac{\lambda t (b-a)^{k-\beta_1+2}}{\Gamma(k-\beta_1+3)} \right. \\
&\left. + \frac{\lambda (b-a)^{k-\beta_1+4}}{(k-\beta_1+4)\Gamma(k-\beta_1+2)} \right] \\
&+ \lambda \int_a^b \left[\sin(t)\, {}_a^C D_s^{\beta_2} u(s) + (t-(s-a)^2)\, {}_a^C D_s^{\beta_1} u(s) \right] ds
\end{aligned}
$$

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For all $\alpha_1, \beta_2, \beta_1$ are real fractional order lies in $(0,1]$ with boundary condition $u(a) - u(b) = e^{b-a} - 1$ and $\lambda \in \mathbb{R}$. while the exact solution is $u(t) = 1 - e^{t-a}$.

Here, the proposed finite difference quadrature-midpoint method is used to obtain its numerical computation on the bounded interval $[a, b] = [1,2]$. Values of the approximate solution for the mentioned equation with different fractional orders α_1, β_2 and β_1 found by taking $N = 10$ and the number of Mittag-Leffler terms $M = 4$ and 10. Table 9 it illustrates a comparison between the approximate and exact solutions for $N = 10, \lambda = \frac{1}{2}$, and fractional orders $(\alpha_1, \beta_2, \beta_1) = (0.6,0.4,0.3)$, while the error profile is depicted in figures 5 and 6 that presents a comparison of absolute errors and approximate solutions for the proposed method. Also, the results, least square error, and the required time for running the programs for different values, i.e., different step sizes h, are presented in Table 10.

Table 9. Numerical results for different values of $M = 4$ and 10 on the interval [1,2].

t_r	Exact solution	Proposed method for example 3 with $N = 10, \lambda = \frac{1}{2}$, and $(\alpha_1, \beta_2, \beta_1) = (0.6,0.4,0.3)$			
		For $M = 4$		For $M = 10$	
		Approximate solution	Absolute error	Approximate solution	Absolute error
1.0	0.0	−0.0006476139	0.0006476139	−0.0007335522	0.00073355226
1.1	−0.1051709	−0.10712899	0.0019580765	−0.10714454	0.0019736268
1.2	−0.2214027	−0.22420307	0.0028003167	−0.2241767	0.0027739382
1.3	−0.3498588	−0.35326106	0.0034022512	−0.35320704	0.0033482277
1.4	−0.4918247	−0.49567638	0.0038516827	−0.49560676	0.0037820604
1.5	−0.6487212	−0.65291331	0.0041920401	−0.6528432	0.0041219291
1.6	−0.8221188	−0.82656393	0.0044451301	−0.82651645	0.0043976537
1.7	−1.0137527	−1.0183707	0.0046180037	−1.0183826	0.0046298845
1.8	−1.2255409	−1.2302459	0.0047049604	−1.2303737	0.0048327337
1.9	−1.4596031	−1.4642902	0.0046871117	−1.4646167	0.0050136384
2.0	−1.7182818	−1.7227991	0.0045172216	−1.7234381	0.0051562671
$L.S.E.$		0.000161675313677		0.000170487106942	
$R.Time/Sec$		0.8630482		0.9414132	

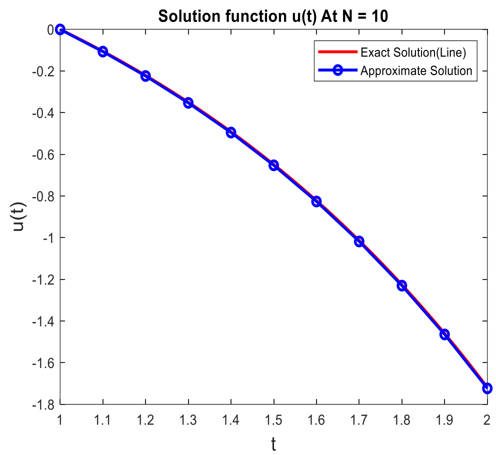

Figure 5. The exact solution (shown by a solid line) and the approximate solutions (shown by bullets) for example 3 with a step size of $h = 0.1$ and $M = 10$.

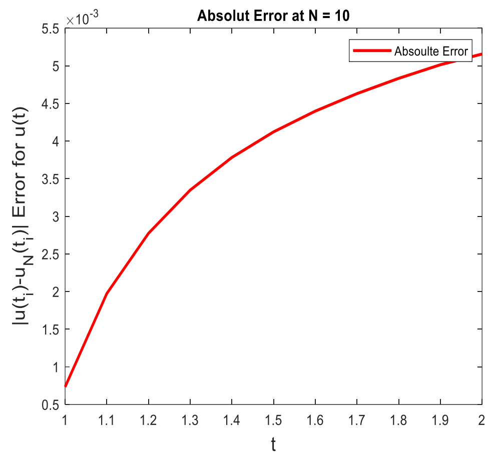

Figure 6. Absolute error plot function $|u(t) - u_N(t)|$ for $N = 10, h = 0.1$ and $M = 10$ for example 3.

Table 10. LSEs of approximate solution for various values of fractional orders and eigenvalues λ when $M = 10$ in example 3 on interval $[a, b] = [1,2]$.

Fractional orders: $(\alpha_1, \beta_2, \beta_1)$		(0.8, 0.5,0.3)		(0.5, 0.3,0.1)		(0.2, 0.3,0.1)	
Eigenvalues: λ		6/5	1/5	6/5	1/5	6/5	1/5
$N = 10$	$L.S.E.$	1.37745×10^{-03}	1.53157×10^{-03}	8.88592×10^{-05}	1.17213×10^{-04}	3.38062×10^{-05}	6.152295×10^{-07}
	$R.Time/Sec$	0.95393	0.94826	0.94322	0.94464	0.97481	0.96859
$N = 100$	$L.S.E.$	1.30419×10^{-05}	8.29523×10^{-06}	2.46491×10^{-07}	1.79466×10^{-07}	1.09317×10^{-08}	3.191955×10^{-10}
	$R.Time/Sec$	34.725	34.388	34.2061	34.3417	34.157	34.3423
$N = 300$	$L.S.E.$	1.09791×10^{-06}	6.25934×10^{-07}	1.097034×10^{-08}	7.17602×10^{-09}	2.91557×10^{-10}	6.127364×10^{-12}
	$R.Time/Sec$	299.046	296.351	295.2252	294.589	297.815	295.8053

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7. CONCLUSION

In this article, the combination of the quadratic midpoint method and forward finite difference scheme is investigated for solving Fredholm integro-differential equations for Caputo fractional orders with variable coefficients. First, we show that IFDEs (1)-(2) are equivalent to linear algebraic systems in matrix forms (13) with a boundary conditions vector (18). Then we use any numerical rule to find the solution to it, which is the solution to our problem (1-2). The special algorithm and computer program were written for this purpose. In addition, we used numerical methods to solve a number of cases pertaining to the suggested equations. Among the numerical schemes used in the literature, the numerical results produced an extraordinary absolute error. The following conclusions are drawn from the tabular representations of the least square error and running time for the comparison of accuracy and speed:

1. With equal step sizes, the AFIFM algorithm gives better accuracy than the trapezium and Simpson methods ([22]), as in running examples 1 and 2.
2. Both the procedure and the step length h affect the accuracy of the findings; that is, the accuracy increases as h is decreased, thereby increasing the number of partitions N.
3. To minimize the error terms on the specified domain, we need to increase the value of N and only a few numbers of M (in stale example 3) can be used for numerical purposes with a high degree of accuracy see all the tables (9) and (10).

Data Availability Statement: Authors can confirm that all relevant data are included in the article.

Conflicts of Interest: The article declares no conflicts of interest.

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Informed Consent Statement: Informed consent was obtained from all subjects involved in the study.

Abbreviations

The following abbreviations are used in this manuscript:

FIFDEs	Fredholm Integro Fractional Differential Equations
FIDEs	Fredholm Integro Differential Equations
IFDEs	Integro Fractional Differential Equations
IDE	Integro Differential Equation
AFIFM	Algorithm of Fractional Integro Fredholm Midpoint
L.S.E.	Least square error
R.Time/Sec	Running Time of any Program per Second

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Conflict of interests.

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مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

الخلاصة

مقدمة:

تقدم هذه المقالة طريقةً لإيجاد حلول عددية لمعادلات فريد هولم التكاملية التفاضلية (FIFDEs) ذات الرتب الكسرية المتعددة التي تساوي واحدًا أو أقل، باستخدام خوارزمية فعّالة.

طرق العمل:

استُخدم تقريب الفروق المحدودة لمشتقة كابوتو باستخدام نقاط التجميع لبناء طريقة نقطة المنتصف لقاعدة التربيع، التي تُشكل أساس هذه الطريقة.

الاستنتاجات:

تُبسط طريقتنا تقييم المعالجات بتحويل معادلات فريد هولم التكاملية التفاضلية إلى معادلات جبرية ذات مصفوفات تشغيلية. بعد حساب مشتقة كابوتو عند نقطة محددة باستخدام طريقة الفروق المحدودة، نستخدم طريقة التربيع، التي تتضمن قاعدة نقطة المنتصف، لإنشاء صيغة فرق محدودة لمعادلتنا الكسرية.

بالإضافة إلى ذلك، تُقدم أمثلة عددية لإثبات صحة هذه الطريقة واستخدامها، بالإضافة إلى مقارنات مع النتائج السابقة. يتم التعبير عن النتائج باستخدام برنامج تم إنشاؤه في.MatLab

الكلمات المفتاحية: حساب الكسور ، المشتقة الكسرية كابوتو، المعادلة التكاملية التفاضلية، قواعد نقاط المنتصف، تقريب الفرق الأمامي.