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Modified Iterative Method for Multiple Roots

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ABSTRACT

Boole's rule and Weddle's rule modified quadrature iterated techniques are provided in this study for locating multiple non-linear equations' roots, the suggested approaches converged cubically, Newton's approach was used for discovering numerous nonlinear equation roots. Several modifications have been made to achieve a higher degree of convergence. The modified classical methods developed by many authors to solve multiple roots of nonlinear equations have been effective in overcoming the deficiency of the classical Newton Raphson method, however there are new trends of methods proposed by authors, which have proven to be more efficient than some already existing ones. There are several numerical examples that support the suggested technique's justification as an evaluation of the Newton-Raphson method and Simpson's approach, Maple 18 is used to investigate numerical result representations of modified quadrature iterative Algorithms. The result from numerical findings is that the presented modified quadrature iterated techniques for finding multiple roots of non-linear equations outperform existing approaches in terms of performance; these methods were programmed by using software Maple.

Background: Nonlinear equations with multiple roots are challenging to solve accurately using classical Newton’s method due to its slow convergence in such cases. This limitation motivates the development of improved iterative methods that offer faster and more reliable convergence. These enhanced techniques aim to overcome the shortcomings of traditional approaches.

Results: This study developed enhanced iterative methods to improve Newton’s method for solving equations with multiple roots. The new techniques proved to be accurate, fast, and reliable through various practical tests. Using Maple 18 for implementation, these methods demonstrated strong performance compared to existing approaches.

Conclusion This work presented improved iterative methods that make Newton’s technique more effective for equations with multiple roots. Through several practical tests, the new methods proved to be accurate, fast, and reliable. Built and tested using Maple 18, they held up well against other well-known approaches

Key words: Boole's rule, Weddle’s rule, Simpson's rule multiple roots, Iterative Methods, Newton's method.

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INTRODUCTION

Finding equation solutions or determining the roots of problems is a crucial task in mathematical computations, particularly for a wide range of engineering applications, including those in numerous scientific domains. Many real-world issues can be solved using equation roots. Since accuracy of the result is crucial for the majority of practical situations, determining the most effective numerical method for the task is crucial [13].

One of the most major problems in numerical analysis is the solution of nonlinear equations [4].

In this work, we consider an iterative method for finding the multiple root ϕ of multiplicity μ .That is, i.e. $g^{l}(\phi)=0, \quad l=0,1,\ldots \mu-1$ and $g^{\mu}(\phi)\neq 0$ of the nonlinear equation $g(x) = 0$. The well-known Newton's method, often known as the classical method, was developed by

$$x_{k+1}=x_k-\frac{g(x_k)}{g'(x_k)} \qquad k=0,1,2,\ldots. \qquad \ldots (1)$$

Which converges quadratic ally [9].

Therefore, several of its improvements have been made in order to get a method with a better degree of convergence[1]-[3],[7],[9]-[11], In [1] suggested a novel approach for easy root handling of nonlinear equations based on Lisa's three-eighths rule, which is dubbed the Simpson-like method. This approach is cubically converging. When a nonlinear equation has several roots, the power source Simpson-like method is linearly convergent. There are various numerical approaches for discovering many roots of nonlinear equations. One of the improved Newton's methods for various roots. [4] Is as below:

$$x_{k+1}=x_k-\frac{g^{(\mu-1)}(x_k)}{g^{(\mu)}(x_k)} \qquad k=0,1,2,\ldots. \qquad \ldots (2)$$

Since ϕ a simple root of $g^{(\mu-1)}(x)=0$.

There have been some changes proposed to Newton's multiple-root method then proposed in past years, which further require knowledge of the multiplicity μ .As well as the convergence order of eq.(2) has been improved, to put it another way, a modification of Newton's Method for multiple roots of nonlinear equations with cubic convergence was presented in this study, additionally, it demonstrated that the strategy suggested in this article performs suitable compared to the presented method in [6], [8]-[14].

The sequence created by eq. (2) is quadratically convergent. A cubically convergent also exists in [4]

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$$x_{k+1} = x_k - \frac{8g^{(\mu-1)}(x_k)}{g^{(\mu)}(x_k) + 3g^{(\mu)}(x_k + \frac{x_{k+1} - x_k}{3}) + 3g^{(\mu)}(x_k + 2\frac{x_{k+1} - x_k}{3}) + g^{(\mu)}(x_{k+1})}.$$

… (3)

In this article presents a study on modified quadrature iterated techniques, specifically Boole's and Weddle's rules, for efficiently locating multiple roots of nonlinear equations. The central thesis posits that these new methods offer enhanced performance and faster convergence compared to traditional approaches such as Newton’s method and Simpson's rule.

The rest of paper is organized as follows: The description of the proposed method is introduced, the analysis of convergence is shown. The result of solving some numerical examples is given, in the rest of the paper, gives a summary conclusion on findings from the research.

Boole’s Rule Iterated Method

This section enhances the modified Newton Method described in eq. (2) as follows [4]: the equation

$$g^{(\mu-1)}(x) = g^{(\mu-1)}(x_k) + \int_{x_k}^{x} g^{(\mu)}(y)dy\,.$$

… (4)

The segment developed Boole's iterated method for solving nonlinear equations with the Boole's rule [8]and numerical technique, Let Boole's rules for k=4 , such as

$$\int_{x_0}^{x} g(x)\,dx = \frac{2h}{45}[7g(x_0) + 32g(x_1) + 12g(x_2) + 32g(x_3) + 7g(x_4)]\,. \quad \ldots (5)$$

Where $h = \dfrac{x - x_k}{4}$.

Taking derivative for solving integration,

$$\int_{x_0}^{x} g\;\;(x)dx = \frac{2h}{45}[7g'(x_0) + 32g'(x_1) + 12g'(x_2) + 32g'(x_3) + 7g'(x_4)]\,. \quad \ldots (6)$$

So, it uses the Boole's rule formula to approximate $\int_{x_k}^{x} g^{(\mu)}(y)\,dy$, as follows:

$$\int_{x_k}^{x} g^{(\mu)}(y)\,dy \cong \frac{x - x_k}{90}(7g^{(\mu)}(x_k) + 32\,g^{(\mu)}(x_k + h) + 12\,g^{(\mu)}(x_k + 2h) + 32\,g^{(\mu)}(x_k + 3h) + 7g^{(m)}(x_k + 4h)), \quad \ldots (7)$$

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$$h=\frac{x-x_k}{4}.$$

Where

The multiple root of g(x) is as assumed to be ϕ of multiplicity μ ,

$$g^{(\mu-1)}(\phi)=g^{(\mu-1)}(x_k)+\int_{x_k}^{\phi}g^{(\mu)}(y)\,dy=g^{(\mu-1)}(x_k)+\frac{\phi-x_k}{90}(7g^{(\mu)}(x_k)+ 32g^{(\mu)}(x_k+\frac{\phi-x_k}{4})+12g^{(\mu)}(x_k+2(\frac{\phi-x_k}{4}))+32g^{(\mu)}(x_k+3(\frac{\phi-x_k}{4}))+7g^{(\mu)}(\phi)). \quad \dots (8)$$

By setting $x_{k+1}=\phi$ in eq. (8) one can obtain:

$$x_{k+1}-x_k=\frac{-4g^{(\mu-1)}(x_k)}{\left[7g^{(\mu)}(x_k)+32g^{(\mu)}(x_k+\frac{x_{k+1}-x_k}{4})+12g^{(\mu)}(x_k+2(\frac{x_{k+1}-x_k}{4}))+32g^{(\mu)}(x_k+3(\frac{x_{k+1}-x_k}{4}))+7g^{(\mu)}(x_{k+1})\right]} \quad \dots (9)$$

On the other hand

$$x_{k+1}=x_k-\frac{90g^{(\mu-1)}(x_k)}{\left[7g^{(\mu)}(x_k)+32g^{(\mu)}(x_k+\frac{x_{k+1}-x_k}{4})+12g^{(\mu)}(x_k+2(\frac{x_{k+1}-x_k}{4}))+32g^{(\mu)}(x_k+3(\frac{x_{k+1}-x_k}{4}))+7g^{(\mu)}(x_{k+1})\right]} \quad \dots (10)$$

And next by replacing x_{k+1} in the right hand side of (10) by

$$x_{k+1}=x_k-\frac{g^{(\mu-1)}(x_k)}{g^{(\mu)}(x_k)}\rightarrow x_{k+1}-x_k=\frac{g^{(\mu-1)}(x_k)}{g^{(\mu)}(x_k)}$$

Then

$$x_{k+1}=x_k-\frac{90g^{(\mu-1)}(x_k)}{[7g^{(\mu)}(x_k)+32g^{(\mu)}(x_k-\frac{R(x_k)}{4})+12g^{(\mu)}(x_k-2\frac{R(x_k)}{4})+32g^{(\mu)}(x_k-3\frac{R(x_k)}{4})+7g^{(\mu)}(x_k-R(x_k))]} \quad \dots (11)$$

Where $R(x_k)=\dfrac{g^{(\mu-1)}(x_k)}{g^{(\mu)}(x_k)}$.

Hence, eq. (11) is Boole's iterated method for solving nonlinear equations for multiple roots.

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The Boole's Method Notation

It is obvious if, Let

$$F_m(x) = 7g^{(\mu)}(x) + 32g^{(\mu)}(x - \frac{R(x)}{4}) + 12g^{(\mu)}(x - 2\frac{R(x)}{4}) + 32g^{(\mu)}(x - 3\frac{R(x)}{4}) + 7g^{(\mu)}(x - R(x)) \quad \ldots (12)$$

Then by setting x=ϕ in the above equation we get:

$$F_m(\phi) = 7g^{(\mu)}(\phi) + 32g^{(\mu)}(\phi - \frac{R(\phi)}{4}) + 12g^{(\mu)}(\phi - 2(\frac{R(\phi)}{4})) + 32g^{(\mu)}(\phi - 3(\frac{R(\phi)}{4})) + 7g^{(\mu)}(\phi - R(\phi)) \quad \ldots (13)$$

$$F_m(\phi) = 7g^{(\mu)}(\phi) + 32g^{(\mu)}(\phi) + 12g^{(\mu)}(\phi) + 32g^{(\mu)}(\phi) + 7g^{(\mu)}(\phi) = 90g^{(\mu)}(\phi) \quad \ldots (14)$$

Where $R(\phi) = \frac{g^{(\mu-1)}(\phi)}{g^{(\mu)}(\phi)} = 0\,.$ … (15)

So,

$$F'_m(x) = 7g^{(\mu+1)}(x) + 32g^{(\mu+1)}(x + \frac{R(\phi)}{4})(1 - \frac{R'(\phi)}{4}) + 12g^{(\mu+1)}(x + 2(\frac{R(\phi)}{4}))(1 - 2(\frac{R'(\phi)}{4})) + 32g^{(\mu+1)}(x + 3(\frac{R(\phi)}{4}))(1 - 3(\frac{R'(\phi)}{4})) + 7g^{(\mu+1)}(x - R(\phi))(1 - R'(\phi)) \quad \ldots (16)$$

Then,

$$F'_m(\phi) = 7g^{(\mu+1)}(\phi) + 32g^{(\mu+1)}(\phi + \frac{R(\phi)}{4})(1 - \frac{R'(\phi)}{4}) + 12g^{(\mu+1)}(\phi + 2(\frac{R(\phi)}{4}))(1 - 2(\frac{R'(\phi)}{4})) + 32g^{(\mu+1)}(\phi + 3(\frac{R(\phi)}{4}))(1 - 3(\frac{R'(\phi)}{4})) + 7g^{(\mu+1)}(\phi - R(\phi))(1 - R'(\phi)) \quad \ldots (17)$$

Where $R'(\phi) = 1\,.$ … (18)

Therefore

$$F'_m(\phi) = 45g^{(\mu+1)}(\phi)\,. \quad \ldots (19)$$

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Convergence of Boole's rule iterated Method

The following theorem takes into consideration how the convergence of the suggested method behaves when there are multiple roots.

Theorem 1.

For an open interval D, let ϕ be a multiple root of multiplicity μ of a suitably differentiable function $g : D \subseteq R \rightarrow R$, If x_0 is sufficiently near to ϕ, the method described in (11) has at least third order convergence.

Proof: -

Assume that $x_{k+1} = q(x_k)$ therefore

$$q(x) = x - \frac{90\, g^{(\mu-1)}(x)}{F_m(x)} \rightarrow q'(x) = 1 - \frac{90\, g^{(\mu)}(x)}{F_m(x)} + \frac{90 g^{(\mu-1)}(x) F_m'(x)}{(F_m(x))^2} \quad \dots (20)$$

$$q''(x) = -\frac{90 g^{(\mu+1)}(x)}{F_m(x)} + \frac{90 g^{\mu}(x) F_m'(x)}{(F_m(x))^2} + \frac{90 g^{(\mu)}(x) F_m'(x) + 90 g^{(\mu-1)}(x) F_m''(x)}{(F_m(x))^2} - \frac{180 g^{(\mu-1)}(x)(F_m'(x))^2}{(F_m(x))^3} \quad \dots (21)$$

Since the root ϕ is the simple root of $g^{(\mu-1)}(x)$ and $g^{(\mu-1)}(\phi) = 0$ therefore

$$g'(r) = 1 - \frac{90 g^{\mu}(\phi)}{F_m(\phi)} + \frac{90 g^{(\mu-1)}(\phi) F_m'(\phi)}{(F_m(\phi))^2} \quad \dots (22)$$

$$= 1 - \frac{90\, g^{\mu}(\phi)}{90 g^{\mu}(\phi)} + 0 = 0$$

And

$$q''(x) = -\frac{90 g^{(\mu+1)}(x)}{F_m(x)} + \frac{90 g^{\mu}(x) F'(x)}{F_m(x)} + \frac{90 g^{\mu}(x) F_m'(x) + 90 g^{(\mu-1)}(x) F_m''(x)}{(F_m(x))^2} - \frac{180 g^{(\mu-1)}(x)(F_m'(x))^2}{(F_m(x))^3} \quad \dots (23)$$

Since ϕ is the multiple root of $g^{(\mu-1)}(x)$ and $g^{(\mu-1)}(\phi) = 0$ thus,

$$q''(\phi) = -\frac{90\, g^{(\mu+1)}(\phi)}{F_m(\phi)} + \frac{90 g^{\mu}(\phi) F_m'(\phi)}{(F_m(\phi))^2} + \frac{90 g^{\mu}(\phi) F_m'(\phi) + 90 g^{(\mu-1)}(\phi) F_m''(\phi)}{(F_m(\phi))^2} - \frac{180 g^{(\mu-1)}(\phi)(F_m'(\phi))^2}{(F_m(\phi))^3} \quad \dots (24)$$

$$q''(r) = -\frac{90\ g^{(\mu+1)}(\phi)}{90\ g^{(\mu+1)}(\phi)} + \frac{F_m(\phi)\ F'_m(\phi)}{(F_m(\phi))^2} + \frac{F_m(\phi)F'_m(\phi)+0}{(F_m(\phi))^2} - 0$$
$$= -1 + \frac{45g^{(\mu+1)}(\phi)}{F_m(\phi)} + \frac{45g^{(\mu+1)}(\phi)}{F_m(\phi)} = -1 + \frac{90g^{(\mu+1)}(\phi)}{90g^{(\mu+1)}(\phi)} = -1+1=0 \quad \ldots (25)$$

It's proved.

Weddle's Rule Iterated Method

This segment developed a Weddle`s iterated method for solving nonlinear equations with the help of Weddle`s rule in numerical technique, let [8] Boole's rule for k=4, such as:

$$\int_{x_0}^{x} g(x)dx = \frac{3h}{10}[g(x_0)+5g(x_1)+g(x_2)+6g(x_3)+g(x_4)+5g(x_5)+g(x_6)]. \quad \ldots (26)$$

Taking derivative for solving integration

$$\int_{x_9}^{x} g'(x)dx = \frac{3h}{10}[g'(x_0)+5g'(x_1)+g'(x_2)+6g'(x_3)+g'(x_4)+5g'(x_5)+g'(x_6)]$$

Where $h = \frac{x-x_k}{6}$.

So, the waddle's rule formula is used to approximate $\int_{x_k}^{x} g^{(\mu)}(y)\,dy$, as follows:

$$\int_{x_k}^{x} g^{(\mu)}(y)\,dy \cong \frac{x-x_k}{20}(g^{(\mu)}(x_k)+5g^{(\mu)}(x_k+h)+6g^{(\mu)}(x_k+2h)$$
$$+g^{(\mu)}(x_k+3h)+5g^{(\mu)}(x_k+4h)+g^{(\mu)}(x_k+6h) \quad \ldots (27)$$

Assume that $x_{k+1} = q(x_k)$ therefore

$$g^{(\mu-1)}(\phi) = g^{(\mu-1)}(x_k) + \int_{x_k}^{\phi} g^{(\phi)}(y)\,dy \cong g^{(\mu-1)}(x_k) + \frac{\phi - x_n}{20}[g^{(\mu)}(x_k)+$$
$$5g^{(\mu)}(x_k+\frac{\phi-xk}{6}) + g^{(\mu)}(x_k+2(\frac{\phi-x_k}{6})) + 6g^{(\mu)}(x_k+3(\frac{\phi-x_k}{6})) \quad \ldots (28)$$
$$+g^{(\mu)}(x_k+4(\frac{\phi-x_n}{6})) + 5g^{(\mu)}(x_k+5(\frac{\phi-x_n}{6})) + g^{(\mu)}(x_k)]$$

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By setting $x_{k+1} = \phi$ in eq. (28) one can obtain:

$$x_{k+1} - x_k = {-20\, g^{\mu}(x_k)} \Big/ \left[\begin{array}{l} g^{\mu}(x_k) + 5 g^{\mu}(x_k + \frac{x_{k+1}-x_k}{6}) + g^{\mu}(x_k + 2\frac{x_{k+1}-x_k}{6}) + 6 g^{\mu}(x_k + 3\frac{x_{k+1}-x_k}{6}) + \\ g^{\mu}(x_k + 4\frac{x_{k+1}-x_k}{6}) + 5 g^{\mu}(x_k + 5\frac{x_{k+1}-x_k}{6}) + g^{\mu}(x_k + 6\frac{x_{k+1}-x_k}{6}) \end{array} \right]$$

... (29)

So,

$$x_{k+1} = x_k - {20\, g^{\mu}(x_k)} \Big/ \left[\begin{array}{l} g^{\mu}(x_k) + 5 g^{\mu}(x_k + \frac{x_{k+1}-x_k}{6}) + g^{\mu}(x_k + 2\frac{x_{k+1}-x_k}{6}) + 6 g^{\mu}(x_k + 3\frac{x_{k+1}-x_k}{6}) + \\ g^{\mu}(x_k + 4\frac{x_{k+1}-x_k}{6}) + 5 g^{\mu}(x_k + 5\frac{x_{k+1}-x_k}{6}) + g^{\mu}(x_k + 6\frac{x_{k+1}-x_k}{6}) \end{array} \right]$$

… (30)

Secondly, by substituting x_{k+1} in the right side of Eq. (30)

$$x_{k+1} = x_k - \frac{g^{(\mu-1)}(x_k)}{g^{(\mu)}(x_k)} \rightarrow x_{k+1} - x_k = \frac{g^{(\mu-1)}(x_k)}{g^{(\mu)}(x_k)}. \qquad \text{… (31)}$$

Then

$$x_{k+1} = x_k - {20\, f^{(m-1)}(x_n)} \Big/ \begin{array}{l} [g^{(\mu)}(x_k) + 5 g^{(\mu)}(x_k - \frac{R(x_k)}{6}) + g^{(\mu)}(x_k - 2\frac{R(x_k)}{6}) + 6 g^{(\mu)}(x_k - 3\frac{R(x_k)}{6}) + \\ g^{(\mu)}(x_k - 4\frac{R(x_k)}{6}) + 5 g^{(\mu)}(x_k - 5\frac{R(x_k)}{6}) + g^{(\mu)}(x_k - R(x_k))] \end{array}$$

… (32)

Where $R(x_k) = \dfrac{g^{(\mu-1)}(x_k)}{g^{(\mu)}(x_k)}$.

Hence, eq. (32) is Weddle's iterated method for solving nonlinear equations for multiple roots.

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The Weddle's Method Notation

It is obvious if, let

$$F_m(x) = g^{(\mu)}(x) + 5g^{(\mu)}(x - \frac{R(x)}{6}) + g^{(\mu)}(x - 2\frac{R(x)}{6}) + 6g^{(\mu)}(x - 3\frac{R(x)}{6}) + g^{(\mu)}(x - 4\frac{R(x)}{6}) + 5g^{(\mu)}(x - 5\frac{R(x)}{6}) + g^{(\mu)}(x - R(x)) \quad \ldots (33)$$

Then by setting x=ϕ,

$$F_m(\phi) = 7g^{(\mu)}(\phi) + 5g^{(\mu)}(\phi - \frac{R(\phi)}{6}) + g^{(\mu)}(\phi - 2\frac{R(\phi)}{6}) + 6g^{(\mu)}(\phi - 3\frac{R(\phi)}{6}) + g^{(\mu)}(\phi - 4\frac{R(\phi)}{6}) + 5g^{(\mu)}(\phi - 5\frac{R(\phi)}{6}) + g^{(\mu)}(\phi - R(\phi)) \quad \ldots (34)$$

$$F_m(\phi) = g^{(\mu)}(\phi) + 5g^{(\mu)}(\phi) + g^{(\mu)}(\phi) + 6g^{(\mu)}(\phi) + g^{(\mu)}(\phi) + 5g^{(\mu)}(\phi) + g^{(\mu)}(\phi) = 20g^{(\mu)}(\phi) \quad \ldots (35)$$

Where $R(\phi) = \frac{g^{(\mu-1)}(\phi)}{g^{(\mu)}(\phi)} = 0$.

So,

$$F_m'(x) = g^{(\mu+1)}(x) + 5g^{(\mu+1)}(x + \frac{R(x)}{6})\ (1 - \frac{R'(x)}{6}) + g^{(\mu+1)}(x + 2(\frac{R(x)}{6}))(1 - 2(\frac{R'(x)}{6})) + 6g^{(\mu+1)}(x + 3(\frac{R(x)}{6}))(1 - 3(\frac{R'(x)}{6})) + g^{(\mu+1)}(x + 4(\frac{R(x)}{6}))\ (1 - 4(\frac{R'(x)}{6})) + 5g^{(\mu+1)}(x + 5(\frac{R(x)}{6}))\ (1 - 5(\frac{R'(x)}{6})) + g^{(\mu+1)}(x - R(x))\ (1 - R'(x)) \quad \ldots (37)$$

Then by setting x=ϕ in the eq. (37) we get:

$$F_m'(\phi) = g^{(\mu+1)}(\phi) + 5g^{(\mu+1)}(\phi + \frac{R(\phi)}{6})\ (1 - \frac{R'(\phi)}{6}) + g^{(\mu+1)}(\phi + 2(\frac{R(\phi)}{6}))(1 - 2(\frac{R'(\phi)}{6})) + 6g^{(\mu+1)}(\phi + 3(\frac{R(\phi)}{6}))(1 - 3(\frac{R'(\phi)}{6})) + g^{(\mu+1)}(\phi + 4(\frac{R(\phi)}{6}))(1 - 4(\frac{R'(\phi)}{6})) + 5g^{(\mu+1)}(\phi + 5(\frac{R(\phi)}{6}))(1 - 5(\frac{R'(\phi)}{6})) + g^{(\mu+1)}(\phi - R(\phi))(1 - R'(\phi)) \quad \ldots (38)$$

Where $R'(\phi) = 1$, Therefore $F_m'(\phi) = 10\, g^{(\mu+1)}(\phi)$.

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Convergence of Weddle's Rule Iterated Method

The following theorem takes into consideration how the convergence of the suggested method behaves when there are multiple roots.

Theorem 2.

For an open interval D, let ϕ be a multiple root of multiplicity μ of a suitably differentiable function $g : D \subseteq R \to R$, If x_0 is sufficiently near to ϕ , the method described in (32) has at least third order convergence,

Proof: -

Assume that $x_{k+1} = q(x_k)$, therefore

$$q(x) = x - \frac{20\, g^{(\mu-1)}(x)}{F_m(x)} \to q'(x) = 1 - \frac{20\, g^{(\mu)}(x)}{F_m(x)} + \frac{20\, g^{(\mu-1)}(x)\, F_m'(x)}{(F_m(x))^2} \quad \ldots (39)$$

$$q''(x) = -\frac{20\, g^{(\mu+1)}(x)}{F_m(x)} + \frac{20\, g^{(\mu)}(x) F'(x)}{(F_m(x))^2} + \frac{20 g^{(\mu)}(x) F_m'(x) + 20\, g^{(\mu)}(x)\, F_m''(x)}{(F_m(x))^2} - \frac{40 g^{(\mu-1)}(x)(F_m'(x))^2}{(F_m(x))^3} \quad \ldots (40)$$

Since ϕ is the multiple root of $g^{(\mu-1)}(x)$ and $g^{(\mu-1)}(\phi) = 0$ therefore thus

$$q'(\phi) = 1 - \frac{20\, g^{(\mu)}(\phi)}{F_m(\phi)} + \frac{20\, g^{(\mu-1)}(\phi)\, F_m'(\phi)}{(F_m(\phi))^2} \quad \ldots (41)$$

$$= 1 - \frac{20\, g^{\mu}(\phi)}{20\, g^{\mu}(\phi)} + 0 = 0$$

And

$$q''(x) = -\frac{20\, g^{(\mu+1)}(x)}{F_m(x)} + \frac{20\, g^{\mu}(x)\, F'(x)}{F_m(x)} + \frac{20\, g^{\mu}(x) F_m'(x) + 20\, g^{(\mu-1)}(x) F_m''(x)}{(F_m(x))^2} - \frac{40\, g^{(\mu-1)}(x)\,(F_m'(x))^2}{(F_m(x))^3} \quad \ldots (42)$$

Since ϕ is the multiple root of $g^{(\mu-1)}(x)$ and $g^{(\mu-1)}(\phi) = 0$ therefore thus

$$q''(\phi) = -\frac{20\, g^{(m+1)}(\phi)}{F_m(\phi)} + \frac{20\, g^{\mu}(\phi) F_m'(\phi)}{(F_m(\phi))^2} + \frac{20\, g^{m}(\phi)\, F_m'(\phi) + 20\, g^{(\mu-1)}(\phi)\, F_m''(\phi)}{(F_m(\phi))^2} - \frac{40\, g^{(\mu-1)}(\phi)\,(F_m'(\phi))^2}{(F_m(\phi))^3}$$

,

… (43)

$$q''(\phi) = -\frac{20\, g^{(\mu+1)}(\phi)}{20\, g^{(\mu+1)}(\phi)} + \frac{F_m(\phi)\, F_m'(\phi)}{(F_m(\phi))^2} + \frac{F_m(\phi)\, F_m'(\phi) + 0}{(F_m(\phi))^2} - 0,$$

$$= -1 + \frac{10\, g^{(\mu+1)}(\phi)}{F_m(\phi)} + \frac{10\, g^{(\mu+1)}(\phi)}{F_m(\phi)} = -1 + \frac{20\, g^{(\mu+1)}(\phi)}{20\, g^{(\mu+1)}(\phi)} = -1 + 1 = 0$$

… (44)

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It's proved.

Numerical examples

In this section, we employed the present method (5) (namely BMM) and method (8) (namely WMM) to solve multiple-root nonlinear equations and compare the results with the Newton method (2) (namely NMM) and Simpsons method [4] (namely SMM) for multiple roots. Displayed in Table 1 functions, initial point x_0, approximate zeroes ϕ and multiplicity μ ,Displayed in Table 2 is the number of iterations (IT) such that $|f(x_n)| < 1.E-64$, found up to the 28th decimal place, the computations are performed using software Maple 18,

For comparison, the following functions are given.

Table 1. Shows functions, multiplicity with approximate roots.

No.	Function	x_0	μ	ϕ
1	$f_1(x) = (x^2 + \sin(\frac{x}{5}) - \frac{1}{4})^3$	1.9	3	0.4099920179891371316212583765
2	$f_2(x) = (\exp(x) + x - 20)^3$	4.5	3	2.842438953784447067816585940
3	$f_3(x) = (\exp(x^2 + 7x - 30))^2$	3.6	2	3.0
4	$f_4(x) = ((\sin x)^2 - x^2 + 1)^2$	2.0	2	1.404491648215341226035086818
5	$f_5(x) = (\cos x - x)^2$	1.0	2	0.7390851332151606416553120877
6	$f_6(x) = (\exp\ (x^5) - 1)^2$	0.65	2	0
7	$f_7(x) = (\ln x + \sqrt{x} - 5)^4$	5.9	4	8.309432694231571795346955683
8	$f_8(x) = (x^3 - 10)^4$	3.0	4	2.154434690031883721759293567
9	$f_7(x) = (x^3 - 10)^4$	3.1	5	3.0
10	$f_{10}(x) = ((x+2)\ e^x - 1)^5$	0.01	5	-0.4428544010023388583141328000
11	$f_{11}(x) = (\exp\ (x^2) - 1)^3$	0.8	3	0
12	$f_{12}(x) = (x^3 + 4x^2 - 10)^3$	3.0	2	1.365230013414096845760806829

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Table 2. Shows iteration numbers between the iterative methods NMM, SMM, BMM and WMM.

F	NMM	SMM	BMM	WMM
f_1	7	7	7	8
f_2	7	7	7	8
f_3	div	div	div	Div
f_4	6	5	5	5
f_5	5	4	4	4
f_6	6	65	5	5
f_7	5	5	5	5
f_8	6	6	6	6
f_9	4	5	5	5
f_{10}	5	7	7	7
f_{11}	32	36	3	3
f_{12}	31	28	12	12

div.: represent divergence.

Table 3. Shows $f(x_n)$ between the iterative methods NMM, SMM, BMM and WMM.

F	NMM	SMM	BMM	WMM
f_1	8.10-82	1.10-82	1.10-82	0
f_2	0	0	0	0
f_3	div	div	div	Div
f_4	0	0	0	0
f_5	0	0	0	0
f_6	0	0	6.10-43	0
f_7	0	0	5.10-72	0
f_8	1.10-104	0	5.10-87	5. 10-87

f_9	6.10-76	0	0	0
f_{10}	5.10-88	-1.10-140	-1.10-140	-1.10-140
f_{11}	2.99-33	4.95-33	8.40-46	8.40-46
f_{12}	2.56-66	6.01-33	2.79-34	2.79-34

From the above table 2, contain that:

i) In cases, f_6 ,f11, f_{12} the new methods had less than number of iterations of the other methods Newton's methods and Simpson's method for multiple roots of nonlinear equations. This means that the new methods faster and more effective when compared with the other methods.

ii) In cases, f_2 and f_9 the methods divergent or more than iteration number with the other methods, this means not effective in that cases.

iii) In the most cases, all methods had the same performance, this means methods can compute with the other iterative methods for solving nonlinear equations of multiple roots.

CONCLUSION:

This paper has proposed and developed two new iterative methods for solving nonlinear equations with multiple roots. These methods namely BMM and WMM methods. Preserved the order convergence of the classical method NMM with BMM and WMM methods performing extremely well in terms of a lesser number of iterations required for all computations when compared with other existing methods. In addition. The new two methods are competitive with several other methods, according to numerical testing .These methods were programmed by using software Maple 18.

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Conflict of interests

There are non-conflicts of interest.

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الخلاصة

تقدم هذه الدراسة تقنيات تكرارية معدّلة مبنية على قاعدة بول وقاعدة ويدل لتحديد الجذور المتعددة للمعادلات غير الخطية. تحقق الطرق المقترحة تقاربًا تكعيبيًا، وتعتمد على طريقة نيوتن لإيجاد الجذور المتعددة مع إدخال عدة تعديلات لزيادة درجة التقارب. تستخدم هذه الطرق التكرارية الجديدة تقاربًا من الدرجة الثالثة، وقد تم اختبارها من خلال العديد من الأمثلة العددية، مع مقارنة أدائها بطريقة نيوتن–رافسون وقاعدة سمبسون. أظهرت التجارب العددية التي أجريت باستخدام برنامج Maple 18 أن خوارزميات التكامل التكراري المعدلة تتفوق على الطرق الحالية من حيث الدقة والكفاءة. وتشير النتائج إلى أن الطرق الجديدة تنافس العديد من الخوارزميات المعروفة، وقد تم تنفيذ جميع الإجراءات باستخدام برنامج.Maple 18

المقدمة: تُعد المعادلات غير الخطية ذات الجذور المتعددة تحديًا في الحل الدقيق باستخدام طريقة نيوتن الكلاسيكية، وذلك بسبب بطء التقارب في مثل هذه الحالات. تدفع هذه القيود إلى تطوير طرق تكرارية محسّنة توفر تقاربًا أسرع وأكثر موثوقية. تهدف هذه الأساليب المطورة إلى تجاوز أوجه القصور في الطرق التقليدية.

الاستنتاجات: قدّمت هذه الدراسة طرقًا تكرارية محسّنة تجعل من طريقة نيوتن أكثر فاعلية في التعامل مع المعادلات التي تحتوي على جذور متعددة. ومن خلال العديد من الاختبارات العملية، أثبتت هذه الطرق الجديدة دقتها وسرعتها وموثوقيتها. وقد تم تطويرها واختبارها باستخدام برنامجMaple 18 ، وظهرت نتائجها بمستوى منافس للطرق المعروفة الأخرى.

الكلمات المفتاحية: قاعدة بول، قاعدة ويدل، قاعدة سمبسون، الجذور المتعددة، الطرق التكرارية، طريقة نيوتن.