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A Bernstein Operational Matrix Approach for Solving Certain Nonlinear Volterra-Fredholm Integro-Differential Equations Involved by Caputo Fractional Derivative

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نهج مصفوفة برنشتاين التشغيلية لحل بعض معادلات فولتيرا–فريدهولم التكاملية التفاضلية غير الخطية التي تتضمن مشتقة كابوتو الكسرية

لافان صديق صالح[1] و شازاد شوقى احمد[2]

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ABSTRACT

Background:

In this article, we present a numerical method that applies the Bernstein polynomial approach in matrix form to solve non-linear integro-fractional differential equations of the Volterra-Fredholm type (NIFDEs of V-F) that involve the Caputo fractional derivative with mixed conditions.

Materials and Methods:

The primary objective of the method is to convert the nonlinear functional equation and its associated conditions into matrix relations using normal and fractional Bernstein polynomials. using a strategy like Newton's method and obtaining the Bernstein coefficients.

Results:

This approach is appealing for computation, and examples and explanations of its use are given. Illustrative examples are used to show the method's correctness and efficiency, and the methodology is validated by comparing the approximate results with exact or reference solutions.

Conclusion:

A table compares both exact and approximate solutions. They also use the least-squares error technique to decrease error terms in the specified domain. As a result, the majority of general codes are written in Python.

Key words:

Bernstein polynomial, Fredholm-Volterra integro-differential equations, Hammerstein type Caputo fractional derivative, operational matrix, approximate solution, Caputo derivative, collocation points.

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1.INTRODUCTION

The study of integrals and derivatives of non-integer order, which are not defined by traditional descriptions of calculus integral and derivative operators, is known as fractional calculus (FC). The three most important contributors to the field of fractional calculus are Oldham [1], Miller with Ross [2], and Podlesny [3], who document the evolution of the discipline. Euler made the first step from the Gamma function by deciding that the result of computing the derivative of the power function has importance for non-integers. Fractional derivatives come in a variety of forms, the most popular of which being the Riemann-Liouville (R-L) and Caputo fractional derivatives [1-3]. Fractional derivatives are used in many fractional-order mathematical models. Over the past 60 years, FC has had a major impact on a wide range of fields, including physics, computer networks, biology, plant epidemiology, nuclear reactor dynamics and mechanics, etc. Fractional calculus has many real-world applications, these being only a handful. Recently, a lot of researchers have concentrated on developing numerical approaches for solving linear and nonlinear functional equations, which are typically the outcome of real-world issues being modeled mathematically. Partial differential equations, integral equations, integro-differential equations, delay differential equations, and partial integro-differential equations are a few examples of these. These equations can be found in a variety of fields, including industrial mathematics, fluid dynamics, biological models, chemical kinetics, ecology, control theory of finance mathematics, and aeronautical systems.

Since integro-fractional differential equations are typically difficult to solve analytically, it is necessary to obtain an efficient approximate solution. Numerous searches for reformed numerical techniques based on the approach of polynomials have been conducted in recent years. Only in the last few years has the collocation technique, one of the mathematical methods for numerically solving various equation types, such as differential equations [4] and integral and integro-differential equations [5-7]. Several methods have been suggested to solve fractional functional equations; for examples, the Adomian decomposition method with modified Bernstein polynomials [8], the Laplace decomposition technique [9], the spectral Petrov-Galerkin method [10], the differential transform method [11], the operational matrix of Chebyshev polynomials [12], the Legendre collocation method [13] and Bernstein polynomial approximations have also been pivotal in various studies [14],[15],[16],[17]. Also, the Volterra-Fredholm integral equations have been solved numerically using the discretization method [18], which makes use of the Bernstein polynomials.

So, in this paper, we introduce the Bernstein polynomials to seek the numerical solution of multi-fractional order equations. With the simple structure and perfect properties [19,20,21], Bernstein polynomials play an important role in the solution of functional equations. This study aims to evaluate the approximate solution using Bernstein polynomials in fractional operational matrix form and collocation techniques to nonlinear Volterra-Fredholm Integro-Fractional Differential Equations of Hammerstein type (NV-FIFDEs) of the following typical meaning (1) and (2):

$$
\begin{aligned}
{}_{a}^{C}D_{x}^{\sigma_n} y(x) &+ \sum_{\ell=1}^{n-1} P_\ell(x)\, {}_{a}^{C}D_{x}^{\sigma_\ell} y(x) + P_n(x) y(x) \\
&= f(x) + \sum_{i=1}^{m_V} \lambda_i^V \int_a^x \mathcal{K}_i^V(x,s) \psi_i^V\left(s, y(s), {}_{a}^{C}D_{s}^{\alpha_i} y(s)\right) ds \\
&+ \sum_{j=1}^{m_F} \lambda_j^F \int_a^b \mathcal{K}_j^F(x,s) \psi_j^F\left(s, y(s), {}_{a}^{C}D_{s}^{\beta_j} y(s)\right) ds \qquad \ldots (1)
\end{aligned}
$$

Together with mixed conditions:

$$\sum_{\rho=0}^{\mu-1} \{ g_{k\rho} y^{(\rho)}(a) + h_{k\rho} y^{(\rho)}(b) \} = \vartheta_k, \quad k = 0,1, \ldots, \mu - 1. \qquad \ldots (2)$$

Where the fractional orders: $\sigma_n > \sigma_{n-1} > \cdots > \sigma_1 > 0$, $\alpha_{m_V} > \alpha_{m_V-1} > \cdots > \alpha_1 > 0$, and $\beta_{m_F} > \beta_{m_F-1} > \cdots > \beta_1 > 0$, and $\mu = \max\{\lceil\sigma_n\rceil, \lceil\alpha_{m_V}\rceil, \lceil\beta_{m_F}\rceil\}$. In addition, the functions $P_\ell(x), f(x) \in C([a,b],\mathbb{R})$, for all $\ell = 1,2, \ldots, n$, and $\mathcal{K}_i^F(x,s), \mathcal{K}_j^V(x,s) \in C(\Xi,\mathbb{R})$, $\Xi = \{(x,s): a \le x \le s \le b\}$ and $\psi_i^V, \psi_j^F: [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are known, with arbitrary real constants $h_{k\rho}$, $g_{k\rho}$ and ϑ_k for all $k, \rho = 0,1, \ldots, \mu - 1$, are given. while $\{\lambda_i^V, \lambda_j^F\}_{i,j=1}^{m_V,m_F}$ are real parameters, for all $i = 1,2, \ldots, m_V, j = 1,2, \ldots, m_F$, (where $n, m_F, m_V \in \mathbb{Z}^+$). Moreover, $y(x)$ are the unknown functions in equation (1) to be determine under conditions (2).

The structure of this paper is as follows: The fundamental background information pertaining to fractional calculus and the Bernstein (normal and fractional) polynomials is provided in Section 2, which covers important ideas, definitions, characteristics, and lemmas. The matrix-form-based collocation method for numerical normal and fractional Bernstein polynomials is presented in Section 3. Section 4 provides numerical examples to illustrate the points. Finally, Section 5 has the conclusion.

2. PRELIMINARIES AND NOTATIONS:

In this section, present the fundamental definitions of normal and fractional Bernstein polynomials, as well as other prerequisites and characteristics of the fractional integral and derivative that are required throughout the article.

2.1. Fractional Calculus:

Definition 1. [22] Let $\rho \in \mathbb{R}$ and let $y(x)$ be a real valued function defined on $\mathrm{I} = [a,b]$. If there is a real number $\rho_0 > \rho$ such that $y(x) = (x - a)^{\rho_0} \varphi(x)$, where φ is a bounded continuous function on I, then the function y is in the space $\mathcal{C}_\rho(\mathrm{I})$. It is said to be in the space $\mathcal{C}_\rho^m(\mathrm{I})$ if and only if $y^{(m)} \in \mathcal{C}_\rho(\mathrm{I})$, with $m \in \mathbb{N}_0$.

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Definition 2. [2,23] Let $\mathrm{I} = [a, b]$ be any closed bounded interval on $\mathbb{R}$ and let $y(x) \in \mathcal{C}_\rho(\mathrm{I}), \rho \geq -1$. Then the $\omega(\in \mathbb{R}^+ \cup \{0\})$-order Riemann-Liouville fractional integral of a function y, is defined as:

$$ {}_a\mathcal{J}_x^\omega y(x) = \begin{cases} \dfrac{1}{\Gamma(\omega)} \displaystyle\int_a^x (x-\xi)^{\omega-1} y(\xi) d\xi & , \quad \omega \in \mathbb{R}^+ \\ y(x) & , \quad \omega = 0 \end{cases} $$

Furthermore, ${}_a\mathcal{J}_x^{\omega_1}\, {}_a\mathcal{J}_x^{\omega_2} y(x) = {}_a\mathcal{J}_x^{\omega_1+\omega_2} y(x) = {}_a\mathcal{J}_x^{\omega_2}\, {}_a\mathcal{J}_x^{\omega_1} y(x)$ for fractional order $\omega_1, \omega_2 \geq 0$.

Definition 3. [3,22] Let $\omega \in \mathbb{R}^+ \cup \{0\}$ and the function $y(x) \in \mathcal{C}_{-1}^m(\mathrm{I})$, $\mathrm{I} = [a, b]$ where $m = \lceil \omega \rceil$. Then, the Riemann-Liouville fractional derivative of order ω is defined as:

$$ {}_a^R\mathcal{D}_x^\omega y(x) = \mathcal{D}_x^m\, {}_a\mathcal{J}_x^{m-\omega} y(x) = \begin{cases} \dfrac{1}{\Gamma(m-\omega)} \dfrac{\partial^m}{\partial x^m} \left(\displaystyle\int_a^x (x-\xi)^{m-\omega-1} y(\xi) d\xi \right), \omega \in \mathbb{R}^+ \\ \dfrac{\partial^m}{\partial x^m} y(x) \qquad , \omega = m \end{cases} $$

In particular, where $\omega = 0$, ${}_a^R\mathcal{D}_x^\omega y(x) = y(x)$.

Definition 4. [23] Let $\omega \in \mathbb{R}^+ \cup \{0\}$ and the function $\phi(t) \in \mathcal{C}_{-1}^m(\mathrm{I})$, $\Upsilon = [a, b]$ where $m - 1 < \omega \leq m, m \in \mathbb{N}$. Then, the Caputo fractional derivative of order ω is defined as:

$$ {}_a^C\mathcal{D}_x^\omega y(x) = {}_a\mathcal{J}_x^{m-\omega}\, \mathcal{D}_x^m y(x) = \begin{cases} \dfrac{1}{\Gamma(m-\omega)} \displaystyle\int_a^x (x-\xi)^{m-\omega-1} \dfrac{\partial^m y(\xi)}{\partial \xi^m} d\xi, m-1 < \omega < m \\ \dfrac{\partial^m y(x)}{\partial x^m} \qquad , \quad \omega = m \end{cases} $$

Hence, ${}_a^C\mathcal{D}_x^0$ is an identity operator as a special case. The ω-Caputo fractional derivative of a constant function (C) equals zero, i.e., ${}_a^C\mathcal{D}_x^\omega C = 0, C \in \mathbb{R}$. The Caputo-type fractional derivative is a linear operator, i.e., ${}_a^C\mathcal{D}_x^\omega [(y_1 + y_2)_x] = {}_a^C\mathcal{D}_x^\omega y_1(x) + {}_a^C\mathcal{D}_x^\omega y_2(x)$. These are the basic characteristics of the Caputo fractional derivative.

Lemma 1. [23] The function $y(x) = (x-a)^\vartheta$ for $\vartheta \geq 0$ has a ω-Caputo derivative $(\omega \geq 0)$, which is created as follows: if $\vartheta \in \{0,1,2, \cdots, \lceil \omega \rceil - 1\}$ the ω-Caputo derivative is vanishes, i.e. ${}_a^C D_x^\omega y(x) = 0$, while if $\vartheta \in \mathbb{N}$ and $\vartheta \geq \lceil \omega \rceil$ or $\vartheta \notin \mathbb{N}$ and $\vartheta > \lceil \omega \rceil - 1$ its:

$$ {}_a^C D_x^\omega y(x) = \frac{\Gamma(\vartheta+1)}{\Gamma(\vartheta-\omega+1)} (x-a)^{\vartheta-\omega} \tag{3} $$

Lemma 2. [23] Let $\omega > 0$ and $\omega \notin \mathbb{N}$ be in such a manner that $m - 1 < \omega \leq m(\in \mathbb{N})$, and let $y(x) = \exp(kx + c)$ for any arbitrary constants $k, c \in \mathbb{R}$. Then

$$ {}_a^C D_x^\omega y(x) = k^m (x-a)^{m-\omega} \exp(c + ak)\, E_{1,m-\omega+1}\big(k(x-a)\big) \tag{4} $$

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Lemma 3. [3] The formation of an association between the ω-Caputo derivative and the ω-Riemann-Liouville integral, $m = \lceil\omega\rceil$ and $y \in C_\rho^m(\mathrm{I})$, computed as:

$${}_a^C D_x^\omega [{}_a J_x^\omega y(x)] = y(x) \; ; \quad a \le x \le b$$

and

$${}_a J_x^\omega \; [{}_a^C D_x^\omega y(x)] = y(x) - \sum_{k=0}^{m-1} \frac{y^{(k)}(a)}{k!} (x-a)^k$$

Lemma 4. [2] The relations between R-L and Caputo derivative can be expression as: ${}_a^C D_x^\omega y(x) = {}_a^R D_x^\omega \left[y(x) - T_{m-1}[y;a]\right]$, $(m = \lceil\omega\rceil)$ and $T_{m-1}[y;a]$ denotes the Taylor polynomial of degree $m-1$ for the function y, centered at a.

2.2. Clenshaw-Curtis Quadrature Formula [20]

`In the interval $[a,b]$, where b is higher than a, reliance the extreme Chebyshev zeros, the N-Cleanshaw-Curti's quadrature rule is developed. Furthermore, we will use Chebyshev polynomials to illustrate the integrand's expansion, which will be shown using the following relation:

$$\int_a^b g(x)dx = \frac{b-a}{2} \sum_{m=0}^{R}{}'' \quad T_m g(x_m) \tag{5}$$

The double prime notation $\left(\sum ''\right)$ in the summing denotes terms with indices $m = 0$ and $m = R$ split by two. The following is the definition of R-shifted Chebyshev collocation points, or x_i:

$$\tau_m = \cos\left(\frac{m\pi}{R}\right), \qquad x_m = \frac{b-a}{2}\tau_m + \frac{b+a}{2}$$

$$T_m = \frac{4}{R} \sum_{\rho=0}^{Q}{}'' \quad v_\rho \cos\left(\frac{m\rho\pi}{Q}\right), \quad v_\rho = \begin{cases} 0 & \text{if } \rho \text{ is odd number} \\ \dfrac{1}{1-\rho^2} & \text{if } \rho \text{ is even number} \end{cases}, \qquad Q \in \mathbb{Z}^+$$

2.3 The Bernstein Polynomials and it's Operational Matrices [15,16,20,21]

Effective polynomials defined on the closed bound interval [0,1] are the Bernstein polynomials. A basis for the power polynomials of degree m is the Bernstein polynomial of degree m. They come in two primary different types:

First Type, the normal r-Bernstein polynomials of degree m over the interval $[0,1]$ are described as follows:

$$B_{r,m}(x) = \binom{m}{r} x^r (1-x)^{m-r}, \qquad r = 0,1,2,\dots,m. \tag{6}$$

Utilizing binomial expansion of $(1-x)^{m-r}$, we can express the r-Bernstein polynomials (6) as:

$$B_{r,m}(x)=\sum_{q=0}^{m-r}\Lambda(r,q,m)x^{q+r},\qquad r=0,1,\dots,m$$

Hence, for all $r=0,1,\dots m$; and $q=0,1,\dots,m-r$:

$$\Lambda(r,q,m)=(-1)^q\binom{m}{r}\binom{m-r}{q}$$

Here, m is a positive integer that has been selected so that $m\geq r$. However, for every $r=0,1,\dots,m(\in\mathbb{Z}^+)$, we may write the normal r-Bernstein polynomials in matrix form:

$$B_{r,m}(x)=A_{r+1}X_m(x) \tag{7}$$

where the $(r+1)$-th row of the matrix is

$$A_{r+1}=\left[\underbrace{0\quad 0\quad \cdots\quad 0}_{r-\text{terms}}\quad \underbrace{(-1)^0\binom{m}{r}\quad (-1)^1\binom{m}{r}\binom{m-r}{1}\quad \cdots\quad (-1)^{m-r}\binom{m}{r}\binom{m-r}{m-r}}_{(m-r+1)-\text{terms}}\right]$$

and

$$X_m(x)=[1\quad x\quad x^2\quad \cdots\quad x^m]^T$$

Now, we define

$$\Phi_m(x)=[B_{0,m}(x)\quad B_{1,m}(x)\quad \cdots\quad B_{m,m}(x)]^T$$

Or in the operational matrix form, for normal r-Bernstein polynomials:

$$\Phi_m(x)=AX_m(x) \tag{8}$$

where A is defined as follows and represents the upper triangular matrix of dimension $(m+1)$:

$$A=\begin{bmatrix} (-1)^0\binom{m}{0} & (-1)^1\binom{m}{0}\binom{m-0}{1} & \cdots & (-1)^{m-0}\binom{m}{0}\binom{m-0}{m-0} \\ \vdots & \vdots & & \vdots \\ 0 & (-1)^0\binom{m}{r} & \cdots & (-1)^{m-r}\binom{m}{r}\binom{m-r}{m-r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (-1)^0\binom{m}{m} \end{bmatrix}_{m+1\times m+1}$$

That is:

$$A=[a_{i,j}]_{i,j=0}^{m},\qquad \text{where }\ a_{i+1,j+1}=\begin{cases} 0 & if\quad i>j \\ (-1)^{j-i}\binom{m}{i}\binom{m-i}{j-i} & if\quad i\leq j \end{cases}$$

where $\det(A)=\prod_{i=0}^{m}\binom{m}{i}$, that is the matrix A is an invertible matrix. Then, we can obtain inverse matrix using the following formula

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$$\{A^{-1}\}_{i,j=0}^{m} = \begin{cases} 0 & if \quad j < i \\ \dfrac{\binom{m-i}{j-i}}{\binom{m}{i}} & if \quad j \geq i \end{cases} \tag{9}$$

that is, for all $x \in [0,1]$, clearly for all $m \in \mathbb{Z}^+$:

$$X_m(x) = A^{-1}\Phi_m(x) \tag{10}$$

Second Type in this subsection, the θ-fractional r-Bernstein polynomials of degree m over the closed bounded interval $[0,1]$ obtain by substituting each x to x^θ:

$$B_{r,m}^{\theta}(x) = \binom{m}{r} x^{r\theta}\,(1 - x^{\theta})^{m-r} \quad ; r = 0,1,2,\dots,m(\in \mathbb{Z}^+) \text{ and } 0 < \theta < 1 \tag{11}$$

Typically, if $\theta = 1$ we obtain the ordinary (normal) r-Bernstein polynomials. On the other hand, we can write the θ-fractional r-Bernstein polynomials in the matrix form, for each $r = 0,1,\dots,m(\in \mathbb{Z}^+)$:

$$B_{r,m}^{\theta}(x) = A_{r+1}X_m^{\theta}(x) \tag{12}$$

where A_{r+1}, express before for all r, and $X_m^{\theta}(x) = [1 \quad x^{\theta} \quad x^{2\theta} \quad \dots \quad x^{m\theta}]^T$. Now, we define for θ-fractional r-Bernstein polynomials:

$$\Phi_m^{\theta}(x) = \left[B_{0,m}^{\theta}(x) \quad B_{1,m}^{\theta}(x) \quad \cdots \quad B_{m,m}^{\theta}(x)\right]^T$$

The operational matrix form, became:

$$\Phi_m^{\theta}(x) = AX_m^{\theta}(x) \tag{13}$$

Where, A is the upper triangular matrix of dimension $(m + 1 \times m + 1)$ and it is invertible defined in equation (9). Thus, for all $x \in [0,1]$ and $0 < \theta < 1$, clearly for all $m \in \mathbb{Z}^+$:

$$X_m^{\theta}(x) = A^{-1}\Phi_m^{\theta}(x) \tag{14}$$

2.4 The Function Approximation Using Bernstein Operational Matrices:

For any two arbitrary functions $y_1, y_2 \in L^2[0,1]$ the norm and inner product are, [24] respectively, defined by

$$\|y_1\|_2 = \sqrt{\langle y_1, y_1\rangle} \quad \text{and} \quad \langle y_1, y_2\rangle = \int_0^1 y_1(x)y_2(x)dx$$

The basic approximation theorem state: The best approximation is unique for any $y \in H$ if H is a Hilbert space and S is a closed subspace of H. The proof can be found in [25].

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2.4.1 The Normal Bernstein Operational Matrix (NBOM):

A function $y \in H = L^2[0,1]$ can be expressed in terms of the normal Berenstein polynomials (NBPs) basis, set $S^N = \{B_{0,m}(x), B_{1,m}(x), \dots, B_{m,m}(x)\}$. Then, in practice, only the first $(m+1)$ terms of NBPs are considered. Hence, there exist unique coefficients $c_0, c_1, \dots, c_m$ such that

$$y(x) \cong y_m(x) = \sum_{r=0}^{m} c_r B_{r,m}(x) = C^T \Phi_m(x), (\text{OR}) = [\Phi_m(x)]^T C \tag{15}$$

where $C^T = [c_0, c_1, \dots, c_m]$, and C^T can be obtained from $\langle y, \Phi_m \rangle = C^T \langle \Phi_m, \Phi_m \rangle$, where

$$\langle y, \Phi_m \rangle = \int_0^1 y(x)[\Phi_m(x)]^T dx = [\langle y, B_{0,m} \rangle \quad \langle y, B_{1,m} \rangle \quad \cdots \quad \langle y, B_{m,m} \rangle]$$

and $\langle \Phi_m, \Phi_m \rangle$ is an $(m+1) \times (m+1)$ matrix that is said to be the dual matrix of Φ_m, and is denoted by $\boldsymbol{Q}_m$**,** and will be introduced in the following. Therefore $C^T = \langle y, \Phi_m \rangle \boldsymbol{Q}_m^{-1}$, and then

$$\begin{aligned} \boldsymbol{Q}_m = \langle \Phi_m, \Phi_m \rangle &= \int_0^1 \Phi_m(x)[\Phi_m(x)]^T dx = \int_0^1 AX_m(x)\,[AX_m(x)]^T dx \\ &= A\left(\int_0^1 X_m(x)\,[X_m(x)]^T dx\right)A^T = \mathrm{A}W_m A^T \end{aligned} \tag{16}$$

Where W_m is a Hilbert matrix, a symmetric matrix of size $(m+1)$, given by

$$W_m = \begin{bmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{m+1} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{m+2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{m+1} & \frac{1}{m+2} & \cdots & \frac{1}{2m+1} \end{bmatrix}_{m+1 \times m+1}$$

On the other hand, Since

$$\begin{aligned} \boldsymbol{Q}_m = \langle \Phi_m, \Phi_m \rangle &= \int_0^1 \Phi_m(x)[\Phi_m(x)]^T dx \\ &= \int_0^1 \begin{bmatrix} B_{0,m}(x) \\ B_{1,m}(x) \\ \vdots \\ B_{m,m}(x) \end{bmatrix} [B_{0,m}(x) \quad B_{1,m}(x) \quad \cdots \quad B_{m,m}(x)]\, dx \\ &= \begin{bmatrix} Q_{1,1} & Q_{1,2} & \cdots & Q_{1,m+1} \\ Q_{2,1} & Q_{2,2} & \cdots & Q_{2,m+1} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{m+1,1} & Q_{m+1,2} & \cdots & Q_{m+1,m+1} \end{bmatrix} \end{aligned} \tag{17}$$

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After applying normal Bernstein polynomial properties and solving product integration in the gamma function, [23], we obtain finally each element in matrix $\boldsymbol{Q}_m$, for all $i,j = 0,1,\dots,m$:

$$Q_{i+1,j+1} = \int_0^1 B_{i,m}(x)B_{j,m}(x)dx = \binom{m}{i}\binom{m}{j}\int_0^1 x^{i+j}(1-x)^{2m-(i+j)}dx$$
$$= \binom{m}{i}\binom{m}{j}\frac{\Gamma(i+j+1)\Gamma(2m-(i+j)+1)}{\Gamma(2m+2)} = \frac{\binom{m}{i}\binom{m}{j}}{(2m+1)\binom{2m}{i+j}} \qquad (18)$$

We can see that equations (16 and 17) are equivalent in application.

2.4.2 The Fractional Bernstein Operational Matrix (FBOM):

Also, the function $y \in H = L^2[0,1]$ can be expressed in terms of the fractional Berenstein polynomials (FBPs) basis, set $S^F = \{B_{0,m}^{\theta}(x), B_{1,m}^{\theta}(x), \dots, B_{m,m}^{\theta}(x)\}$. Then, in practice only the first $(m+1)$ terms of FBPs are considered. Hence, there exist unique coefficients $\mathbb{d}_0, \mathbb{d}_1, \dots, \mathbb{d}_m$ such that

$$y(x) \cong y_m^{\theta}(x) = \sum_{r=0}^{m} \mathbb{d}_r B_{r,m}^{\theta}(x) = \mathbb{D}^T\Phi_m^{\theta}(x), (\mathrm{OR}) = \left[\Phi_m^{\theta}(x)\right]^T\mathbb{D} \qquad (19)$$

where $\mathbb{D}^T = [\mathbb{d}_0, \mathbb{d}_1, \dots, \mathbb{d}_m]$, and $\mathbb{D}^T$ can be obtained from $\langle y, \Phi_m^{\theta}\rangle = \mathbb{D}^T\langle\Phi_m^{\theta}, \Phi_m^{\theta}\rangle$, where

$$\langle y, \Phi_m^{\theta}\rangle = \int_0^1 y(x)\left[\Phi_m^{\theta}(x)\right]^T dx = \left[\langle y, B_{0,m}^{\theta}\rangle \quad \langle y, B_{1,m}^{\theta}\rangle \quad \cdots \quad \langle y, B_{m,m}^{\theta}\rangle\right]$$

and $\langle\Phi_m^{\theta}, \Phi_m^{\theta}\rangle$ is an $(m+1)\times(m+1)$ matrix that is said to be the dual matrix of Φ_m^{θ}, and is denoted by $\boldsymbol{Q}_m^{\theta}$, and will be introduced in the following depending on θ-fractional order. Therefore, $\mathbb{D}^T = \langle y, \Phi_m^{\theta}\rangle\left[\boldsymbol{Q}_m^{\theta}\right]^{-1}$, and then

$$\boldsymbol{Q}_m^{\theta} = \langle\Phi_m^{\theta}, \Phi_m^{\theta}\rangle = \int_0^1 \Phi_m^{\theta}(x)\left[\Phi_m^{\theta}(x)\right]^T dx = \int_0^1 AX_m^{\theta}(x)\left[AX_m^{\theta}(x)\right]^T dx$$
$$= A\left(\int_0^1 X_m^{\theta}(x)\left[X_m^{\theta}(x)\right]^T dx\right)A^T = \mathrm{A}\mathrm{W}_m^{\theta}A^T \qquad (20)$$

Where W_m^{θ} is a Hilbert matrix,a symmetric matrix depending on θ-fractional orders given by

$$W_m^\theta = \begin{bmatrix} 1 & \frac{1}{\theta+1} & \cdots & \frac{1}{m\theta+1} \\ \frac{1}{\theta+1} & \frac{1}{2\theta+1} & \cdots & \frac{1}{(m+1)\theta+1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{m\theta+1} & \frac{1}{(m+1)\theta+1} & \cdots & \frac{1}{2m\theta+1} \end{bmatrix}_{m+1\times m+1}$$

On the other hand, since

$$\boldsymbol{Q}_m^\theta = \langle \Phi_m^\theta, \Phi_m^\theta \rangle = \int_0^1 \begin{bmatrix} B_{0,m}^\theta(x) \\ B_{1,m}^\theta(x) \\ \vdots \\ B_{m,m}^\theta(x) \end{bmatrix} \begin{bmatrix} B_{0,m}^\theta(x) & B_{1,m}^\theta(x) & \cdots & B_{m,m}^\theta(x) \end{bmatrix} dx$$

$$= \begin{bmatrix} Q_{1,1}^\theta & Q_{1,2}^\theta & \cdots & Q_{1,m+1}^\theta \\ Q_{2,1}^\theta & Q_{2,2}^\theta & \cdots & Q_{2,m+1}^\theta \\ \vdots & \vdots & \ddots & \vdots \\ Q_{m+1,1}^\theta & Q_{m+1,2}^\theta & \cdots & Q_{m+1,m+1}^\theta \end{bmatrix} \tag{21}$$

After applying normal Bernstein polynomial properties and solving product integration in gamma

function, [23], we obtain finally each element in matrix $\boldsymbol{Q}_m^\theta = \left[Q_{i+1,j+1}^\theta\right]_{i,j=0}^m$. Thus, for all $i,j = 0,1,\dots,m$ we can obtain:

$$Q_{i+1,j+1}^\theta = \int_0^1 B_{i,m}^\theta(x) B_{j,m}^\theta(x) dx = \binom{m}{i}\binom{m}{j} \int_0^1 x^{\theta(i+j)} (1 - x^\theta)^{2m-(i+j)} dx$$

$$= \binom{m}{i}\binom{m}{j} \frac{\Gamma\left(i + j + \frac{1}{\theta}\right) \Gamma(2m - (i+j) + 1)}{\theta\, \Gamma\left(2m + \frac{1}{\theta} + 1\right)} \tag{22}$$

2.5 The Operational Matrix of the Kernels:

We can approximate a function's Volterra kernels $\mathcal{K}_i^V(x,s) \in L^2([0,1]\times[0,1])$ and Fredholm kernels $\mathcal{K}_j^F(x,s) \in L^2([0,1]\times[0,1])$, for each $i = 1,2,\dots,m_V$ and $j = 1,2,\dots,m_F$, by double Fourier expansion using the normal and fractional Bernstein polynomials as follows:

By **NBOMs**:

$$\mathcal{K}_i^V(x,s) \cong \sum_{p=0}^{m}\sum_{q=0}^{m} {}^i k_{pq}^V\, B_{p,m}(x) B_{q,m}(s) = [\Phi_m(x)]^T\ {}^i K^V\ \Phi_m(s),$$
$$\text{for each } i = 1,2,\dots,m_V$$

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$$\mathcal{K}_j^F(x,s) \cong \sum_{p=0}^{m}\sum_{q=0}^{m} {}^j k_{pq}^F\, B_{p,m}(x)B_{q,m}(s) = [\Phi_m(x)]^T\ {}^jK^F\ \Phi_m(s),$$
$$\text{for each } j = 1,2,\dots,m_F$$

where

$${}^iK^V = [\ {}^ik_{pq}^V]_{p,q=0}^{m} = \begin{bmatrix} {}^ik_{00}^V & {}^ik_{01}^V & \cdots & {}^ik_{0m}^V \\ {}^ik_{10}^V & {}^ik_{11}^V & \cdots & {}^ik_{1m}^V \\ \vdots & & \ddots & \\ {}^ik_{m0}^V & {}^ik_{m1}^V & \cdots & {}^ik_{mm}^V \end{bmatrix} \quad {}^jK^F = [\ {}^jk_{pq}^F]_{p,q=0}^{m} = \begin{bmatrix} {}^jk_{00}^F & {}^jk_{01}^F & \cdots & {}^jk_{0m}^F \\ {}^jk_{10}^F & {}^jk_{11}^F & \cdots & {}^jk_{1m}^F \\ \vdots & & \ddots & \\ {}^jk_{m0}^F & {}^jk_{m1}^F & \cdots & {}^jk_{mm}^F \end{bmatrix}$$

and the elements ${}^ik_{pq}^V$ and ${}^jk_{pq}^F$ are can be calculated by definition of inner product, respectively, as:

$${}^ik_{pq}^V = \frac{\langle B_{p,m}(x), \langle \mathcal{K}_i^V(x,s), B_{q,m}(s)\rangle\rangle}{\langle B_{p,m}(x), B_{p,m}(x)\rangle\langle B_{q,m}(s), B_{q,m}(s)\rangle} \qquad {}^jk_{pq}^F = \frac{\langle B_{p,m}(x), \langle \mathcal{K}_j^F(x,s), B_{q,m}(s)\rangle\rangle}{\langle B_{p,m}(x), B_{p,m}(x)\rangle\langle B_{q,m}(s), B_{q,m}(s)\rangle}$$

For each $i = 1,2,\dots,m_V$ and $j = 1,2,\dots,m_F$. Thus, by applying equations (16 or 17) and (18) we obtain

$$\left.\begin{aligned} {}^iK^V &= \boldsymbol{Q}_m^{-1}\ \langle \Phi_m(x), \langle \mathcal{K}_i^V(x,s), \Phi_m(s)\rangle\rangle\, \boldsymbol{Q}_m^{-1} \\ {}^jK^F &= \boldsymbol{Q}_m^{-1}\ \langle \Phi_m(x), \langle \mathcal{K}_j^F(x,s), \Phi_m(s)\rangle\rangle\, \boldsymbol{Q}_m^{-1} \end{aligned}\right\} \tag{23}$$

Apply the inner product definition with numerical computation integration by Clenshaw-Curtis Quadrature Formula (5) to calculate the middle terms for each i and j.

By **FBOMs,** apply the same procedure as for NBOMs and using equations (20 or 21) and (22) with apply the inner product definition with numerical computation integration by Clenshaw-Curtis Quadrature Formula (5) to calculate the middle terms for each i and j. we obtain, respectively:

$$\mathcal{K}_i^V(x,s) \cong [\Phi_m^\theta(x)]^T\ {}^iK^V\ \Phi_m^\theta(s), \qquad \text{for each } i = 1,2,\dots,m_V$$
$$\mathcal{K}_j^F(x,s) \cong [\Phi_m^\theta(x)]^T\ {}^jK^F\ \Phi_m^\theta(s), \qquad \text{for each } j = 1,2,\dots,m_F$$

while,

$$\left.\begin{aligned} {}^iK^V &= [\boldsymbol{Q}_m^\theta]^{-1}\ \langle \Phi_m^\theta(x), \langle \mathcal{K}_i^V(x,s), \Phi_m^\theta(s)\rangle\rangle\, [\boldsymbol{Q}_m^\theta]^{-1} \\ {}^jK^F &= [\boldsymbol{Q}_m^\theta]^{-1}\ \langle \Phi_m^\theta(x), \langle \mathcal{K}_j^F(x,s), \Phi_m^\theta(s)\rangle\rangle\, [\boldsymbol{Q}_m^\theta]^{-1} \end{aligned}\right\} \tag{24}$$

2.6 The Product Bernstein Operational Matrix:

A general formula for computing the operational matrix of the product of normal Bernstein polynomials (**NBOMs**) of degree m is introduced in the first section. $\hat{C}$ is a $(m+1)\times(m+1)$

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operational matrix of product wherever at any time, assuming that C is an arbitrary $(m+1)\times 1$ vector, defined in equation (15).

$$C^T\,\Phi_m(x)\,[\Phi_m(x)]^T \cong [\Phi_m(x)]^T\,\hat{C}$$

Now, we try to find the form of $\hat{C}$. From equations (8 and 15), since $\sum_{r=0}^{m} c_r B_{r,m}(x) = C^T\Phi_m(x)$, we have:

$$\begin{aligned} &C^T\,\Phi_m(x)\,[\Phi_m(x)]^T = C^T\,\Phi_m(x)[AX_m(x)]^T = C^T\,\Phi_m(x)X_m(x)^T A^T \\ &= [C^T\,\Phi_m(x) \quad xC^T\Phi_m(x) \quad x^2C^T\Phi_m(x) \quad \cdots \quad x^mC^T\Phi_m(x)]A^T \\ &= \left[\sum_{r=0}^{m} c_r B_{r,m}(x) \quad \sum_{r=0}^{m} c_r x B_{r,m}(x) \quad \sum_{r=0}^{m} c_r x^2 B_{r,m}(x) \quad \cdots \quad \sum_{r=0}^{m} c_r x^m B_{r,m}(x)\right] A^T \end{aligned}$$

Now, $\{x^k B_{r,m}(x)\}$ will be estimated in terms of $\Phi_m(x)$ for all $r,k = 0,1,\dots,m$. Thus, we define $\mathrm{u}_{k,r} = [\mathrm{u}_0^{k,r},\mathrm{u}_1^{k,r},\dots,\mathrm{u}_m^{k,r}]^T$ or $\mathrm{u}_{k,r}^T = [\mathrm{u}_0^{k,r} \quad \mathrm{u}_1^{k,r} \quad \cdots \quad \mathrm{u}_m^{k,r}]$. Call equation (15), we can write

$$x^k B_{r,m}(x) \cong \mathrm{u}_{k,r}^T\Phi_m(x), \qquad \text{for all } r,k = \overline{0{:}m}$$

So, $\langle x^k B_{r,m},\Phi_m\rangle = \mathrm{u}_{k,r}^T\langle\Phi_m,\Phi_m\rangle$ thus, $\mathrm{u}_{k,r} = \boldsymbol{Q}_m^{-1}\left[\langle x^k B_{r,m},\Phi_m\rangle\right]^T$ that is by definition of the inner product on $L^2[0,1]$ we have

$$\mathrm{u}_{k,r} = \boldsymbol{Q}_m^{-1}\int_0^1 x^k B_{r,m}(x)\,\Phi_m(x)dx = \boldsymbol{Q}_m^{-1}\begin{bmatrix} \int_0^1 x^k B_{r,m}(x)\,B_{0,m}(x)dx \\ \int_0^1 x^k B_{r,m}(x)\,B_{1,m}(x)dx \\ \vdots \\ \int_0^1 x^k B_{r,m}(x)\,B_{m,m}(x)dx \end{bmatrix}$$

We can evaluate the integral that is formulated as follows, for all ℓ, r and $k = 0,1,\dots,m$:

$$\int_0^1 x^k B_{r,m}(x)\,B_{\ell,m}(x)dx = \frac{\binom{m}{r}\binom{m}{\ell}}{(2m+k+1)\binom{2m+k}{k+r+\ell}}$$

Then, we have

$$\mathrm{u}_{k,r} = \boldsymbol{Q}_m^{-1}\frac{\binom{m}{r}}{2m+k+1}\begin{bmatrix} \frac{\binom{m}{0}}{\binom{2m+k}{k+r+0}} \\ \frac{\binom{m}{1}}{\binom{2m+k}{k+r+1}} \\ \vdots \\ \frac{\binom{m}{m}}{\binom{2m+k}{k+r+m}} \end{bmatrix}, \qquad \text{for all } r,k = \overline{0{:}m}$$

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Therefore, for each k, we have

$$\sum_{r=0}^{m} c_r x^k B_{r,m}(x) \cong \sum_{r=0}^{m} c_r \left(\mathfrak{u}_{k,r}^{T}\Phi_m(x)\right) = \sum_{r=0}^{m} c_r \left(\begin{bmatrix}\mathfrak{u}_0^{k,r} & \mathfrak{u}_1^{k,r} & \cdots & \mathfrak{u}_m^{k,r}\end{bmatrix}\begin{bmatrix}B_{0,m}(x)\\ B_{1,m}(x)\\ \vdots \\ B_{m,m}(x)\end{bmatrix}\right)$$

$$= \sum_{r=0}^{m} c_r \left(\sum_{j=0}^{m} \mathfrak{u}_j^{k,r} B_{j,m}(x)\right) = \sum_{j=0}^{m} B_{j,m}(x)\left(\sum_{r=0}^{m} c_r \mathfrak{u}_j^{k,r}\right)$$

$$= \begin{bmatrix}B_{0,m}(x) & B_{1,m}(x) & \cdots & B_{m,m}(x)\end{bmatrix}\begin{bmatrix}\sum_{r=0}^{m} c_r \mathfrak{u}_0^{k,r}\\ \sum_{r=0}^{m} c_r \mathfrak{u}_1^{k,r}\\ \vdots \\ \sum_{r=0}^{m} c_r \mathfrak{u}_m^{k,r}\end{bmatrix}$$

Thus:

$$\sum_{r=0}^{m} c_r x^k B_{r,m}(x) = [\Phi_m(x)]^T \begin{bmatrix}[\mathfrak{u}_0^{k,0} & \mathfrak{u}_0^{k,1} & \cdots & \mathfrak{u}_0^{k,m}]\\ [\mathfrak{u}_1^{k,0} & \mathfrak{u}_1^{k,1} & \cdots & \mathfrak{u}_1^{k,m}]\\ & \vdots & & \\ [\mathfrak{u}_m^{k,0} & \mathfrak{u}_m^{k,1} & \cdots & \mathfrak{u}_m^{k,m}]\end{bmatrix}\begin{bmatrix}c_0\\ c_1\\ \vdots\\ c_m\end{bmatrix}$$

$$= [\Phi_m(x)]^T[\tilde{\mathfrak{u}}_{k,0} \quad \tilde{\mathfrak{u}}_{k,1} \quad \cdots \quad \tilde{\mathfrak{u}}_{k,m}]C = [\Phi_m(x)]^T[\mathfrak{A}_{k+1}C]$$

where $\mathfrak{A}_{k+1} = [\tilde{\mathfrak{u}}_{k,0} \quad \tilde{\mathfrak{u}}_{k,1} \quad \cdots \quad \tilde{\mathfrak{u}}_{k,m}]$. That is $\mathfrak{A}_{k+1}\left(k = \overline{0:m}\right)$ is an $(m+1)\times(m+1)$ matrix that has vector $\tilde{\mathfrak{u}}_{k,r}\left(r = \overline{0:m}\right)$ for each column. While, if we define

$\tilde{C} = [\mathfrak{A}_1 C \quad \mathfrak{A}_2 C \quad \cdots \quad \mathfrak{A}_{m+1} C]$ therefore we can write:

$$C^T\,\Phi_m(x)\,[\Phi_m(x)]^T \cong [\Phi_m(x)]^T\tilde{C}A^T = [\Phi_m(x)]^T\,\hat{C} \qquad \dots (25)$$

Consequently, we get the normal Bernstein operational matrix of the product $\hat{C} = \tilde{C}A^T$.

A general formula for computing the operational matrix of the product of fractional Bernstein polynomials (**FBOMs**) of degree m is introduced in the first section. $\widehat{\mathbb{D}}$ is a $(m+1)\times(m+1)$ operational matrix of product wherever at any time, assuming that $\mathbb{D}$ is an arbitrary $(m+1)\times 1$ vector, defined in equation (19).

$$\mathbb{D}^T\,\Phi_m^{\theta}(x)\left[\Phi_m^{\theta}(x)\right]^T \cong \left[\Phi_m^{\theta}(x)\right]^T\,\widehat{\mathbb{D}}$$

Now, we try to find the form of $\widehat{\mathbb{D}}$. From equations (8 and 15), since $\sum_{r=0}^{m} c_r B_{r,m}^{\theta}(x) = \mathbb{D}^T\Phi_m^{\theta}(x)$,

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we have:

$$\mathbb{D}^T\,\Phi_m^\theta(x)\,\left[\Phi_m^\theta(x)\right]^T = \mathbb{D}^T\,\Phi_m^\theta(x)\mathrm{X}_m^\theta(x)^T A^T =$$
$$= \left[\mathbb{D}^T\,\Phi_m^\theta(x) \quad x^\theta\mathbb{D}^T\Phi_m^\theta(x) \quad x^{2\theta}\mathbb{D}^T\Phi_m^\theta(x) \quad \cdots \quad x^{m\theta}\mathbb{D}^T\Phi_m^\theta(x)\right]A^T$$
$$= \left[\sum_{r=0}^{m} \mathbb{d}_r B_{r,m}^\theta(x) \quad \sum_{r=0}^{m} \mathbb{d}_r x^\theta B_{r,m}^\theta(x) \quad \sum_{r=0}^{m} \mathbb{d}_r x^{2\theta} B_{r,m}^\theta(x) \quad \cdots \quad \sum_{r=0}^{m} \mathbb{d}_r x^{m\theta} B_{r,m}^\theta(x)\right] A^T$$

Now, $\{x^{k\theta}B_{r,m}^\theta(x)\}$ will be estimated in terms of $\Phi_m^\theta(x)$ for all $r,k = 0,1,\dots,m$. Thus, we define $\mathbb{u}_{k,r} = [\mathbb{u}_0^{k,r}, \mathbb{u}_1^{k,r}, \dots, \mathbb{u}_m^{k,r}]^T$ or $\mathbb{u}_{k,r}^T = \left[\mathbb{u}_0^{k,r} \quad \mathbb{u}_1^{k,r} \quad \cdots \quad \mathbb{u}_m^{k,r}\right]$. Call equation (19), we can write

$$x^{k\theta}B_{r,m}^\theta(x) \cong \mathbb{u}_{k,r}^T\Phi_m^\theta(x), \qquad \text{for all } r,k = \overline{0:m}$$

So, $\langle x^{k\theta}B_{r,m}^\theta, \Phi_m^\theta\rangle = \mathbb{u}_{k,r}^T\langle\Phi_m^\theta, \Phi_m^\theta\rangle$ thus, $\mathbb{u}_{k,r} = \left[\boldsymbol{Q}_m^\theta\right]^{-1}\left[\langle x^{k\theta}B_{r,m}^\theta, \Phi_m^\theta\rangle\right]^T$ that is by definition of inner product on $L^2[0,1]$ we have

$$\mathbb{u}_{k,r} = \left[\boldsymbol{Q}_m^\theta\right]^{-1}\int_0^1 x^{k\theta}B_{r,m}^\theta(x)\,\Phi_m^\theta(x)dx = \left[\boldsymbol{Q}_m^\theta\right]^{-1}\begin{bmatrix}\int_0^1 x^{k\theta}B_{r,m}^\theta(x)\,B_{0,m}^\theta(x)dx \\ \int_0^1 x^{k\theta}B_{r,m}^\theta(x)\,B_{1,m}^\theta(x)dx \\ \vdots \\ \int_0^1 x^{k\theta}B_{r,m}^\theta(x)\,B_{m,m}^\theta(x)dx\end{bmatrix}$$

We can evaluate the integral that is formulated as follows, for all ℓ, r and $k = 0,1,\dots,m$:

$$\int_0^1 x^{k\theta}B_{r,m}^\theta(x)\,B_{\ell,m}^\theta(x)dx = \binom{m}{r}\binom{m}{\ell}\int_0^1 x^{(k+r+\ell)\theta}\left(1-x^\theta\right)^{2m-\ell-r}dx = \frac{\binom{m}{r}}{\theta}\frac{\binom{m}{\ell}}{\Gamma\begin{bmatrix}2m+k+\frac{1}{\theta}+1 \\ k+r+\ell+\frac{1}{\theta}\end{bmatrix}}$$

Since we defined the symbol $\Gamma\begin{bmatrix}\eta \\ \xi\end{bmatrix}$ by $\frac{\Gamma(\eta)}{\Gamma(\xi)\Gamma(\eta-\xi)}$. Then, we have

$$\mathbb{u}_{k,r} = \left[\boldsymbol{Q}_m^\theta\right]^{-1}\frac{\binom{m}{r}}{\theta}\begin{bmatrix}\dfrac{\binom{m}{0}}{\Gamma\begin{bmatrix}2m+k+\frac{1}{\theta}+1 \\ k+r+\frac{1}{\theta}+0\end{bmatrix}} \\ \dfrac{\binom{m}{1}}{\Gamma\begin{bmatrix}2m+k+\frac{1}{\theta}+1 \\ k+r+\frac{1}{\theta}+1\end{bmatrix}} \\ \vdots \\ \dfrac{\binom{m}{m}}{\Gamma\begin{bmatrix}2m+k+\frac{1}{\theta}+1 \\ k+r+\frac{1}{\theta}+m\end{bmatrix}}\end{bmatrix}, \qquad \text{for all } r,k = \overline{0:m}$$

Therefore, for each k, we have

$$\sum_{r=0}^{m} \mathbb{d}_r x^{k\theta} B_{r,m}^{\theta}(x) \cong \sum_{r=0}^{m} \mathbb{d}_r \left(\mathbb{u}_{k,r}^{T} \Phi_m^{\theta}(x)\right) = \sum_{r=0}^{m} \mathbb{d}_r \left(\begin{bmatrix} \mathbb{u}_0^{k,r} & \mathbb{u}_1^{k,r} & \cdots & \mathbb{u}_m^{k,r} \end{bmatrix} \begin{bmatrix} B_{0,m}^{\theta}(x) \\ B_{1,m}^{\theta}(x) \\ \vdots \\ B_{m,m}^{\theta}(x) \end{bmatrix} \right)$$

$$= \sum_{r=0}^{m} \mathbb{d}_r \left(\sum_{j=0}^{m} \mathbb{u}_j^{k,r} B_{j,m}^{\theta}(x) \right) = \sum_{j=0}^{m} B_{j,m}^{\theta}(x) \left(\sum_{r=0}^{m} \mathbb{d}_r \mathbb{u}_j^{k,r} \right)$$

$$= \begin{bmatrix} B_{0,m}^{\theta}(x) & B_{1,m}^{\theta}(x) & \cdots & B_{m,m}^{\theta}(x) \end{bmatrix} \begin{bmatrix} \sum_{r=0}^{m} \mathbb{d}_r \mathbb{u}_0^{k,r} \\ \sum_{r=0}^{m} \mathbb{d}_r \mathbb{u}_1^{k,r} \\ \vdots \\ \sum_{r=0}^{m} \mathbb{d}_r \mathbb{u}_m^{k,r} \end{bmatrix}$$

$$= \left[\Phi_m^{\theta}(x)\right]^T \begin{bmatrix} [\mathbb{u}_0^{k,0} & \mathbb{u}_0^{k,1} & \cdots & \mathbb{u}_0^{k,m}] \\ [\mathbb{u}_1^{k,0} & \mathbb{u}_1^{k,1} & \cdots & \mathbb{u}_1^{k,m}] \\ & \vdots & & \\ [\mathbb{u}_m^{k,0} & \mathbb{u}_m^{k,1} & \cdots & \mathbb{u}_m^{k,m}] \end{bmatrix} \begin{bmatrix} \mathbb{d}_0 \\ \mathbb{d}_1 \\ \vdots \\ \mathbb{d}_m \end{bmatrix}$$

$$= \left[\Phi_m^{\theta}(x)\right]^T \begin{bmatrix} \widetilde{\mathbb{u}}_{k,0} & \widetilde{\mathbb{u}}_{k,1} & \cdots & \widetilde{\mathbb{u}}_{k,m} \end{bmatrix} \mathbb{D} = \left[\Phi_m^{\theta}(x)\right]^T [\mathbb{U}_{k+1}\mathbb{D}]$$

where $\mathbb{U}_{k+1} = [\widetilde{\mathbb{u}}_{k,0} \quad \widetilde{\mathbb{u}}_{k,1} \quad \cdots \quad \widetilde{\mathbb{u}}_{k,m}]$. That is $\mathbb{U}_{k+1}(k = \overline{0:m})$ is an $(m+1) \times (m+1)$ matrix that has vector $\widetilde{\mathbb{u}}_{k,r}(r = \overline{0:m})$ for each column. While, if we define $\widetilde{\mathbb{D}} = [\mathbb{U}_1\mathbb{D} \quad \mathbb{U}_2\mathbb{D} \quad \cdots \quad \mathbb{U}_{m+1}\mathbb{D}]$ therefore we can write:

$$\mathbb{D}^T \Phi_m^{\theta}(x) \left[\Phi_m^{\theta}(x)\right]^T \cong \left[\Phi_m^{\theta}(x)\right]^T \widetilde{\mathbb{D}} A^T = [\Phi_m(x)]^T \,\widehat{\mathbb{D}}. \qquad \dots(26)$$

Consequently, we get the fractional Bernstein operational matrix of the product $\widehat{\mathbb{D}} = \widetilde{\mathbb{D}} A^T$.

Lemma 5. Suppose that $y(x) \cong C^T \Phi_m(x)$ or $\cong [\Phi_m(x)]^T C$ and $\hat{\mathbb{C}}$ is the $(m+1) \times (m+1)$ normal operational matrix of product defined in equation (25) depending on vector c. The approximate function for $\ell = 1,2,3, \dots$ multiple of $y(x)$ using NBOMs, given

$$[y(x)]^{\ell} \cong [\Phi_m(x)]^T \left[\hat{\mathbb{C}}\right]^{\ell-1} C. \qquad (27)$$

Proof: apply induction in this prove. For $\ell = 1$, by equation (15) we obtain (27). Also, for $\ell = 2$ with using equation (25) we obtain

$$[y(x)]^2 = \{C^T \Phi_m(x) [\Phi_m(x)]^T\} C = [\Phi_m(x)]^T \left[\,\hat{\mathbb{C}}\right]^{2-1} C \,.$$

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Suppose that equation (27) it is true for $\ell-1, \ell \in \mathbb{Z}^{+}$, that is $[y(x)]^{\ell-1} \cong [\Phi_m(x)]^T [\hat{\mathrm{C}}]^{\ell-2} C$, with using equation (25). Thus

$$[y(x)]^{\ell} = y(x)[y(x)]^{\ell-1} \cong \{C^T \Phi_m(x)\}\left\{[\Phi_m(x)]^T [\hat{\mathrm{C}}]^{\ell-1} C\right\} = \{C^T \Phi_m(x)[\Phi_m(x)]^T\}[\hat{\mathrm{C}}]^{\ell-1} C$$
$$= [\Phi_m(x)]^T\, \hat{\mathrm{C}} [\hat{\mathrm{C}}]^{\ell-1} C = [\Phi_m(x)]^T [\hat{\mathrm{C}}]^{\ell-1} C.$$

Lemma 6. Suppose that $y(x) \cong \mathbb{D}^T \Phi_m^{\theta}(x)$ or $\cong \left[\Phi_m^{\theta}(x)\right]^T \mathbb{D}$ and $\widehat{\mathbb{D}}$ be the $(m+1)\times(m+1)$ fractional operational matrix of product defined in equation (26) depending on vector $\mathbb{d}$. The approximate function for $\ell = 1,2,3, \ldots$ multiple of $y(x)$ using FBOMs, given

$$[y(x)]^{\ell} \cong \left[\Phi_m^{\theta}(x)\right]^T [\widehat{\mathbb{D}}]^{\ell-1} \mathbb{D}. \tag{28}$$

Proof: By the same stages as in Lemma 5 with respect to $\Phi_m^{\theta}(x)$.

2.7 BOMs for Fractional Derivative:

In this section, we obtain the OMs for the fractional derivative based on normal and fractional Bernstein polynomials, that is NBOMs and FBOMs for arbitrary derivatives in Caputo sense. **First for NBOMs**: According to equation (8) and applying the lemma (1) for all fractional orders $\lceil\omega\rceil - 1 < \omega \leq \lceil\omega\rceil$ and taking $m \gg \lceil\omega\rceil$, we have

$$
\begin{aligned}
{}^{C}D_x^{\omega}\Phi_m(x) &= \mathrm{A}\ {}^{C}D_x^{\omega}\, X_m(x) = A\left[{}^{C}D_x^{\omega}\, 1 \quad {}^{C}D_x^{\omega}\, x \quad {}^{C}D_x^{\omega} x^2 \quad \cdots \quad {}^{C}D_x^{\omega}\, x^m\right]^T \\
&= A\left[\underbrace{0 \ \ \cdots \ \ 0}_{\lceil\omega\rceil-\text{terms}} \quad \underbrace{\frac{\Gamma(\lceil\omega\rceil+1)x^{\lceil\omega\rceil-\omega}}{\Gamma(\lceil\omega\rceil-\omega+1)} \ \ \cdots \ \ \frac{\Gamma(m+1)x^{m-\omega}}{\Gamma(m-\omega+1)}}_{(m-\lceil\omega\rceil+1)-\text{terms}}\right]^T \\
&= A\begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \frac{\Gamma(\lceil\omega\rceil+1)}{\Gamma(\lceil\omega\rceil-\omega+1)}x^{\lceil\omega\rceil-\omega} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \frac{\Gamma(m+1)}{\Gamma(m+1-\omega)}x^{m-\omega}
\end{bmatrix}
\end{aligned}
$$

$$
= A\begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \frac{\Gamma(\lceil\omega\rceil+1)}{\Gamma(\lceil\omega\rceil-\omega+1)} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \frac{\Gamma(m+1)}{\Gamma(m-\omega+1)}
\end{bmatrix}
\begin{bmatrix}
x^{-\omega} \\ x^{1-\omega} \\ \vdots \\ x^{\lceil\omega\rceil-\omega} \\ \vdots \\ x^{m-\omega}
\end{bmatrix}
$$

$$= A\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dfrac{\Gamma(\lceil\omega\rceil+1)}{\Gamma(\lceil\omega\rceil-\omega+1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \dfrac{\Gamma(m+1)}{\Gamma(m-\omega+1)} \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & x^{-\omega} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x^{-\omega} \end{bmatrix}\begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^m \end{bmatrix}$$

Define the following symbols

$$\psi^{\omega} = \begin{array}{c} \lceil\omega\rceil\text{-column} \\ \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dfrac{\Gamma(\lceil\omega\rceil+1)}{\Gamma(\lceil\omega\rceil-\omega+1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \dfrac{\Gamma(m+1)}{\Gamma(m-\omega+1)} \end{bmatrix} \end{array} \begin{array}{l} \lceil\omega\rceil\text{-row} \end{array} \qquad (29)$$

$$\bar{X}_m^{\omega}(x) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & x^{-\omega} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x^{-\omega} \end{bmatrix} \qquad (30)$$

Thus,

$${}^{C}D_x^{\omega}\Phi_m(x) = A\psi^{\omega}\bar{X}_m^{\omega}(x)X_m(x) \qquad \ldots (31)$$

Using the same process and taking transform in equation (8), we can find this matrix below:

$${}^{C}D_x^{\omega}[\Phi_m(x)]^T = [X_m(x)]^T\bar{X}_m^{\omega}(x)\psi^{\omega}A^T \qquad \ldots (32)$$

As a special case, if fractional order ω be lie in (0,1). So, matrices (29) and (30) becomes

$$\psi^{\omega} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \dfrac{\Gamma(\lceil\omega\rceil+1)}{\Gamma(\lceil\omega\rceil-\omega+1)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \dfrac{\Gamma(m+1)}{\Gamma(m-\omega+1)} \end{bmatrix};\ \bar{X}_m^{\omega}(x) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & x^{-\omega} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x^{-\omega} \end{bmatrix} \qquad (33)$$

Second for FBOMs: According to equation (12) and applying the lemma (1) for all fractional order $0 < \omega \leq 1$ and take $m \gg 1$, with property that $\theta > \omega$, after apply same steps as before we obtain:

$${}^{C}D_x^{\omega}\Phi_m^{\theta}(x) = A\left[{}^{C}D_x^{\omega}\,1 \quad {}^{C}D_x^{\omega}\,x^{\theta} \quad {}^{C}D_x^{\omega}x^{2\theta} \quad \cdots \quad {}^{C}D_x^{\omega}\,x^{m\theta}\right]^T$$

$$= A\left[0 \quad \frac{\Gamma(\theta+1)}{\Gamma(\theta+1-\omega)}x^{\theta-\omega} \quad \frac{\Gamma(2\theta+1)}{\Gamma(2\theta+1-\omega)}x^{2\theta-\alpha} \quad \cdots \quad \frac{\Gamma(m\theta+1)}{\Gamma(m\theta+1-\omega)}x^{m\theta-\omega}\right]^T$$

Therefore, we get the operational matrix of this part as follows

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$$ {}^{C}D_x^{\omega}\Phi_m^{\theta}(x) = A\psi_\omega^\theta \bar{X}_m^\omega(x) X_m^\theta(x) \qquad \dots (34)$$

$$\psi_\omega^\theta = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(\theta+1)}{\Gamma(\theta+1-\omega)} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\Gamma(2\theta+1)}{\Gamma(2\theta+1-\omega)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\Gamma(m\theta+1)}{\Gamma(m\theta+1-\omega)} \end{bmatrix},$$

$$\bar{X}_m^\omega(x) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & x^{-\omega} & 0 & \cdots & 0 \\ 0 & 0 & x^{-\omega} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x^{-\omega} \end{bmatrix}$$

Also, for transpose the OPMs formed as:

$$ {}^{C}D_x^{\omega}\left[\Phi_m^{\theta}(x)\right]^T = \left[X_m^\theta(x)\right]^T \bar{X}_m^\omega(x)\psi_\omega^\theta A^T \qquad \dots (35)$$

The following lemmas are combines the two sections, 2.6 and 2.7, to state the BOMs (normal and fractional) for product and fractional together.

Lemma 7. Suppose that $y(x) \cong C^T\Phi_m(x)$ and $\hat{\mathrm{C}}$ is the $(m+1)\times(m+1)$ normal operational matrix of product defined in equation (25) depending on vector c. The approximate function ω-fractional multiple functions of $y(x)$ using NBOMs, for all $\ell = 1,2,3,\dots$, formed as

$$ {}^{C}D_x^{\omega}\{[y(x)]^\ell\} = [X_m(x)]^T \bar{X}_m^\omega(x)\psi^\omega A^T\left[\hat{\mathrm{C}}\right]^{\ell-1} C \qquad (36)$$

Proof: Applying the ω-Caputo definition to lemma 5 (equation 27) and then using equation (32) we obtain equation (36).

Lemma 8. Suppose that $y(x) \cong \mathbb{D}^T\Phi_m^\theta(x)$ and $\hat{\mathbb{D}}$ be the $(m+1)\times(m+1)$ fractional operational matrix of product defined in equation (26) depending on vector $\mathbb{d}$. The approximate function $\omega(\in(0,1))$-fractional multiple functions of $y(x)$ using FBOMs, for all $\ell = 1,2,3,\dots$, formed as

$$ {}^{C}D_x^{\omega}\{[y(x)]^\ell\} = \left[X_m^\theta(x)\right]^T \bar{X}_m^\omega(x)\psi_\omega^\theta A^T\left[\hat{\mathbb{D}}\right]^{\ell-1}\mathbb{D} \qquad (37)$$

Proof: Applying the ω-Caputo definition to lemma 6 (equation 28) and then using equation (35) we obtain equation (37).

2.8 BOMs for Ordinary Derivative:

Assume that Ω_m is a square matrix of dimension $(m+1)$ operational matrix differentiation, expressed

$$\frac{d}{dx}\Phi_m(x) \cong \Omega_m\,\Phi_m(x)$$

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As a result, take the first derivative for normal Bernstein in matrix form using equation (8):

$$\frac{d}{dx}\Phi_m(x) \cong A\frac{d}{dx}\,[X_m(x)] = A\begin{bmatrix} 0 \\ 1 \\ 2x \\ \vdots \\ mx^{m-1} \end{bmatrix} = A\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m \end{bmatrix}\begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^{m-1} \end{bmatrix} = A\Lambda_m\overline{X}_m(x)$$

where $\dim(\Lambda_m) = (m+1)\times m$ and $\dim\left(\overline{X}_m(x)\right) = m\times 1$, with

$$\Lambda_m = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m \end{bmatrix}, \qquad \overline{X}_m(x) = \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^{m-1} \end{bmatrix}$$

Now we try to express the column vector $\overline{X}_m(x)$ in terms of m-normal Bernstein $\{B_{r,m}(x)\}_{r=0}^{m}$, from equation (10), $X_m(x) = A^{-1}\Phi_m(x)$, we can be write the powers for each $k \in \mathbb{Z}^+ \cup \{0\}$ as:

$$x^k \cong A_{[k]}^{-1}\Phi_m(x)$$

where $A_{[k]}^{-1}$ is k-row of matrix $A^{-1}$for each $k = 1,2,\dots,m,$ as defined in equation (9). Suppose that $\mathrm{Z}_m = \begin{bmatrix} A_{[1]}^{-1} & A_{[2]}^{-1} & \cdots & A_{[m]}^{-1} \end{bmatrix}^T$ that is $\overline{X}_m(x) = \mathrm{Z}_m\Phi_m(x)$. So,

$$\frac{d}{dx}\Phi_m(x) = A\Lambda_m\mathrm{Z}_m\Phi_m(x) = \Omega_m\Phi_m(x) \tag{38}$$

Therefore, the **NBOMs** the first derivative is equation (38) where $\Omega_m = A\Lambda_m\mathrm{Z}_m$. Utilizing, for each $k = 0,1,2,\dots$, we can form the following formula

$$\frac{d^k}{dx^k}y(x) \cong C^T\frac{d^k}{dx^k}\Phi_m(x) = C^T[\Omega_m]^k\Phi_m(x), \qquad [\Omega_m]^0 = I \tag{39}$$

While the **FBOMs** of the first derivative has only one equation, since for all fractional orders $0 < \omega \le 1$ and $m \gg 1$, with the property that $\theta > \omega$, and $0 < \theta < 1$, has $\mu = \max\{\lceil\sigma_n\rceil, \lceil\alpha_{m_V}\rceil, \lceil\beta_{m_F}\rceil\} = 1$, thus, we have the equation of derivative zero.

2.9 BOMs for the Integral Parts:

In this section, we consider the Volterra and Fredholm integral parts with special nonlinear Hammerstein terms $\psi_i^V\left(s, y(s), {}_a^C D_s^{\alpha_i}y(s)\right) = {}^C D_s^{\alpha_i}[y(s)]^{p_i}$ and $\psi_j^F\left(s, y(s), {}_a^C D_s^{\beta_j}y(s)\right) = {}^C D_s^{\beta_j}[y(s)]^{q_j}$, for all p_i and q_j are positive integer numbers greater than or equal to one for each i and j, respectively. We will represent the parts from equation (1) as symbols below:

The Volterra integral parts:

$$VI\left(\{\alpha_i, p_i\}_{i=1}^{m_V}, x\right) = \int_0^x \mathcal{K}_i^V(x,s)\,{}^C D_s^{\alpha_i}[y(s)]^{p_i}ds \tag{40}$$

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and the Fredholm integral parts:

$$FI\left(\{\beta_j, q_j\}_{j=1}^{m_F}, x\right) = \int_0^1 \mathcal{K}_j^F(x,s)\ {}^C D_s^{\beta_j}[y(s)]^{q_j} ds \tag{41}$$

Prior to starting the estimated solution, the unknown function $y(x)$ and all of its component parts of equations (40 and 41): ${}^C D_s^{\alpha_i}\{[y(x)]^{p_i}\}$ and ${}^C D_s^{\beta_j}\{[y(x)]^{q_j}\}$, for all $i = 1,2,\dots,m_V$ and $j = 1,2,\dots,m_F$ must be converted into matrix form. To do this, we need to establish the following new subsections:

2.9.1 The Volterra Term:

First for **NBOMs**, apply lemma 7 for $\ell = p_i$ and $\omega = \alpha_i$, for all $i = 1,2,\dots,m_V$ we can obtain by transform the fractional part in matrix form as

$$ {}^C D_s^{\alpha_i}\{[y(s)]^{p_i}\} = [X_m(s)]^T \bar{X}_m^{\alpha_i}(s)\psi^{\alpha_i}A^T[\hat{\mathrm{C}}]^{p_i-1}C \tag{42}$$

Recall Volterra kernel in section 2.5, for normal Bernstein polynomials with using equation (8) and putting with equation (42) into Volterra integral part (40) we have

$$\begin{aligned} VI(\{\alpha_i, p_i\}_{i=1}^{m_V}, x) &= \int_0^x \mathcal{K}_i^V(x,s)\ {}^C D_s^{\alpha_i}[y(s)]^{p_i} ds \\ &= [\Phi_m(x)]^T\ {}^i K^V A\left\{\int_0^x \mathrm{X}_m(s)[X_m(s)]^T \bar{X}_m^{\alpha_i}(s) ds\right\}\psi^{\alpha_i}A^T[\hat{\mathrm{C}}]^{p_i-1}C \\ &= [\Phi_m(x)]^T\ {}^i K^V A\bar{\gamma}(\alpha_i, x)\psi^{\alpha_i}A^T[\hat{\mathrm{C}}]^{p_i-1}C \end{aligned} \tag{43}$$

where

$$\bar{\gamma}(\alpha_i, x) = \int_0^x \mathrm{X}_m(s)[X_m(s)]^T \bar{X}_m^{\alpha_i}(s) ds$$

$$= \begin{bmatrix} 0 & \cdots & 0 & \dfrac{x^{\lceil\alpha_i\rceil+1-\alpha_i}}{\lceil\alpha_i\rceil+1-\alpha_i} & \dfrac{x^{\lceil\alpha_i\rceil+2-\alpha_i}}{\lceil\alpha_i\rceil+2-\alpha_i} & \cdots & \dfrac{x^{m+1-\alpha_i}}{m+1-\alpha_i} \\ 0 & \cdots & 0 & \dfrac{x^{\lceil\alpha_i\rceil+2-\alpha_i}}{\lceil\alpha_i\rceil+2-\alpha_i} & \dfrac{x^{\lceil\alpha_i\rceil+3-\alpha_i}}{\lceil\alpha_i\rceil+3-\alpha_i} & \cdots & \dfrac{x^{m+2-\alpha_i}}{m+2-\alpha_i} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \dfrac{x^{\lceil\alpha_i\rceil+m+1-\alpha_i}}{\lceil\alpha_i\rceil+m+1-\alpha_i} & \dfrac{x^{\lceil\alpha_i\rceil+m+2-\alpha_i}}{\lceil\alpha_i\rceil+m+2-\alpha_i} & \cdots & \dfrac{x^{2m+1-\alpha_i}}{2m+1-\alpha_i} \end{bmatrix} \tag{44}$$

$\lceil\alpha_i\rceil$-Zero column

Second for **FBOMs**, applying lemma 8 for $\ell = p_i$ and $\omega = \alpha_i$, for all $i = 1,2,\dots,m_V$ we can obtain by transforming the fractional part in matrix form as

$$ {}^C D_x^{\alpha_i}\{[y(x)]^{p_i}\} = \left[X_m^{\theta}(x)\right]^T \bar{X}_m^{\alpha_i}(x)\psi_{\alpha_i}^{\theta}A^T[\widehat{\mathbb{D}}]^{p_i-1}\mathbb{D} \tag{45}$$

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Recall Volterra kernel in section 2.5, for θ-fractional Bernstein polynomial with using equation (13) and putting with equation (45) into the Volterra integral part (40) we have

$$VI(\{\alpha_i,p_i\}_{i=1}^{m_V},x)=\left[\Phi_m^\theta(x)\right]^T\ {}^iK^V\ A\left\{\int_0^x X_m^\theta(s)\left[X_m^\theta(s)\right]^T\bar{X}_m^{\alpha_i}(s)ds\right\}\psi_{\alpha_i}^\theta A^T\left[\widehat{\mathbb{D}}\right]^{p_i-1}\mathbb{D}$$
$$=\left[\Phi_m^\theta(x)\right]^T\ {}^iK^V\ A\bar{\gamma}^\theta(\alpha_i,x)\psi_{\alpha_i}^\theta A^T\left[\widehat{\mathbb{D}}\right]^{p_i-1}\mathbb{D} \tag{46}$$

where

$$\bar{\gamma}^\theta(\alpha_i,x)=\int_0^x X_m^\theta(s)\left[X_m^\theta(s)\right]^T\bar{X}_m^{\alpha_i}(s)ds$$
$$=\begin{bmatrix} 0 & \frac{x^{\theta+1-\alpha_i}}{\theta+1-\alpha_i} & \frac{x^{2\theta+1-\alpha_i}}{2\theta+1-\alpha_i} & \cdots & \frac{x^{m\theta+1-\alpha_i}}{m\theta+1-\alpha_i} \\ 0 & \frac{x^{2\theta+1-\alpha_i}}{2\theta+1-\alpha_i} & \frac{x^{3\theta+1-\alpha_i}}{3\theta+1-\alpha_i} & \cdots & \frac{x^{(m+1)\theta+1-\alpha_i}}{(m+1)\theta+1-\alpha_i} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{x^{(m+1)\theta+1-\alpha_i}}{(m+1)\theta+1-\alpha_i} & \frac{x^{(m+2)\theta+1-\alpha_i}}{(m+2)\theta+1-\alpha_i} & \cdots & \frac{x^{2m\theta+1-\alpha_i}}{2m\theta+1-\alpha_i} \end{bmatrix} \tag{47}$$

2.9.2 The Fredholm Term:

For **NBOMs,** applying lemma 7 for $\ell=q_j$ and $\omega=\beta_j$, for all $j=1,2,\dots,m_F$ we can obtain by transforming the fractional part in matrix form as

$${}^CD_s^{\beta_j}\{[y(s)]^{q_j}\}=[X_m(s)]^T\bar{X}_m^{\beta_j}(s)\psi^{\beta_j}A^T\left[\hat{\mathbb{C}}\right]^{q_j-1}C \tag{48}$$

Recall Fredholm kernel in section 2.5, for normal Bernstein polynomials with using equation (8) and putting it with equation (48) into Fredholm integral part (41) we have

$$FI\left(\{\beta_j,q_j\}_{j=1}^{m_F},x\right)=\int_0^1\mathcal{K}_j^F(x,s)\ {}^CD_s^{\beta_j}[y(s)]^{q_j}ds$$
$$=[\Phi_m(x)]^T\ {}^jK^F A\left\{\int_0^1 \mathrm{X}_m(s)[X_m(s)]^T\bar{X}_m^{\beta_j}(s)ds\right\}\psi^{\beta_j}A^T\left[\hat{\mathbb{C}}\right]^{q_j-1}C$$
$$=[\Phi_m(x)]^T\ {}^jK^F A\bar{\gamma}(\beta_j)\psi^{\beta_j}A^T\left[\hat{\mathbb{C}}\right]^{q_j-1}C \tag{49}$$

where

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$$\bar{\gamma}(\beta_j) = \int_0^1 X_m(s)[X_m(s)]^T \bar{X}_m^{\beta_j}(s)ds =$$

$$\begin{bmatrix} 0 & \cdots & 0 & \frac{1}{\lceil\beta_j\rceil+1-\beta_j} & \frac{1}{\lceil\beta_j\rceil+2-\beta_j} & \cdots & \frac{1}{m+1-\beta_j} \\ 0 & \cdots & 0 & \frac{1}{\lceil\beta_j\rceil+2-\beta_j} & \frac{1}{\lceil\beta_j\rceil+3-\beta_j} & \cdots & \frac{1}{m+2-\beta_j} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{1}{\lceil\beta_j\rceil+m+1-\beta_j} & \frac{1}{\lceil\beta_j\rceil+m+2-\beta_j} & \cdots & \frac{1}{2m+1-\beta_j} \end{bmatrix} \quad (50)$$

$\lceil \beta_j \rceil$-Zero column

For **FBOMs**, apply lemma 8 for $\ell = q_j$ and $\omega = \beta_j$, for all $j = 1,2,\ldots,m_F$ we can obtain by transforming the fractional part in matrix form as

$$^{C}D_x^{\beta_j}\{[y(x)]^{q_j}\} = \left[X_m^{\theta}(x)\right]^T \bar{X}_m^{\beta_j}(x)\psi_{\beta_j}^{\theta} A^T \left[\widehat{\mathbb{D}}\right]^{q_j-1}\mathbb{D} \quad (51)$$

Recall Fredholm kernel in section 2.5, for θ-fractional Bernstein polynomial with using equation (13) and putting it equation (51) into Fredholm integral part (41) we have

$$FI\left(\{\beta_j, q_j\}_{j=1}^{m_F}, x\right) = \left[\Phi_m^{\theta}(x)\right]^T\ {}^{j}K^F\ A\left\{\int_0^1 X_m^{\theta}(s)\left[X_m^{\theta}(s)\right]^T \bar{X}_m^{\beta_j}(s)ds\right\}\psi_{\beta_j}^{\theta} A^T\left[\widehat{\mathbb{D}}\right]^{q_j-1}\mathbb{D}$$
$$= \left[\Phi_m^{\theta}(x)\right]^T\ {}^{j}K^F\ A\bar{\gamma}^{\theta}(\beta_j)\psi_{\beta_j}^{\theta} A^T\left[\widehat{\mathbb{D}}\right]^{q_j-1}\mathbb{D} \quad (52)$$

where

$$\bar{\gamma}^{\theta}(\beta_j) = \int_0^1 X_m^{\theta}(s)\left[X_m^{\theta}(s)\right]^T \bar{X}_m^{\beta_j}(s)ds$$

$$= \begin{bmatrix} 0 & \frac{1}{\theta+1-\beta_j} & \frac{1}{2\theta+1-\beta_j} & \cdots & \frac{1}{m\theta+1-\beta_j} \\ 0 & \frac{1}{2\theta+1-\beta_j} & \frac{1}{3\theta+1-\beta_j} & \cdots & \frac{1}{(m+1)\theta+1-\beta_j} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{1}{(m+1)\theta+1-\beta_j} & \frac{1}{(m+2)\theta+1-\beta_j} & \cdots & \frac{1}{2m\theta+1-\beta_j} \end{bmatrix} \quad (53)$$

3. THE NUMERICAL METHOD:

Here, we present a numerical method for solving equation (1) with the mixed conditions observed in equation (2). We employ the operational matrices approach, which is based on the normal and fractional Bernstein polynomials, to achieve this goal. We approximate the functions

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$y(x), \mathcal{K}_i^V(x,s), \mathcal{K}_j^F(x,s)$ with ${}_a^C D_x^{\sigma_\ell} y(x)$, ${}^C D_s^{\alpha_i}[y(s)]^{p_i}$ and ${}^C D_s^{\beta_j}\{[y(s)]^{q_j}\}$ in terms of BOMs using the method given in section 2, respectively.

3.1 NBOMs Procedure: Utilizing equations (15), (32), (43) and (49), equation (1) can be converted into operational matrix form as follows:

$$[X_m(x)]^T \bar{X}_m^{\sigma_n}(x)\psi^{\sigma_n} A^T C + \sum_{\ell=1}^{n-1} P_\ell(x)[X_m(x)]^T \bar{X}_m^{\sigma_\ell}(x)\psi^{\sigma_\ell} A^T C + P_n(x)[X_m(x)]^T A^T C$$
$$= f(x) + \sum_{i=1}^{m_V} \lambda_i^V [X_m(x)]^T A^T \ {}^i K^V A \bar{\gamma}(\alpha_i, x)\psi^{\alpha_i} A^T [\hat{\mathrm{C}}]^{p_i-1} C$$
$$+ \sum_{j=1}^{m_F} \lambda_j^F [X_m(x)]^T A^T \ {}^j K^F A \bar{\gamma}(\beta_j)\psi^{\beta_j} A^T [\hat{\mathrm{C}}]^{q_j-1} C \qquad (54)$$

Moreover, by taking into account equations (39), (15) and (8), we can write the given mixed boundary conditions (2) as follows:

$$C^T \sum_{\rho=0}^{\mu-1} [\Omega_m]^\rho A\{g_{k\rho} X_m(0) + \hbar_{k\rho} X_m(1)\} = \vartheta_k, \quad k = 0,1,\dots,\mu-1.$$

In matrix form, we can write as:

$$\boldsymbol{Y}_k = \sum_{\rho=0}^{\mu-1} [\Omega_m]^\rho A\{g_{k\rho} X_m(0) + \hbar_{k\rho} X_m(1)\}$$

That is for each $k = 0,1,\dots,\mu-1$ and $\mu = \max\{\lceil \sigma_n \rceil, \lceil \alpha_{m_V} \rceil, \lceil \beta_{m_F} \rceil\}$ we can form as follows

$$C^T \boldsymbol{Y}_k = [\boldsymbol{\vartheta}_k] \qquad \dots (55)$$

$$\boldsymbol{Y}_k = [y_{k0} \quad y_{k1} \quad y_{k2} \quad \cdots \quad y_{km}] \ and \ \boldsymbol{\vartheta}_k = [\vartheta_0 \quad \vartheta_1 \quad \cdots \quad \vartheta_{k-1}]^T.$$

Let us define

$$\mathcal{R}_m(x) = \left\{[X_m(x)]^T \bar{X}_m^{\sigma_n}(x)\psi^{\sigma_n} + \sum_{\ell=1}^{n-1} P_\ell(x)[X_m(x)]^T \bar{X}_m^{\sigma_\ell}(x)\psi^{\sigma_\ell} + P_n(x)[X_m(x)]^T\right\} A^T C$$
$$- f(x) - \sum_{i=1}^{m_V} \lambda_i^V [X_m(x)]^T A^T \ {}^i K^V A \bar{\gamma}(\alpha_i, x)\psi^{\alpha_i} A^T [\hat{\mathrm{C}}]^{p_i-1} C$$
$$- \sum_{j=1}^{m_F} \lambda_j^F [X_m(x)]^T A^T \ {}^j K^F A \bar{\gamma}(\beta_j)\psi^{\beta_j} A^T [\hat{\mathrm{C}}]^{q_j-1} C$$

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Then we set

$$\mathcal{R}_m(\mathcal{X}_r) = 0 \tag{56}$$

where $\mathcal{X}_r$ are Newton-Cotes nodes

$$\mathcal{X}_r = \frac{2r-1}{2(m+1)} \quad ; r = 1,2,\dots,m-\mu+1 \tag{57}$$

We can standardize the equation (56):

$$\begin{aligned} f(\mathcal{X}_r) = &\left\{ [X_m(\mathcal{X}_r)]^T \bar{X}_m^{\sigma_n}(\mathcal{X}_r)\psi^{\sigma_n} + \sum_{\ell=1}^{n-1} P_\ell(\mathcal{X}_r)[X_m(\mathcal{X}_r)]^T \bar{X}_m^{\sigma_\ell}(\mathcal{X}_r)\psi^{\sigma_\ell} \right. \\ &\left. + P_n(\mathcal{X}_r)[X_m(\mathcal{X}_r)]^T \right\} A^T C \\ &- \sum_{i=1}^{m_V} \lambda_i^V [X_m(\mathcal{X}_r)]^T A^T \; {}^iK^V A \bar{\gamma}(\alpha_i, \mathcal{X}_r)\psi^{\alpha_i} A^T \left[\hat{\mathrm{C}}\right]^{p_i-1} C \\ &- \sum_{j=1}^{m_F} \lambda_j^F [X_m(\mathcal{X}_r)]^T A^T \; {}^jK^F A \bar{\gamma}(\beta_j)\psi^{\beta_j} A^T \left[\hat{\mathrm{C}}\right]^{q_j-1} C \end{aligned} \tag{58}$$

After solving nonlinear system (58) with conditions (55) by the Newton-Raphson method we get all normal m-Bernstein coefficients C. Then we have the approximate solution $y_m(x)$ of equation (1) in form (15) by applying the following algorithm.

The Algorithm (ANBOM):

Step 1:

- **a.** Input the number of truncated m- normal Bernstein polynomials such that $m \gg [\omega]$, $\omega = \max\{\sigma_n, \alpha_{m_V}, \beta_{m_F}\}$.
- **b.** Assume $h = \frac{b-a}{m}$, $m \in \mathbb{N}$.

Step 2: Determine the matrix A from the equation (8), and vectors $X_m(x)$ with $\Phi_m(x)$ respectively.

Step 3: Compute the matrices W_m and $\boldsymbol{Q_m}$ from equation (16) or (17,18).

Step 4: Evaluate the matrices ${}^iK^V$ and ${}^jK^F$ for each $i = 1,2,\dots,m_V$ and $j = 1,2,\dots,m_F$ from equation (23), respectively.

Step 5: Construct the matrices $\bar{X}_m^{\sigma_\ell}(x)$ and ψ^{σ_ℓ} for all $\ell = 1,2,\dots,n-1,n$, ψ^d for $d = \alpha_i, \beta_j$ for each $i = 1,2,\dots,m_V$ and $j = 1,2,\dots,m_F$ from equations (30 and 29), respectively.

Step 6: Assess the matrices $\bar{\gamma}(\beta_j)$ and $\bar{\gamma}(\alpha_i, x)$ for each $i = 1,2,\dots,m_V$ and $j = 1,2,\dots,m_F$ from the equations (50) and (44) respectively.

Step 7: Evaluate the matrices $\hat{\mathrm{C}}$ from equation (25), The product of the normal Bernstein operational matrix, section 2.6.

Step 8: Construct the μ-row conditional matrix $\boldsymbol{Y}_k$ and $\boldsymbol{\vartheta}_k$ for all $k = 0,1,\dots,\mu-1$, from equation (55).

Step 9: Evaluate at each collocation point (57), $\mathcal{X}_r, r = 1,2,\dots,m-\mu+1$, the formula (56).

Step 10: Combine steps 8 and 9 to obtain the nonlinear system (58 with 55) and solve it by the Newton-Raphson method to get all normal m-Bernstein coefficients C.

Step 11: The approximate solution $y_m(x)$ of equation (1), $y(x)$ is obtained by putting C's in equation (15).

3.2 FBOMs Procedure: Utilizing equations (19), (34), (46) and (52), equation (1) can be converted into operational matrix form as follows:

$$\left[X_m^\theta(x)\right]^T \bar{X}_m^{\sigma_n}(x)\psi_{\sigma_n}^\theta A^T \mathbb{D} + \sum_{\ell=1}^{n-1} P_\ell(x)\left[X_m^\theta(x)\right]^T \bar{X}_m^{\sigma_\ell}(x)\psi_{\sigma_\ell}^\theta A^T \mathbb{D} + P_n(x)\left[X_m^\theta(x)\right]^T A^T \mathbb{D}$$
$$= f(x) + \sum_{i=1}^{m_V} \lambda_i^V \left[X_m^\theta(x)\right]^T A^T \ {}^iK^V \ A\bar{\gamma}^\theta(\alpha_i, x)\psi_{\alpha_i}^\theta A^T \left[\widehat{\mathbb{D}}\right]^{p_i-1}\mathbb{D}$$
$$+ \sum_{j=1}^{m_F} \lambda_j^F \left[X_m^\theta(x)\right]^T A^T \ {}^jK^F \ A\bar{\gamma}^\theta(\beta_j)\psi_{\beta_j}^\theta A^T \left[\widehat{\mathbb{D}}\right]^{q_j-1}\mathbb{D}$$

Moreover, we have only one mixed boundary condition (2) form as follow:

$$\boldsymbol{Y}_0^\theta = A\{g_{00}X_m^\theta(0) + h_{00}X_m^\theta(1)\}$$
$$\mathbb{D}^T \boldsymbol{Y}_0^\theta = [\boldsymbol{\vartheta}_0] \qquad (59)$$

Let us define

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$$\mathcal{R}_m^{\theta}(x)=\left\{\left[X_m^{\theta}(x)\right]^T\bar{X}_m^{\sigma_n}(x)\psi_{\sigma_n}^{\theta}+\sum_{\ell=1}^{n-1}P_\ell(x)\left[X_m^{\theta}(x)\right]^T\bar{X}_m^{\sigma_\ell}(x)\psi_{\sigma_\ell}^{\theta}+P_n(x)\left[X_m^{\theta}(x)\right]^T\right\}A^T\mathbb{D}$$
$$-f(x)-\sum_{i=1}^{m_V}\lambda_i^V\left[X_m^{\theta}(x)\right]^TA^T\ {}^iK^V\ A\bar{\gamma}^{\theta}(\alpha_i,x)\psi_{\alpha_i}^{\theta}A^T\left[\widehat{\mathbb{D}}\right]^{p_i-1}\mathbb{D}$$
$$-\sum_{j=1}^{m_F}\lambda_j^F\left[X_m^{\theta}(x)\right]^TA^T\ {}^jK^F\ A\bar{\gamma}^{\theta}(\beta_j)\psi_{\beta_j}^{\theta}A^T\left[\widehat{\mathbb{D}}\right]^{q_j-1}\mathbb{D} \tag{60}$$

Then we set

$$\mathcal{R}_m^{\theta}(\mathcal{X}_r)=0,\qquad r=1,2,\dots,m \tag{61}$$

where $\mathcal{X}_r$ are Newton-Cotes nodes equation (57). We can standardize the equation (61):

$$f(\mathcal{X}_r)$$
$$=\left\{\left[X_m^{\theta}(\mathcal{X}_r)\right]^T\bar{X}_m^{\sigma_n}(\mathcal{X}_r)\psi_{\sigma_n}^{\theta}+\sum_{\ell=1}^{n-1}P_\ell(\mathcal{X}_r)\left[X_m^{\theta}(\mathcal{X}_r)\right]^T\bar{X}_m^{\sigma_\ell}(\mathcal{X}_r)\psi_{\sigma_\ell}^{\theta}+P_n(\mathcal{X}_r)\left[X_m^{\theta}(\mathcal{X}_r)\right]^T\right\}A^T\mathbb{D}$$
$$-\sum_{i=1}^{m_V}\lambda_i^V\left[X_m^{\theta}(\mathcal{X}_r)\right]^TA^T\ {}^iK^V\ A\bar{\gamma}^{\theta}(\alpha_i,\mathcal{X}_r)\psi_{\alpha_i}^{\theta}A^T\left[\widehat{\mathbb{D}}\right]^{p_i-1}\mathbb{D}$$
$$-\sum_{j=1}^{m_F}\lambda_j^F\left[X_m^{\theta}(\mathcal{X}_r)\right]^TA^T\ {}^jK^F\ A\bar{\gamma}^{\theta}(\beta_j)\psi_{\beta_j}^{\theta}A^T\left[\widehat{\mathbb{D}}\right]^{q_j-1}\mathbb{D} \tag{62}$$

After solving nonlinear system (62) with one boundary condition (59) by the Newton-Raphson method we get all fractional m-Bernstein coefficients $\mathbb{D}$. Then we have the approximate solution $y_m^{\theta}(x)$ of equation (1) in form (19).

The Algorithm (AFBOM):

Step 1:

a. Input the number of truncated m- fractional Bernstein polynomials such that $m \gg 1$.

b. Assume $h=\frac{b-a}{m}$, $m \in \mathbb{N}$.

Step 2: Determine the matrix A from the equation (8), and vectors $X_m^{\theta}(x)$ with $\Phi_m^{\theta}(x)$ from equation (13), respectively.

Step 3: Compute the matrices W_m^{θ} and $\boldsymbol{Q}_m^{\theta}$ from equation (20) or (21,22).

Step 4: Evaluate the matrices ${}^iK^V$ and ${}^jK^F$ for each $i=1,2,\dots,m_V$ and $j=1,2,\dots,m_F$ from equation (24), respectively.

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Step 5: Construct the matrices $\bar{X}_m^{\sigma_\ell}(x)$ and $\psi_{\sigma_\ell}^{\theta}$ for all $\ell = 1,2,\dots,n-1,n$, ψ_d^{θ} for $d = \alpha_i, \beta_j$ for each $i = 1,2,\dots,m_V$ and $j = 1,2,\dots,m_F$ from equations (30 and 34), respectively.

Step 6: Assess the matrices $\bar{\gamma}^{\theta}(\beta_j)$ and $\bar{\gamma}^{\theta}(\alpha_i, x)$ for each $i = 1,2,\dots,m_V$ and $j = 1,2,\dots,m_F$ from the equations (53) and (47) respectively.

Step 7: Evaluate the matrices $\hat{\mathbb{D}}$ from equation (26), The product of the fractional Bernstein operational matrix, section 2.6.

Step 8: Construct the 1-row conditional matrix from equation (59).

Step 9: Evaluate at each collocation point (57), $\mathcal{X}_r, r = 1,2,\dots,m-\mu+1$, the formula (60).

Step 10: Combine steps 8 and 9 to obtain a nonlinear system (62 with 59) and solve it by the Newton-Raphson method to get all fractional m-Bernstein coefficients $\mathbb{D}$.

Step 11: The approximate solution $y_m^{\theta}(x)$ of equation (1), $y(x)$ is obtained by putting $\mathbb{D}$'s in equation (19).

4. NUMERICAL TESTS:

This section uses the numerical results of a few descriptive problems to demonstrate the feasibility of the current method. Also, these outcomes are explained in tables and figures. Every one of them was done on a computer with Python version 3.8.8 (2021) software. The values of least square errors (LSE.) and the running time are listed in tables. In addition, apply the absolute error function at each point x_i; $E(x_i)$ is equal to $|y(x_i) - y_m(x_i)|$, or $|y(x_i) - y_m^{\theta}(x_i)|$ for all $i = 0,1,\dots,N \in \mathbb{N}$ to each test problems at the chosen interval points. We used graphics to illustrate it.

Test Problem 1*:* Consider the nonlinear Fredholm-Volterra integro-differential equation over [0,1] of multi-fractional order, which is offered by:

$$^{C}D_x^{0.6}y(x) - x^2\, {}^{C}D_x^{0.5}y(x) + 2xy(x)$$

$$= f(x) + \frac{1}{1000}\int_0^x (xs-1)\, {}^{C}D_s^{0.8}[y(s)]^2 ds + \frac{1}{1000}\int_0^1 (x^2+2s)\, {}^{C}D_s^{0.4}[y(s)]^3 ds$$

where

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$$f(x)=\frac{-2}{\Gamma(2.4)}x^{1.4}+\frac{2}{\Gamma(2.5)}x^{3.5}+2x-2x^3+\left(\frac{1}{8800\,\Gamma(2.2)}x^{\frac{11}{5}}\right)(11x^2-16)$$
$$-\left(\frac{2}{9100\,\Gamma(4.2)}x^{\frac{21}{5}}\right)(21x^2-26)+\frac{1}{1000\Gamma(2.6)}\left[\frac{30}{13}x^2+\frac{10}{3}\right]$$
$$-\frac{1}{1000\Gamma(4.6)}\left[\frac{360}{23}x^2+\frac{180}{7}\right]+\frac{1}{1000\Gamma(6.6)}\left[\frac{1200}{11}x^2+\frac{3600}{19}\right]$$

Subject to the boundary source: $2\mathrm{y}(0)-\mathrm{y}(1)=2$, having the exact solution $y(x)=1-x^2$.

Here, apply the algorithm **ANBOM**, from consider problem we have: $n=2$, $\sigma_2=0.6$, $\sigma_1=0.5$, $\alpha_1=0.8$, $\beta_1=0.4$, thus $\mu=\max\{\lceil 0.6\rceil,\lceil 0.8\rceil,\lceil 0.4\rceil\}=1$ and we have $m_V=m_F=1$, $P_2(x)=2x$, $P_1(x)=-x^2$, while the kernels that $\mathcal{K}_1^V(x,s)=xs-1$, $\mathcal{K}_1^{\mathrm{F}}(\mathrm{x},s)=x^2+2\mathrm{s}$ and $p_1=2$; $q_1=3$, with eigenvalue parameters $\lambda_1^{\mathrm{V}}=\lambda_1^{\mathrm{F}}=\frac{1}{1000}$.

Hence $\mu=1$ so take $m=3$, the standard Newton-Cotes collocation points $\mathcal{X}_r, r=1,2,3$ are $\left\{x_1=\frac{1}{8}, x_2=\frac{3}{8}, x_3=\frac{5}{8}\right\}$ and the fundamental matrix equation of the given NIFDEs of V-F is derived from equation (58), written as:

$$f(x)=\{[X_3(x)]^T\bar{X}_3^{\sigma_2}(x)\psi^{\sigma_2}+p_1(x)[X_3(x)]^T\bar{X}_3^{\sigma_1}(x)\psi^{\sigma_1}+p_2(x)[X_3(x)]^T\}A^TC$$
$$-\lambda_1^V[X_3(x)]^TA^T\,{}^1K^VA\bar{\gamma}(\alpha_1,x)\psi^{\alpha_1}A^T\left[\hat{\mathrm{C}}\right]^{p_1-1}C$$
$$-\lambda_1^F[X_3(x)]^TA^T\,{}^1K^FA\bar{\gamma}(\beta_1)\psi^{\beta_1}A^T\left[\hat{\mathrm{C}}\right]^{q_1-1}C \tag{63}$$

where

$$X_3(x)=\begin{bmatrix}1 & x & x^2 & x^3\end{bmatrix}^T, C^T=\begin{bmatrix}c_0 & c_1 & c_2 & c_3\end{bmatrix}, A=\begin{bmatrix}1 & -3 & 3 & 1\\ 0 & 3 & -6 & 3\\ 0 & 0 & 3 & -3\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$${}^1K^V=\begin{bmatrix}-1 & -1 & -1 & -1\\ -1 & -8/9 & -7/9 & -2/3\\ -1 & -7/9 & -5/9 & -1/3\\ -1 & -2/3 & -1/3 & 0\end{bmatrix},\qquad {}^1K^F=\begin{bmatrix}0 & 2/3 & 4/3 & 2\\ 0 & 2/3 & 4/3 & 2\\ 1/3 & 1 & 5/3 & 7/3\\ 1 & 5/3 & 7/3 & 3\end{bmatrix}$$

and

$$\bar{X}_3^{\sigma_2}(x)=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & x^{-\frac{3}{5}} & 0 & 0\\ 0 & 0 & x^{-\frac{3}{5}} & 0\\ 0 & 0 & 0 & x^{-\frac{3}{5}}\end{bmatrix},\psi^{\sigma_2}=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & \frac{1}{\Gamma(1.4)} & 0 & 0\\ 0 & 0 & \frac{2}{\Gamma(2.4)} & 0\\ 0 & 0 & 0 & \frac{6}{\Gamma(3.4)}\end{bmatrix}$$

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$$\bar{X}_3^{\sigma_1}(x)=\begin{bmatrix}0&0&0&0\\0&x^{-\frac{1}{2}}&0&0\\0&0&x^{-\frac{1}{2}}&0\\0&0&0&x^{-\frac{1}{2}}\end{bmatrix},\quad \psi^{\sigma_1}=\begin{bmatrix}0&0&0&0\\0&\frac{1}{\Gamma(1.5)}&0&0\\0&0&\frac{2}{\Gamma(2.5)}&0\\0&0&0&\frac{6}{\Gamma(3.5)}\end{bmatrix}$$

$$\psi^{\alpha_1}=\begin{bmatrix}0&0&0&0\\0&\frac{1}{\Gamma(1.2)}&0&0\\0&0&\frac{2}{\Gamma(2.2)}&0\\0&0&0&\frac{6}{\Gamma(3.2)}\end{bmatrix},\quad \psi^{\beta_1}=\begin{bmatrix}0&0&0&0\\0&\frac{1}{\Gamma(1.6)}&0&0\\0&0&\frac{2}{\Gamma(2.6)}&0\\0&0&0&\frac{6}{\Gamma(3.6)}\end{bmatrix}$$

and

$$W_3=\begin{bmatrix}1&\frac{1}{2}&\frac{1}{3}&\frac{1}{4}\\\frac{1}{2}&\frac{1}{3}&\frac{1}{4}&\frac{1}{5}\\\frac{1}{3}&\frac{1}{4}&\frac{1}{5}&\frac{1}{6}\\\frac{1}{4}&\frac{1}{5}&\frac{1}{6}&\frac{1}{7}\end{bmatrix},\boldsymbol{Q}_3=\begin{bmatrix}\frac{1}{7}&\frac{1}{14}&\frac{1}{35}&\frac{1}{140}\\\frac{1}{14}&\frac{3}{35}&\frac{9}{140}&\frac{1}{35}\\\frac{1}{35}&\frac{9}{140}&\frac{3}{35}&\frac{1}{14}\\\frac{1}{140}&\frac{1}{35}&\frac{1}{14}&\frac{1}{7}\end{bmatrix},\boldsymbol{Q}_3^{-1}=\begin{bmatrix}16&-24&16&-4\\-24&\frac{208}{3}&\frac{-172}{3}&16\\16&\frac{-172}{3}&\frac{208}{3}&-24\\-4&16&-24&16\end{bmatrix}$$

$$\bar{\gamma}(\alpha_1,x)=\begin{bmatrix}0&\frac{x^{1.2}}{1.2}&\frac{x^{2.2}}{2.2}&\frac{x^{3.2}}{3.2}\\0&\frac{x^{2.2}}{2.2}&\frac{x^{3.2}}{3.2}&\frac{x^{4.2}}{4.2}\\0&\frac{x^{3.2}}{3.2}&\frac{x^{4.2}}{4.2}&\frac{x^{5.2}}{5.2}\\0&\frac{x^{4.2}}{4.2}&\frac{x^{5.2}}{5.2}&\frac{x^{6.2}}{6.2}\end{bmatrix},\quad \bar{\gamma}(\beta_1)=\begin{bmatrix}0&\frac{1}{1.6}&\frac{1}{2.6}&\frac{1}{3.6}\\0&\frac{1}{2.6}&\frac{1}{3.6}&\frac{1}{4.6}\\0&\frac{1}{3.6}&\frac{1}{4.6}&\frac{1}{5.6}\\0&\frac{1}{4.6}&\frac{1}{5.6}&\frac{1}{6.6}\end{bmatrix}$$

and

$$\hat{C}=[\mathfrak{A}_1C\quad \mathfrak{A}_2C\quad \mathfrak{A}_3C\quad \mathfrak{A}_4C]A^T=\begin{bmatrix}\hat{c}_{00}&\hat{c}_{01}&\hat{c}_{02}&\hat{c}_{03}\\\hat{c}_{10}&\hat{c}_{11}&\hat{c}_{12}&\hat{c}_{13}\\\hat{c}_{20}&\hat{c}_{21}&\hat{c}_{22}&\hat{c}_{23}\\\hat{c}_{30}&\hat{c}_{31}&\hat{c}_{32}&\hat{c}_{33}\end{bmatrix}$$

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First row:

$$\hat{c}_{00}=\frac{13c_0}{14}+\frac{2c_1}{21}-\frac{2c_2}{105}-\frac{c_3}{210}$$
$$\hat{c}_{01}=\frac{2c_0}{21}-\frac{2c_1}{35}-\frac{3c_2}{70}+\frac{c_3}{210}$$
$$\hat{c}_{02}=\frac{c_2}{70}-\frac{3c_1}{70}-\frac{2c_0}{105}+\frac{c_3}{21}$$
$$\hat{c}_{03}=\frac{c_1}{210}-\frac{c_0}{210}+\frac{c_2}{21}-\frac{c_3}{21}$$

Second row:

$$\hat{c}_{10}=\frac{5c_1}{14}-\frac{23c_0}{42}+\frac{6c_2}{35}+\frac{2c_3}{105}$$
$$\hat{c}_{11}=\frac{5c_0}{14}+\frac{18c_1}{35}+\frac{6c_2}{35}-\frac{3c_3}{70}$$
$$\hat{c}_{12}=\frac{6c_0}{35}+\frac{6c_1}{35}-\frac{9c_2}{70}-\frac{3c_3}{14}$$
$$\hat{c}_{13}=\frac{2c_0}{105}-\frac{3c_1}{70}-\frac{3c_2}{14}+\frac{5c_3}{21}$$

Third row:

$$\hat{c}_{20}=\frac{5c_0}{21}-\frac{3c_1}{14}-\frac{3c_2}{70}+\frac{2c_3}{105}$$
$$\hat{c}_{21}=\frac{6c_2}{35}-\frac{9c_1}{70}-\frac{3c_0}{14}+\frac{6c_3}{35}$$
$$\hat{c}_{22}=\frac{6c_1}{35}-\frac{3c_0}{70}+\frac{18c_2}{35}+\frac{5c_3}{14}$$
$$\hat{c}_{23}=\frac{2c_0}{105}+\frac{6c_1}{35}+\frac{5c_2}{14}-\frac{23c_3}{42}$$

Fourth row:

$$\hat{c}_{30}=\frac{c_1}{21}-\frac{c_0}{21}+\frac{c_2}{210}-\frac{c_3}{210}$$
$$\hat{c}_{31}=\frac{c_0}{21}+\frac{c_1}{70}-\frac{3c_2}{70}-\frac{2c_3}{105}$$
$$\hat{c}_{32}=\frac{c_0}{210}-\frac{3c_1}{70}-\frac{2c_2}{35}+\frac{2c_3}{21}$$
$$\hat{c}_{33}=\frac{2c_2}{21}-\frac{2c_1}{105}-\frac{c_0}{210}+\frac{13c_3}{14}$$

Since all of the matrices have been input and computed in matrix equation (63), furthermore calculating the formula at each point $x=\mathcal{X}_r=\frac{2r-1}{2(3+1)}$; $r=1,2,3$; hence, in accordance with condition (55) and the system of the $(m+1)$ nonlinear algebraic equations with the unknown coefficients $c_r(r=0,1,2,3)$ is:

$$\boldsymbol{C}^T\ \boldsymbol{Y_0}=[\boldsymbol{\vartheta}_0],\qquad \text{That is}\qquad [\boldsymbol{Y}_0:\boldsymbol{\vartheta}_0]=[2\ \ 0\ \ 0\ -1\ :\ 2]$$

consequently, solve it by the Newton-Raphson method to get all normal m-Bernstein coefficients C. Thus, the approximate solution $y_m(x)$ of equation (1), $y(x)$ is obtain by putting C's in equation (15). After running the python program which was written for this purpose using all steps in the algorithm (ANBOM), we obtain:

$$C=[0.999996547661906, 0.999988744076861, 0.666662138343389, -0.00000690467618822165]^T$$

The next step is to find an approximate solution $y_m(x)$of the m-truncated normal Bernstein series in Equation (15). As a result, for $m\ =\ 3$, the approximate solution to the problem can be obtained as

$$y(x)\cong y_3(x)=-2.36351552302949\times10^{-5}x^3-0.999956406397354x^2$$
$$-2.34107825622942\times10^{-5}x+0.999996547664853$$

For both $\boldsymbol{m\ =\ 5}$ ***and*** $\boldsymbol{m=\ 10}$, by adhering to the aforementioned procedures and executing our

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specialized Python program, we may obtain the approximate answer to the problem:

$$y_5(x) = -7.68207834234147 \times 10^{-6}x^5 + 1.87118545014187 \times 10^{-5}x^4 - 1.84170104091663 \times 10^{-5}x^3 - 0.999990630568039x^2 - 2.56397101772166 \times 10^{-6}x + 0.99999941822669$$

Table 1. (Test example 1) Comparison of the normal Bernstein polynomial solutions $y_m(x)$ with the exact result $y(x)$ for $m = 3$,5 and 10, respectively, according to least square and absolute errors with running time.

x	Exact Solution	$y_m(x)$ Approximate Solutions			Absolute Error		
		$m = 3$	$m = 5$	$m = 10$	$m = 3$	$m = 5$	$m = 10$
0.0	1.00	0.999996547665	0.999999418227	1.0000000001	$3.45233514 \times 10^{-06}$	$5.81773310 \times 10^{-07}$	$1.00924602 \times 10^{-10}$
0.1	0.99	0.989994618888	0.989999238901	0.990000000096	$5.38111253 \times 10^{-06}$	$7.61098737 \times 10^{-07}$	$9.56572598 \times 10^{-11}$
0.2	0.96	0.959993420171	0.959999160354	0.960000000086	$6.57982879 \times 10^{-06}$	$8.39645616 \times 10^{-07}$	$8.57927062 \times 10^{-11}$
0.3	0.91	0.909992809705	0.909999127924	0.910000000082	$7.19029486 \times 10^{-06}$	$8.72076449 \times 10^{-07}$	$8.17045320 \times 10^{-11}$
0.4	0.84	0.839992645678	0.839999113418	0.840000000069	$7.35432168 \times 10^{-06}$	$8.86582276 \times 10^{-07}$	$6.90065782 \times 10^{-11}$
0.5	0.75	0.74999278628	0.749999105899	0.750000000058	$7.21372017 \times 10^{-06}$	$8.94101172 \times 10^{-07}$	$5.82293102 \times 10^{-11}$
0.6	0.64	0.639993089699	0.639999102463	0.640000000054	$6.91030126 \times 10^{-06}$	$8.97536732 \times 10^{-07}$	$5.43347589 \times 10^{-11}$
0.7	0.51	0.509993414124	0.509999099023	0.510000000038	$6.58587588 \times 10^{-06}$	$9.00976574 \times 10^{-07}$	$3.75438569 \times 10^{-11}$
0.8	0.36	0.359993617745	0.359999081089	0.360000000026	$6.38225498 \times 10^{-06}$	$9.18910827 \times 10^{-07}$	$2.59117727 \times 10^{-11}$
0.9	0.19	0.189993558751	0.189999014549	0.190000000018	$6.44124947 \times 10^{-06}$	$9.85450630 \times 10^{-07}$	$1.79019299 \times 10^{-11}$
1.0	0.00	$-6.9046702 \times 10^{-06}$	$-1.1635466203 \times 10^{-06}$	$2.018494039 \times 10^{-10}$	$6.90467029 \times 10^{-06}$	$1.16354662 \times 10^{-06}$	$2.01849403 \times 10^{-10}$
LSE.		$\mathbf{4.6301665 \times 10^{-10}}$	$\mathbf{8.7553804 \times 10^{-12}}$	$\mathbf{8.7621554 \times 10^{-20}}$			
R. Time (Sec.)		**31.233407**	**84.224483**	**523.202559**			

$$y_{10}(x) = 3.06433185670585 \times 10^{-6}x^{10} - 1.44815920464225 \times 10^{-5}x^9 + 2.90990572864303e \times 10^{-5}x^8 - 3.24091730519172 \times 10^{-5}x^7 + 2.18677719203697 \times 10^{-5}x^6 - 9.17837996894377 \times 10^{-6}x^5 + 2.36818232224323e \times 10^{-6}x^4 - 3.57317887278441 \times 10^{-7}x^3 - 0.999999971845824x^2 - 9.34297972321474 \times 10^{-10}x + 1.00000000010092$$

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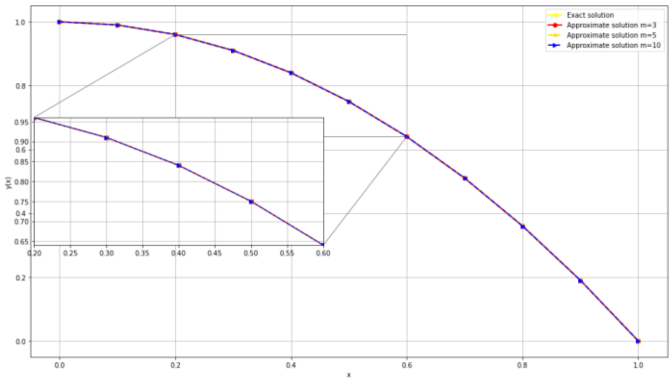

Figure 1a. Analysis of the exact and our method result of test problem 1.

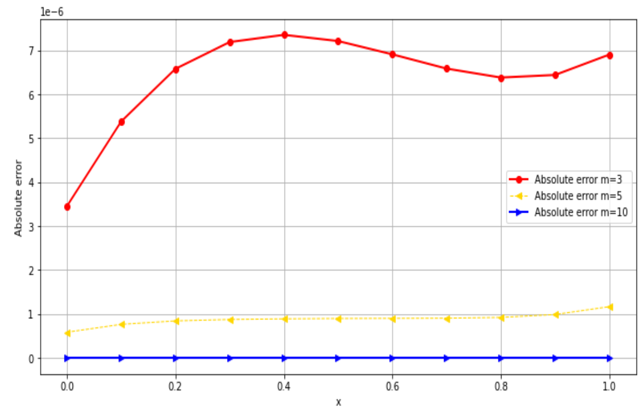

Figure 1b. Absolute error of test problem 1.

Test Problem 2: Consider the multi-fractional order NIDEs of V-F type over closed bounded interval [0,1], which is given by:

$$\begin{aligned} {}^{C}D_x^{0.7}y(x) + x\ {}^{C}D_x^{0.1}y(x) + \cos x\, y(x) \\ = f(x) + 1\times 10^{-4}\int_0^x e^x\ {}^{C}D_s^{0.4}[y(s)]^2 ds + 1\times 10^{-4}\int_0^1 s^2 x\ {}^{C}D_s^{0.2}[y(s)]^4 ds \end{aligned}$$

$$\begin{aligned} f(x) = \frac{1}{\Gamma(1.3)}x^{0.3} + \frac{1}{\Gamma(1.9)}x^{1.9} + 2\cos x + x\cos x - 0.0001\left(\frac{4}{\Gamma(2.6)}e^x x^{1.6} + \frac{2}{\Gamma(3.6)}e^x x^{2.6}\right) \\ - 0.0001x\left(\frac{160}{19\Gamma(1.8)} + \frac{10}{\Gamma(2.8)} + \frac{240}{29\Gamma(3.8)} + \frac{60}{17\Gamma(4.8)}\right) \end{aligned}$$

Subject to the boundary condition: $y(0) + 4y(1) = 14$. The exact solution of this equation is $y(x) = 2 + x$.

Here, from considering the problem we have: $n = 2$, $\sigma_2 = 0.7$, $\sigma_1 = 0.1$, $\alpha_1 = 0.4$, $\beta_1 = 0.2$, thus $\mu = \max\{\lceil 0.7\rceil, \lceil 0.4\rceil, \lceil 0.2\rceil\} = 1$ and we have $m_V = m_F = 1$, $P_2(x) = \cos x$, $P_1(x) = x$, while the kernels that $\mathcal{K}_1^V(x,s) = e^x$, $\mathcal{K}_1^F(x,s) = s^2 x$ and $p_1 = 2$; $q_1 = 4$, with eigenvalue parameters $\lambda_1^V = \lambda_1^F = 1\times 10^{-4}$.

Hence $\mu = 1$ so take $m = 2$, the standard Newton-Cotes collocation points $\mathcal{X}_r, r = 1,2$ are $\left\{ x_1 = \frac{1}{6}, x_2 = \frac{3}{6}\right\}$ and the fundamental matrix equation of the given NIFDEs of V-F is derived from equation (58), written as:

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$$f(x) = \{[X_2(x)]^T \bar{X}_2^{\sigma_2}(x)\psi^{\sigma_2} + p_1(x)[X_2(x)]^T \bar{X}_2^{\sigma_1}(x)\psi^{\sigma_1} + p_2(x)[X_2(x)]^T\} A^T C$$
$$- \lambda_1^V [X_2(x)]^T A^T \; {}^1K^V A \bar{\gamma}(\alpha_1, x)\psi^{\alpha_1} A^T [\hat{\mathrm{C}}]^{p_1 - 1} C$$
$$- \lambda_1^F [X_2(x)]^T A^T \; {}^1K^F A \bar{\gamma}(\beta_1)\psi^{\beta_1} A^T [\hat{\mathrm{C}}]^{q_1 - 1} C \qquad (64)$$

After entering all of the matrices:

$$\{A, X_2(x), \bar{X}_2^{\sigma_2}(x), \psi^{\sigma_2}, \bar{X}_2^{\sigma_1}(x), \psi^{\sigma_1}, \; {}^1K^V, \; {}^1K^F, \bar{\gamma}(\alpha_1, x), \psi^{\alpha_1}, \bar{\gamma}(\beta_1), \psi^{\beta_1} \text{ and } \hat{\mathrm{C}}\}$$

into matrix equation (64) and apply the iterative step in algorithm ANBOM. Furthermore, calculating the formula at each point $x = \mathcal{X}_r = (2r - 1)/6$; $r = 1,2$; hence, in accordance with condition (55): relates to the system of the three nonlinear algebraic equations with the unknown coefficients $c_r, (r = 0,1,2)$: $C^T \boldsymbol{Y_0} = [\boldsymbol{\vartheta}_0]$, That is $[\boldsymbol{Y}_0 : \boldsymbol{\vartheta}_0] = [1 \quad 0 \quad 4 \quad : \quad 14]$. While computing the result by Python program designed for this, the augmented matrix of this basic matrix equation with the boundary conditions rows we obtained the augmented matrix for the nonlinear system, consequently,we solve it by the Newton-Raphson method to get all normal 2-Bernstein coefficients C. Thus, the approximate solution $y_2(x)$ of equation (1), $y(x)$ is obtain by putting C's in equation (15), that is:

$$C = [2.00000540670947 \quad 2.50000018902471 \quad 2.99999864832263]^T$$

The next step is to find an approximate solution $y_2(x)$of the truncated normal Bernstein series in Equation (15). As a result, for $m = 2$, the approximate solution to the problem can be obtained as

$$y(x) \cong y_2(x) = 3.67698267211836 \times 10^{-6} x^2 + 0.999989564630495x + 2.00000540670947$$

and

$$y(x) \cong y_4(x) = 8.38463698471514 \times 10^{-9} x^4 - 2.56044501156794 \times 10^{-8} x^3 + 2.04753973775951 \times 10^{-8} x^2 + 0.999999993643064x + 2.00000000248108$$

and

$$y(x) \cong y_{10}(x) = 1.52478156323355 \times 10^{-6} x^{10} - 5.56467276169315 \times 10^{-6} x^9 + 8.09333198503737 \times 10^{-6} x^8 - 5.70323823012586 \times 10^{-6} x^7 + 1.58097191160778 \times 10^{-6} x^6 + 4.02308614866342 \times 10^{-7} x^5 - 4.70555050924304 \times 10^{-7} x^4 + 1.69754343914974 \times 10^{-7} x^3 - 3.63285153071047 \times 10^{-8} x^2 + 1.00000000654073x + 1.99999999768381$$

The precise solution $y(x)$ and the approximate solution $y_2(x)$ for $m = 2, 4$ and 10 are compared in Table 2 based on running time and least square with absolute errors, respectively.

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Table 2. (Test example 2) Comparison of the normal Bernstein polynomial solutions $y_m(x)$ with the exact result $y(x)$ for $m = 2,4$ and 10, respectively, according to least square and absolute errors with running time.

x	Exact Solution	$y_m(x)$ Approximate Solutions			Absolute Error		
		$m = 2$	$m = 4$	$m = 10$	$m = 2$	$m = 4$	$m = 10$
0.0	2.0	2.00000540671	2.00000000248	1.99999999768	$5.40670946 \times 10^{-06}$	$2.481082894 \times 10^{-09}$	$2.316191016 \times 10^{-09}$
0.1	2.1	2.10000439994	2.10000000203	2.0999999981	$4.399942342 \times 10^{-06}$	$2.025377199 \times 10^{-09}$	$1.897594970 \times 10^{-09}$
0.2	2.2	2.20000346671	2.20000000184	2.19999999832	$3.466714872 \times 10^{-06}$	$1.837291652 \times 10^{-09}$	$1.681093042 \times 10^{-09}$
0.3	2.3	2.30000260703	2.30000000179	2.29999999846	$2.607027055 \times 10^{-06}$	$1.793383219 \times 10^{-09}$	$1.538347671 \times 10^{-09}$
0.4	2.4	2.40000182088	2.40000000179	2.39999999856	$1.820878892 \times 10^{-06}$	$1.79033365 \times 10^{-09}$	$1.438084318 \times 10^{-09}$
0.5	2.5	2.50000110827	2.50000000174	2.49999999863	$1.108270382 \times 10^{-06}$	$1.744947297 \times 10^{-09}$	$1.364925061 \times 10^{-09}$
0.6	2.6	2.6000004692	2.60000000159	2.59999999868	$4.692015256 \times 10^{-07}$	$1.594151477 \times 10^{-09}$	$1.318190889 \times 10^{-09}$
0.7	2.7	2.69999990367	2.70000000129	2.69999999872	$9.632767783 \times 10^{-08}$	$1.294996110 \times 10^{-09}$	$1.282300488 \times 10^{-09}$
0.8	2.8	2.79999941168	2.80000000082	2.79999999874	$5.883172269 \times 10^{-07}$	$8.246567873 \times 10^{-10}$	$1.263472437 \times 10^{-09}$
0.9	2.9	2.89999899323	2.90000000018	2.89999999878	$1.006767123 \times 10^{-06}$	$1.804267846 \times 10^{-10}$	$1.219877976 \times 10^{-09}$
1.0	3.0	2.99999864832	2.99999999938	3.00000000058	$1.351677366 \times 10^{-06}$	$6.202709457 \times 10^{-10}$	$5.7904792072 \times 10^{-10}$
LSE.		$\mathbf{7.5366723\ e-11}$	$\mathbf{2.8415607\ e-17}$	$\mathbf{2.4890985\ e-17}$			
R. Time (Sec.)		$\mathbf{28.321521}$	$\mathbf{94.988791}$	$\mathbf{2033.952366}$			

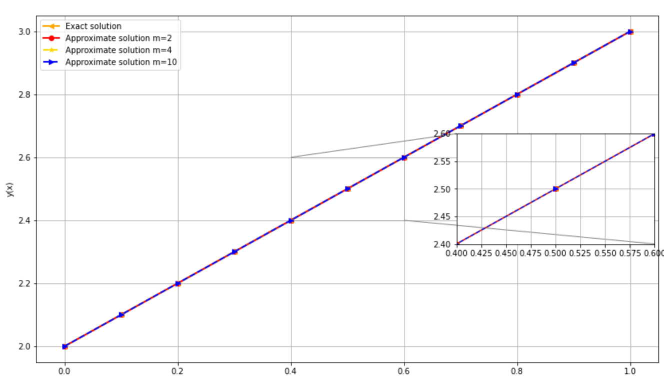

Figure 2a. Analysis of the exact and our method result of test problem 2.

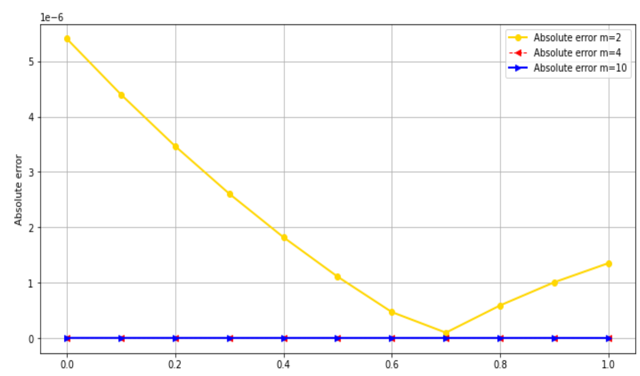

Figure 2b. Absolute error of test problem 2.

Test Problem 3: Consider the multi-fractional order NIDEs of V-F type over closed bounded interval [0,1], which is given by:

$$
\begin{aligned}
&{}^{C}D_x^{0.4}y(x)+\cosh(x)\,y(x)\\
&\qquad=\frac{-1}{\Gamma(1.6)}x^{0.6}+\cosh(x)-x\cosh(x)+\frac{3}{100\Gamma(2.8)}e^{x}x^{1.8}\\
&\qquad+\left(\frac{3}{280\,\Gamma(1.8)}-\frac{3e^{x}}{50\Gamma(3.8)}\right)x^{2.8}+\left(\frac{3e^{x}}{50\Gamma(4.8)}-\frac{3}{190\,\Gamma(2.8)}\right)x^{3.8}+\frac{1}{80\Gamma(3.8)}x^{4.8}\\
&\qquad+\frac{2}{135\Gamma(1.7)}-\frac{1}{185\Gamma(2.7)}+0.01\int_0^x(s+e^{x})\,{}^{C}D_s^{0.2}[y(s)]^3ds\\
&\qquad+0.02\int_0^1 s\,{}^{C}D_s^{0.3}[y(s)]^2ds
\end{aligned}
$$

Subject to the boundary condition: $4y(0)-y(1)=4$. The exact solution of this equation is $y(x)=1-x$.

Here, apply the algorithm **AFBOM** for $\theta=0.5$, from the considered problem we have: $n=1$, $\sigma_1=0.4, \alpha_1=0.2, \beta_1=0.3$, thus $\mu=\max\{\lceil 0.4\rceil,\lceil 0.2\rceil,\lceil 0.3\rceil\}=1$ and we have $m_V=m_F=1$, $P_1(x)=\cosh(x)$, while the kernels that $\mathcal{K}_1^V(x,s)=s+e^{x}, \mathcal{K}_1^{\mathrm{F}}(\mathrm{x},\mathrm{s})=\mathrm{s}$ and $p_1=3;\ q_1=2$, with eigenvalue parameters $\lambda_1^{\mathrm{V}}=0.01, \lambda_1^{\mathrm{F}}=0.02$.

Hence $\mu=1$ so take $m=2$, the standard Newton-Cotes collocation points $\mathcal{X}_r, r=1,2$ are $\{x_1=1/6, x_2=3/6\}$ and the fundamental matrix equation of the given NIFDEs of V-F type is derived from equation (62), written as:

$$
\begin{aligned}
f(x)=&\left\{\left[X_2^{\theta}(x)\right]^T\bar{X}_2^{\sigma_1}(x)\psi_{\sigma_1}^{\theta}+p_1(x)\left[X_2^{\theta}(x)\right]^T\right\}A^T\mathbb{D}\\
&-\lambda_1^V\left[X_2^{\theta}(x)\right]^T A^T\,{}^1K^V\,A\bar{\gamma}^{\theta}(\alpha_1,x)\psi_{\alpha_1}^{\theta}A^T\left[\widehat{\mathbb{D}}\right]^{p_1-1}\mathbb{D}\\
&-\lambda_1^F\left[X_2^{\theta}(x)\right]^T A^T\,{}^1K^F\,A\bar{\gamma}^{\theta}(\beta_1)\psi_{\beta_1}^{\theta}A^T\left[\widehat{\mathbb{D}}\right]^{q_1-1}\mathbb{D}
\end{aligned}
\tag{65}
$$

where,

$$
X_2^{\theta}(x)=\begin{bmatrix}1 & x^{\theta} & x^{2\theta}\end{bmatrix}^T,\ \mathbb{D}^T=\begin{bmatrix}\mathbb{d}_0 & \mathbb{d}_1 & \mathbb{d}_2\end{bmatrix},\ A=\begin{bmatrix}1 & -2 & 1\\ 0 & 2 & -2\\ 0 & 0 & 1\end{bmatrix}
$$

$$
{}^1K^V=\begin{bmatrix}1107824/936685 & 323709/273742 & 951256/435813\\ 350603/582036 & 426929/708849 & 305298/190541\\ 723073/273786 & 2631147/996410 & 3404175/934987\end{bmatrix},
$$

$$
{}^1K^F=\begin{bmatrix}0 & 0 & 943251/943177\\ 0 & 0 & 567908/567947\\ 0 & 0 & 356758/356751\end{bmatrix}
$$

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$$\bar{X}_2^{\sigma_1}(x)=\begin{bmatrix}0&0&0\\0&x^{-\frac{2}{5}}&0\\0&0&x^{-\frac{2}{5}}\end{bmatrix},\psi_{\sigma_1}^{\theta}=\begin{bmatrix}0&0&0\\0&\frac{\Gamma(1.5)}{\Gamma(1.1)}&0\\0&0&\frac{1}{\Gamma(1.6)}\end{bmatrix}$$

$$\psi_{\alpha_1}^{\theta}=\begin{bmatrix}0&0&0\\0&\frac{\Gamma(1.5)}{\Gamma(1.3)}&0\\0&0&\frac{1}{\Gamma(1.8)}\end{bmatrix},\qquad \psi^{\beta_1}=\begin{bmatrix}0&0&0\\0&\frac{\Gamma(1.5)}{\Gamma(1.2)}&0\\0&0&\frac{1}{\Gamma(1.7)}\end{bmatrix}$$

and,

$$W_2^{\theta}=\begin{bmatrix}1&\frac{2}{3}&\frac{1}{2}\\\frac{2}{3}&\frac{1}{2}&\frac{2}{5}\\\frac{1}{2}&\frac{2}{5}&\frac{1}{3}\end{bmatrix},\boldsymbol{Q}_2^{\theta}=AW_2^{\theta}A^T=\begin{bmatrix}\frac{1}{15}&\frac{1}{15}&\frac{1}{30}\\\frac{1}{15}&\frac{2}{15}&\frac{2}{15}\\\frac{1}{30}&\frac{2}{15}&\frac{1}{3}\end{bmatrix},\left[\boldsymbol{Q}_2^{\theta}\right]^{-1}=\begin{bmatrix}36&-24&6\\-24&\frac{57}{2}&-9\\6&-9&6\end{bmatrix}$$

$$\bar{\gamma}^{\theta}(\alpha_1,x)=\begin{bmatrix}0&\frac{10x^{1.3}}{13}&\frac{5x^{1.8}}{9}\\0&\frac{5x^{1.8}}{9}&\frac{10x^{2.3}}{23}\\0&\frac{10x^{2.3}}{23}&\frac{5x^{2.8}}{14}\end{bmatrix},\qquad \bar{\gamma}^{\theta}(\beta_1)=\begin{bmatrix}0&\frac{5}{6}&\frac{10}{17}\\0&\frac{10}{17}&\frac{5}{11}\\0&\frac{5}{11}&\frac{10}{27}\end{bmatrix}$$

with

$$\widehat{\mathbb{D}}=[\mathbb{U}_1\mathbb{D}\quad \mathbb{U}_2\mathbb{D}\quad \mathbb{U}_3\mathbb{D}]A^T=\begin{bmatrix}\hat{d}_{00}&\hat{d}_{01}&\hat{d}_{02}\\\hat{d}_{10}&\hat{d}_{11}&\hat{d}_{12}\\\hat{d}_{20}&\hat{d}_{21}&\hat{d}_{22}\end{bmatrix}$$

First row:

$$\hat{d}_{00}=\frac{53d_0}{70}+\frac{9d_1}{35}-\frac{d_2}{70}$$
$$\hat{d}_{01}=\frac{9d_0}{35}-\frac{2d_1}{35}-\frac{d_2}{5}$$
$$\hat{d}_{02}=-\frac{d_0}{70}-\frac{d_1}{5}+\frac{3d_2}{14}$$

Second row:

$$\hat{d}_{10}=-\frac{17d_0}{70}+\frac{4d_1}{35}+\frac{9d_2}{70}$$
$$\hat{d}_{11}=\frac{4d_0}{35}+\frac{18d_1}{35}+\frac{13d_2}{35}$$
$$\hat{d}_{12}=\frac{9d_0}{70}+\frac{13d_1}{35}-\frac{d_2}{2}$$

Third row:

$$\hat{d}_{20}=\frac{3d_0}{70}-\frac{d_1}{35}-\frac{d_2}{70}$$
$$\hat{d}_{21}=-\frac{d_0}{35}-\frac{2d_1}{35}+\frac{3d_2}{35}$$
$$\hat{d}_{22}=-\frac{d_0}{70}+\frac{3d_1}{35}+\frac{13d_2}{14}$$

Since all of the matrices have been input and computed in matrix equation (65), furthermore calculating the formula at each point $x=\mathcal{X}_r=(2r-1)/6;\ r=1,2;$ which corresponds to the system of the $3-$ nonlinear algebraic equations with the unknown coefficients $d_r(r=0,1,2)$ and according to the condition (59) that is:

$$\mathbb{D}^T\boldsymbol{Y}_0^{\theta}=[\boldsymbol{\vartheta}_0],\qquad \text{That is}\qquad \left[\boldsymbol{Y}_0^{\theta}:\boldsymbol{\vartheta}_0\right]=[4\quad 0\quad -1\quad :\quad 4]$$

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consequently, solve it by the Newton-Raphson method to get all fractional Bernstein coefficients $\mathbb{D}$. Thus, the approximate solution $y_2^{\theta}(x)$ of equation (1) with respect to fractional $\theta = 0.5$, $y(x)$ is obtained by putting $\mathbb{D}$'s in equation (19). Using all steps in the algorithm (**AFBOM**), we execute the Python program designed for this purpose and get:

$$\mathbb{D} = [0.998735525319421 \quad 0.998879173680240 \quad -0.00505789872231711]^T$$

The next step is to find an approximate solution $y_2(x)$of the truncated fractional Bernstein series in Equation (19). As a result, for $m = 2$, the approximate solution to the problem can be obtained as

$$y(x) \cong y_2^{\theta}(x) = 0.000287296721638741x^{0.5} - 1.00408072076338x + 0.998735525319421$$

For both $m = 4$ and $m = 10$, by adhering to the aforementioned procedures and executing our specialized Python program, we may obtain the approximate answer to the problem:

$$y_4^{\theta}(x) = 4.84516290262249 \times 10^{-5}x^{0.5} - 1.00076196663514x + 0.00170054502343575x^{1.5} - 0.000628523292331984x^{2.0} + 1.00011950224166$$

$$y_{10}^{\theta}(x) = 0.0853106883557455x^{0.5} - 2.34490942490022x^{1.0} + 9.4384425209781x^{1.5} - 38.1447982359375x^{2.0} + 97.2471139823056x^{2.5} - 161.779123081532x^{3.0} + 175.46938026208x^{3.5} - 119.653428822921x^{4.0} + 46.5696979394812x^{4.5} - 7.88762655292904x^{5.0} + 1.00001975832697$$

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Table 3. (Test example 3) Comparison of the fractional Bernstein polynomial solutions $y_m^{\theta}(x)$ for $\theta = 0.5$ with the exact result $y(x)$ for $m = 2$,4 and 10, respectively, according to least square and absolute errors with running time.

x	Exact Solution	$y_m^{\theta}(x)$ Approximate Solutions			Absolute Error		
		$m = 2$	$m = 4$	$m = 10$	$m = 2$	$m = 4$	$m = 10$
0.0	1.0	0.998735525319	1.00011950224	1.00001975833	0.0012644746	0.00011950224	$1.975832697 \times 10^{-05}$
0.1	0.9	0.898418304444	0.900106118051	0.900188188294	0.0015816955	0.0001061180	0.00018818829
0.2	0.8	0.798047864167	0.800115737581	0.800105658172	0.0019521358	0.0001157375	0.00010565817
0.3	0.7	0.697668667986	0.700140311266	0.700078868346	0.0023313320	0.0001403112	$7.88683457 \times 10^{-05}$
0.4	0.6	0.597284939415	0.600175003005	0.600061315859	0.0027150605	0.0001750030	$6.1315858729 \times 10^{-05}$
0.5	0.5	0.496898314398	0.500216882035	0.500045938788	0.0031016856	0.0002168820	$4.5938788041 \times 10^{-05}$
0.6	0.4	0.396509631945	0.400263926252	0.400041252068	0.0034903680	0.0002639262	$4.1252067749 \times 10^{-05}$
0.7	0.3	0.296119390468	0.300314631356	0.300038886193	0.0038806095	0.0003146313	$3.888619264 \times 10^{-05}$
0.8	0.2	0.195727914708	0.200367821448	0.200028505215	0.0042720852	0.0003678214	$2.8505215095 \times 10^{-05}$
0.9	0.1	0.0953354302338	0.10042254445	0.10002780364	0.0046645697	0.0004225444	$2.7803639516 \times 10^{-05}$
1.0	0.0	−0.00505789872	0.0004780089667	$7.90333078851 \times 10^{-05}$	0.0050578987	0.0004780089	$7.903330788 \times 10^{-05}$
LSE.		**0.000123171**	$\mathbf{8.4726709 \times 10^{-07}}$	$\mathbf{7.0104820 \times 10^{-08}}$			
R. Time (Sec.)		**26.261228**	**81.761285**	**767.977398**			

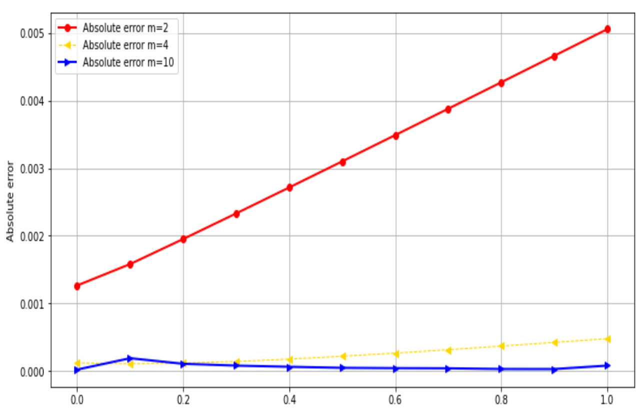

Figure 3a. Absolute error of test problem 3.

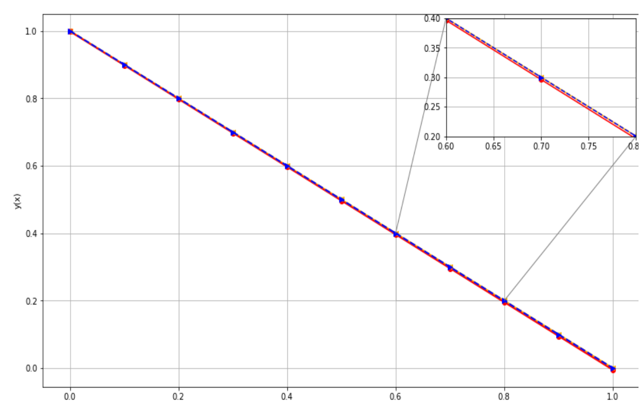

Figure 3b. Analysis of the exact and our method result of test problem 3.

Test Problem 4: Consider the multi-fractional order NIDEs of V-F type over closed bounded interval [0,1], which is given by:

$$ {}_0^C D_x^{0.5} y(x) + x^2\, y(x) = f(x) + 0.0006 \int_0^x \cos(x)\, {}_0^C D_s^{0.6} [y(s)]^2 ds + 0.07 \int_0^1 e^x\, {}_0^C D_s^{0.7} [y(s)]^3 ds $$

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where

$$f(x) = x^2 - x^2 e^x + \lim_{\bar{N}\to\infty} \sum_{\hbar=0}^{\bar{N}} \left[\frac{-x^{\hbar+0.5}}{\Gamma(\hbar+1.5)} + \frac{3\cos(x)\, x^{\hbar+1.4}(1-2^{\hbar})}{2500\Gamma(\hbar+2.4)} - \frac{21e^x(-1+2^{\hbar+1}-3^{\hbar})}{100\Gamma(\hbar+2.3)} \right]$$

Subject to the boundary condition: $2y(0) + y(1) = 1 - e$. Which has the exact solution $y(x) = 1 - e^x$.

In this problem we have: $n = 1$, $\sigma_1 = 0.5, \alpha_1 = 0.6, \beta_1 = 0.7$, thus $\mu = \max\{\lceil 0.5\rceil, \lceil 0.6\rceil, \lceil 0.7\rceil\} = 1$ and we have $m_V = m_F = 1$, $P_1(x) = x^2$, while the kernels that $\mathcal{K}_1^V(x,s) = \cos(x)$, $\mathcal{K}_1^F(x,s) = e^x$ and $p_1 = 2$; $q_1 = 3$, with eigenvalue parameters $\lambda_1^V = 0.0006, \lambda_1^F = 0.07$.

By using the first algorithm (ANBOM): Table 4 provides the numerical data that we obtained, and figures 4a and 4b display the absolute error for our method for both analytical and normal Bernstein polynomial solutions.

Table 4. (Test example 4) Comparison of the normal Bernstein polynomial solutions $y_m(x)$ with the exact result $y(x)$ for $m = 2,4$ and 10, respectively, according to least squares and absolute errors with running time.

x	Exact Solution	$y_m(x)$ Approximate Solutions			Absolute Error		
		$m = 2$	$m = 4$	$m = 10$	$m = 2$	$m = 4$	$m = 10$
0.0	0.00000000	−0.0733437688	−0.0016412529	0.000521502470	0.0733437688	0.00164125294	0.000521502470
0.1	−0.105170918	−0.1536743770	−0.1056910239	−0.1050966177	0.04850345893	0.00052010583	$7.4300361534 \times 10^{-05}$
0.2	−0.221402758	−0.2494481543	−0.2211594402	−0.2215684918	0.02804539617	0.000243317948	0.000165733679
0.3	−0.349858807	−0.3606651008	−0.3490170930	−0.35023839581	0.01080629323	0.000841714501	0.000379588235
0.4	−0.491824697	−0.4873252164	−0.4904371858	−0.49240470177	0.00449948120	0.00138751183	0.000580004133
0.5	−0.648721270	−0.6294285012	−0.6467955338	−0.64949067305	0.01929276948	0.00192573681	0.000769402357
0.6	−0.822118800	−0.7869749551	−0.8196705648	−0.82306443415	0.03514384524	0.00244823550	0.000945633765
0.7	−1.013752707	−0.9599645782	−1.0108433185	−1.0148587428	0.05378812925	0.00290938896	0.00110603534
0.8	−1.225540928	−1.1483973704	−1.2222974466	−1.2267893081	0.07714355809	0.003243481898	0.00124837963
0.9	−1.459603111	−1.3522733318	−1.4562192131	−1.4609649535	0.10732977938	0.003383898056	0.00136184242
1.0	−1.718281828	−1.5715924623	−1.7149974941	−1.7193230049	0.14668936615	0.003284334346	0.00104117648
LSE.		**0.052144503**	**$5.6581754 \times 10^{-05}$**	**$7.9920774 \times 10^{-06}$**			
R. Time (Sec.)		**37.286733**	**89.409467**	**768.523834**			

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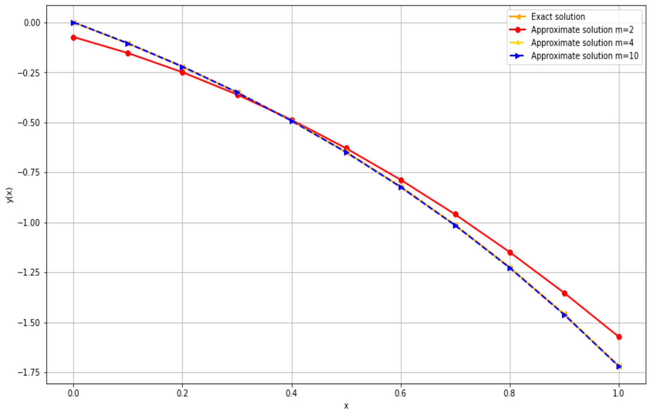

Figure 4a. Analysis of the exact and our method result of test problem 4.

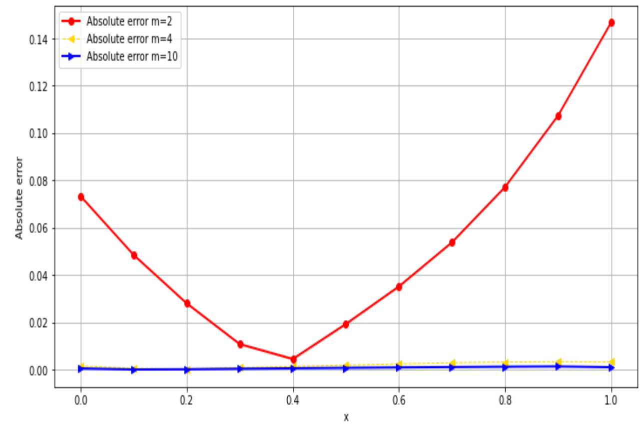

Figure 4b. Absolute error of test problem 4.

By using the second algorithm (AFBOM) for $\theta = 0.8$: Figures 5a and 5b display the absolute error for our method for fractional Bernstein polynomials and analytical solutions, and table 5 provides the numerical values we obtained.

Table 5. (Test example 4) Comparison of the fractional Bernstein polynomial solutions $y_m^{\theta}(x)$ for $\theta = 0.8$ with the exact result $y(x)$ for $m = 2$,4 and 10, respectively, according to least square and absolute errors with running time.

x	Exact Solution	$y_m^{\theta}(x)$ Approximate Solutions			Absolute Error		
		$m = 2$	$m = 4$	$m = 10$	$m = 2$	$m = 4$	$m = 10$
0.0	0	−0.11054043462	−0.00436501901	0.00090003164	0.110540434624	0.004365019011	0.000900031648
0.1	−0.1051709180	−0.1817216612	−0.1072746931	−0.10508597103	0.076550743137	0.002103775094	$8.49470376573 \times 10^{-05}$
0.2	−0.2214027581	−0.2705843213	−0.2195634286	−0.22150608967	0.049181563148	0.001839329500	0.000103331515
0.3	−0.3498588075	−0.3768675613	−0.3456583658	−0.35016670566	0.027008753724	0.004200441688	0.000307898090
0.4	−0.4918246976	−0.4986586005	−0.4856699759	−0.49234152601	0.006833902898	0.006154721726	0.000516828377
0.5	−0.6487212707	−0.6345977778	−0.6405496273	−0.64944179831	0.014123492854	0.008171643301	0.000720527619
0.6	−0.8221188003	−0.7836655084	−0.8119084971	−0.82303914718	0.038453291921	0.010210303303	0.000920346790
0.7	−1.013752707	−0.9450605709	−1.0018514982	−1.01485816697	0.068692136504	0.011901209275	0.001105459495
0.8	−1.2255409285	−1.1181309800	−1.2128664064	−1.2268125037	0.107409948461	0.012674522014	0.001271575257
0.9	−1.4596031111	−1.3023320107	−1.4477489164	−1.4610305200	0.157271100453	0.011854194693	0.001427408884
1.0	−1.7182818284	−1.4971991307	−1.7095499619	−1.7200800633	0.221082697708	0.008731866481	0.001798234837
LSE.		**0.11281960**	**0.0007724624**	$\mathbf{1.0666150 \times 10^{-05}}$			
R. Time (Sec.)		**40.185433**	**96.989010**	**1008.412208**			

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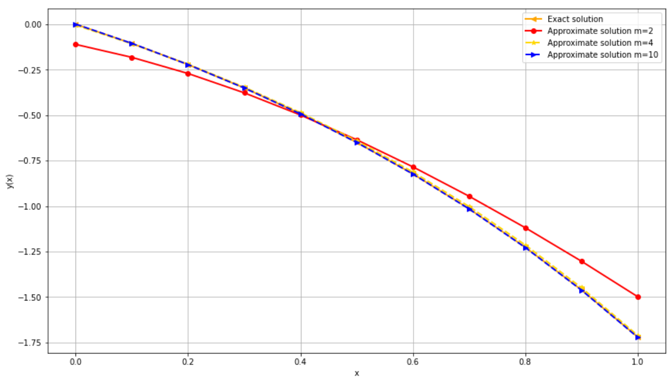

Figure 5a. Analysis of the exact and our method result of test problem 5.

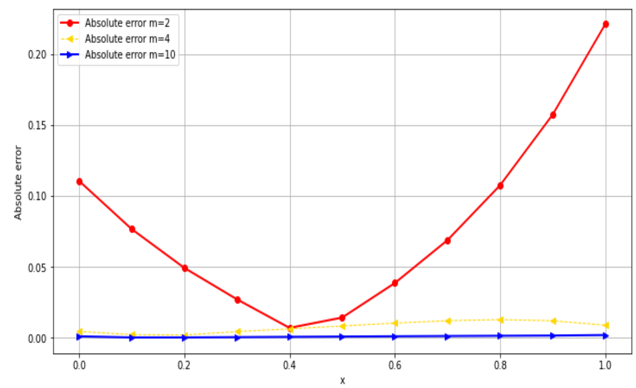

Figure 5b. Absolute error of test problem 5.

5.CONCLUSION

In this work, we proposed a numerical solution for nonlinear IFDEs of Hammerstein Volterra-Fredholm type by the operational matrices of normal and fractional m-Bernstein polynomials. Along with the collocation method, we use the operational matrices of the product, power, fractional derivative, and Volterra-Fredholm kernels by normal and θ-fractional m-Bernstein polynomials to convert the nonlinear FIDEs of Hammerstein Volterra-Fredholm type into an algebraic system of nonlinear equations that are easily solved using Newton's method. Both the normal and fractional OBMs are incredibly straightforward and appealing. To illustrate the approach, several examples are provided, and the technique's precision is evaluated by minimizing the least square error within the specified domain and running time. As the value of m (the order of Bernstein polynomials) increases, the error rate decreases, bringing the solution closer to the exact one. Our assertion that the solution's Berstein coefficients may be determined very quickly using computer code written in Python V.3.8.8 (2021) is supported by the numerical examples provided in this research and the comparison findings.

Future directions: We can improve the normal and fractional Bernstein collocation approach and solve multi-term high fractional orders of linear Fredholm-Volterra integro differential equations, including those with delays, efficiently by utilizing the residual error function. Additionally, by using this method, it is possible to obtain accurate error estimation, which improves our results' accuracy even more.

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Conflict of interests

There are non-conflicts of interest.

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الخلاصة

في هذه المقالة، نقدم طريقة عددية تطبق نهج متعدد الحدود بيرنشتاين في شكل مصفوفة لحل المعادلات التفاضلية الكسرية التكاملية غير الخطية من نوع فولتيرا–فريدهولم (NIFDEs of V–F) التي تنطوي على المشتق الكسري كابوتو مع ظروف مختلطة.

طرق العمل:

الهدف الأساسي من هذه الطريقة هو تحويل المعادلة الوظيفية غير الخطية وشروطها المرتبطة بها إلى علاقات مصفوفية باستخدام متعددات برنشتاين العادية والكسرية. باستخدام استراتيجية مثل طريقة نيوتن والحصول على معاملات برنشتاين.

الستنتاجات:

هذا النهج جذابٌ للحوسبة، ويُقدَّم فيه أمثلةٌ وشروحٌ لاستخدامه. تُستخدم أمثلةٌ توضيحيةٌ لإثبات صحة الطريقة وكفاءتها، ويُوثَّق نجاح المنهجية بمقارنة النتائج التقريبية بالحلول الدقيقة أو المرجعية.

الكلمات المفتاحية:

متعددة حدود برنشتاين، معادلات فريدهولم–فولتيرا التكاملية التفاضلية، مشتقة كابوتو الكسرية من نوع هامرشتاين، مصفوفة العمليات، الحل التقريبي، مشتقة كابوتو، نقاط التجميع.

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