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Bernstein Polynomial Approximation for Fredholm Integro-Differential Equations of Fractional Order with Numerous Constant Delays

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قريب متعدد الحدود لبرنشتاين لمعادلات فريدهولم التكاملية التفاضلية من الرتبة الكسرية مع تأخيرات ثابتة متعددة

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ABSTRACT

Background:

This study presents a useful new framework that uses Bernstein polynomials to improve a spectral collocation technique for numerically solving Fredholm integro-differential equations of fractional orders with variable coefficients and multi-time constant delay (FIFDEs-Delays) under boundary conditions.

Materials and Methods:

The approximate solutions are assumed to be in the form of the truncated Bernstein polynomial series. This novel approach is based on the use of a matrix technique to convert the display equation with conditions into an algebraic linear system of equations with unknown Bernstein coefficients.

Results:

This approach improves the accuracy of the solutions found while simultaneously simplifying the problem. The solution of this system determines the coefficients of the assumed solution. In addition, the integral operators employed in this technique were quantitatively evaluated using the Clenshaw-Curtis formula.

Conclusion:

we provide specific examples to showcase the accuracy of the method, and we employ the least-squares error methodology to minimize error terms within the given domain. Ultimately, the most common application suggested for the numerical approaches is implemented in a Python program.

Key words:

Fredholm integral equations, fractional derivative, constant delay types, Bernstein polynomial, Caputo derivative, matrix technique, standard collocation points.

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1. INTRODUCTION

Fractional calculus (FC) is the branch of calculus that allows operations such as deriving a function to $1/2$ orders by extending the derivative of a function to non-integer order. The term "fractional" is used to describe this kind of derivative [1]. The fractional calculus may be considered an old but inventive subject that has been explored to the current day, starting with the different conjectures of Leibniz (1695, 1697) and L. Euler (1730). Leibniz proposed the idea of generalizing the concept of derivative, which is defined as $d^m y/dx^m = D^m y(x)$, to non-integer order (m), namely to the order $1/2$, in his conversation with Bernoulli, L'Hopital, and Wallis. Eulerbegan by pointing out that his Gamma Function provided a meaning for non-integers in the result of the power function's derivative computation. The fact that fractional calculus may be thought of as a superset of integer-order calculus is one of its main benefits. Therefore, fractional calculus may be able to achieve things that integer-order calculus cannot. See [1-4] for further information on the historical evolution of fractional calculus. Because of its many applications in different scientific fields, fractional calculus (integral and derivative) has recently attracted significant attention in the past few decades. These days, a fractional integral and derivative are required to characterize a viscoelastic process. Polymer physics, thermodynamics, corrosion electrochemistry, optics, electrical networks, biophysics, and the behavior of viscoplastic materials are among the other fields in which FC has found utility in engineering, physics, finance, and hydrology (see [2], [3], [5], [6], [7]). The derivatives of some unknown functions at this moment rely on the values of the functions at earlier times in a significant family of functional differential equations known as delay differential equations (DDEs). One of the reasons for their significance is that, if the independent variable represents time, they explain processes with "after-effects" or time lag. Consequently, it goes without saying that delay differential equations have several uses in the fields of biology, physics, engineering, mechanics, and economics, particularly in the theory of automatic control [8]. In reality, nevertheless, these functional differential equations appear in a wide range of mathematical modeling domains, such as chemical kinetics [9], epidemiology, population dynamics ([10]), metal cutting, lasers, traffic models [11], control systems [12], etc. We specifically mention Bellman and Cooke [13], Hale [14], Driver [15], and El'sgol'ts and Norkin [16] from the several works that now outline the application areas for DDEs. Furthermore, there is a lot of interest in integro-fractional differential equations (IFDEs) for both the Volterra and Fredholm types in a variety of application fields, such as engineering, physical, and biological problems (see [17-21]). In this area, Volterra, Fredholm, and mixed Volterra-Fredholm integro-differential equations are acknowledged as important equation types. As in (see [1], [3], [22-26]), several analytical techniques have been devised to solve particular instances of these equations. However, in many situations, it can be quite difficult to discover precise analytical solutions for fractional differential and integral equations. Approximate methods have therefore become more significant since they provide workable answers to this problem and yield trustworthy results.

The solutions of integral and integro-differential equations with integer orders can be approximated using a variety of techniques. These techniques include weighted residual methods [27], Legendre polynomial approaches [28], the collocation method [29], and Bernstein

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polynomials (see [30], [31]). These methods offer useful strategies for handling the complexity of these equations and finding approximations of solutions. The techniques for solving integral and integro-differential equations can be modified for the solution of integro-fractional differential equations.

In this article, the aims to solve linear Fredholm integro-fractional differential equations of constants delay type (FIFDEs-Delays) with variable coefficients, use the Bernstein polynomial for how to solve Equation (1.1) with boundary and historical conditions (1.2), for all $a \le x \le b$.

$$ {}_a^C D_x^{\beta} y(x) + \sum_{\ell=1}^{n} p_\ell(x)\, {}_a^C D_x^{\alpha_\ell} y(x-\tau_\ell) + p_0(x) y(x-\tau_0) $$

$$ = f(x) + \sum_{j=0}^{m} \lambda_j \int_a^b k_j(x,t)\, y(t-\tau_j^*)\, dt\,, \qquad a \le x \le b \qquad (1.1) $$

Together with boundary conditions and historical functions:

$$ \underline{Boundary\ conditions} \qquad\qquad \underline{Historical\ functions} $$

$$ \sum_{\rho=0}^{\mu-1} \left\{ g_{k\rho} y^{(\rho)}(a) + h_{k\rho} y^{(\rho)}(b) \right\} = \vartheta_k \ ; k = 0,1,\dots,\mu-1 \quad , \qquad y(x) = \varphi(x) \in C^{\mu-1}[a,b] \qquad (1.2) $$

Where $\mu = \max\{\omega^{\beta}, \omega_\ell^{\alpha} : \ell = \overline{1:n}\}$ with historical property : for all, $x \in [\bar{a}, a]$ such that $\bar{a} = a - \max\{\tau_{\ell,} \tau_j^* : j = \overline{0:m}\ and\ \ell = \overline{0:n}\}$ where $y(x)$ is the unknown function which is the solution of equation (1.1), the functions $k_j : S \times \mathbb{R} \longrightarrow \mathbb{R}, (S = \{(x,t) : a \le x < t \le b\}), j = 0,1,2,\dots,m$ and $f, p_\ell : [a,b] \longrightarrow \mathbb{R};\ for\ all\ \ell = 0,1,2,\dots,n$ continuous functions. In addition $\alpha_\ell \in \mathbb{R}^+$ for all $\omega^{\beta} - 1 < \beta \le \omega^{\beta}, \omega^{\beta} = \lceil \beta \rceil\ ;\ \omega_\ell^{\alpha} - 1 < \alpha_\ell \le \omega_\ell^{\alpha}, \omega_\ell^{\alpha} = \lceil \alpha_\ell \rceil$ for all $\ell = 1,2,\dots,n$ with property $\beta > 0, \alpha_n > \alpha_{n-1} > \cdots > \alpha_1 > 0$ and positive constant time-lags (delay), τ_j^*, τ_ℓ for all $j = 0,1,2,\dots,m$ and $\ell = 0,1,2,\dots,n$, while $m \in \mathbb{Z}^+ \cup \{0\}$ and $n \in \mathbb{Z}^+$. This paper presents the generalized Bernstein approach for the solutions of FIFDEs-Delays equation (1.1). The residual error estimate, collocation mesh points, for the problem and the technique will also be provided. Moreover, we obtain an adjusted approximate solution of equation (1.1) with (1.2) in the reduced generalized Bernstein series form.

This study's structure is categorized as follows: A summary of generalized Bernstein polynomials, derivatives, fractional integrals, and other features required for this investigation is provided in Section 2. Section 3 introduces the problem statement and the approximation technique is expressed using a decent approach, Section 4 presents a few solved numerical instances. In Section 5, we provide a succinct conclusion.

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2. PRELIMINARIES AND NOTATIONS

The fundamental principles and characteristics of the fractional calculus that will be utilized throughout this study are introduced in this part. We direct those interested to Podlubny in [1] and Kilbas et al. and Miller et al. (see [2] and [4]), respectively, for more details.

2.1. Basic Definitions and Lemmas:

Definitions 2.1. Every real-valued function, including y defined on $[a,b]$, is contained in the defined space $C_\gamma[a,b]$, $\gamma \in \mathbb{R}$. If a real number r> γ exists and $\hat{y} \in C[a,b]$, then $y(x)$ can be expressed as $(x-a)^r\hat{y}(x)$. If and only if $y^{(n)} \in C_\gamma[a,b], n \in \mathbb{N}_0$, it is in the space $C_\gamma^n[a,b]$ (see [32], [33]).

Definitions 2.2. For a function $y \in C_\gamma[a,b], \gamma \geq -1$, the Reimann-Liouville fractional integral operator ${}_aJ_x^\beta$ of order $\beta \geq 0$ is defined by (see [1], [33])

$${}_aJ_x^\beta\, y(x) = \begin{cases} \dfrac{1}{\Gamma(\beta)} \displaystyle\int_a^x (x-t)^{\beta-1} y(t)dt \;, & \beta > 0 \\ y(x) \;, & \beta = 0 \end{cases} \qquad (2.1)$$

Hence ${}_aJ_x^\beta$, has a semi-group property, that is for all $\alpha, \beta \geq 0$, on the closed interval $[a,b]$:

$${}_aJ_x^\beta \;\; {}_aJ_x^\alpha y(x) = \; {}_aJ_x^{\beta+\alpha} y(x) = \; {}_aJ_x^\alpha \;\; {}_aJ_x^\beta y(x) \qquad (2.2)$$

And $\beta - RL$ operator for γ-power function is:

$${}_aJ_x^\beta (x-a)^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\beta+1)}(x-a)^{\gamma+\beta} \qquad (2.3)$$

Definition 2.3. The following is the definition of the Caputo fractional derivative of order $\beta \geq 0$ of $y(x) \in C_{-1}^n[a,b], n \in \mathbb{N}$ (see [1], [33],[34]) :

$${}_a^C D_x^\beta\, y(x) = \begin{cases} {}_aJ_x^{n-\beta}\, y^{(n)}(x) & , n-1 < \beta < n \\ \dfrac{d^n y(x)}{dx^n} & , \quad \beta = n \end{cases} \qquad (2.4)$$

For all $c_1, c_2 \in \mathbb{R}$ and $u, v \in C_{-1}^n(I), n = \lceil \beta \rceil, I = [a,b]$. The β-Caputo fractional derivative satisfies the property of linearity, that is:

$${}_a^C D_x^\beta \; (c_1 u(x) + c_2 v(x)) = c_1 \, {}_a^C D_x^\beta \; u(x) + c_2 \, {}_a^C D_x^\beta \; v(x), \qquad x \in I. \qquad (2.5)$$

Also, the β-Caputo fractional derivative for any constant $A \in \mathbb{R}$, is negligible, i.e, ${}_a^C D_x^\beta \; A = 0$. Furthermore, the basic relation between β-RL integral operator $({}_a^C D_x^\beta)$ and the β-Caputo differential operator $({}_a^C D_x^\beta)$ is left inverse while not the right, so expressed in the following formulas, $m = \lceil \beta \rceil$:

$${}_a^C D_x^\beta \;\; {}_aJ_x^\beta u(x) = u(x) \qquad (2.6)$$

$$ {}_aJ_x^{\beta}\ {}_a^{C}D_x^{\beta}\ u(x) = \sum_{k=0}^{m-1} \frac{u^{(k)}(x)}{k!}(x-a)^k, \qquad x > a. \qquad (2.7) $$

Definition 2.4. The following is the definition of the general $(N+1)$-Bernstein polynomials $B_{r,N}(x)$ of degree N when $x \in [a,b]$ (see [35], [36]):

$$ B_{r,N}(x) = \frac{1}{(b-a)^N}\binom{N}{r}\ (x-a)^r(b-x)^{N-r}\ \ for\ all\ \ r = 0,1,\dots,N \qquad (2.8) $$

In a specific instance, $[a,b] = [0,1]$ is expressed as

$$ B_{r,N}(x) = \binom{N}{r}\ (x)^r(1-x)^{N-r}\ \ for\ all\ \ r = 0,1,\dots,N \qquad (2.9) $$

Equation (2.8) 's expression can be rewrite using the binomial expansion to

$$ B_{r,N}(x) = \sum_{i=r}^{N} \frac{(-1)^{i-r}}{(b-a)^i}\binom{N}{r}\binom{N-r}{i-r}(x-a)^i \qquad (2.10) $$

By the binomial property, It can be entered into the form

$$ B_{r,N}(x) = \sum_{i=r}^{N} \frac{(-1)^{i-r}}{(b-a)^i}\binom{N}{i}\binom{i}{r}(x-a)^i \qquad (2.11) $$

The Bernstein polynomial's first derivative can be expressed as follows:

$$ B'_{r,N}(x) = \sum_{i=r}^{N} \frac{(-1)^{i-r}}{(b-a)^i}\binom{N}{i}\binom{i}{r}(i)(x-a)^{i-1} \qquad (2.12) $$

and the integration:

$$ \int_a^b B_{r,N}(x)dx = \frac{b-a}{N+1} \qquad (2.13) $$

Lemma 2.5. The $\beta-$ fractional derivative for the N^{th} degree Bernstein polynomials in the Caputo sense is supplied, if $\beta \in \mathbb{R}^+\backslash\mathbb{N}$, then [36]:

$$ {}_a^{C}D_x^{\beta}B_{r,N}(x) = \sum_{z=\lceil\beta\rceil}^{N} \frac{(-1)^{(z-r)}}{(b-a)^z}\binom{N}{z}\binom{z}{r}\frac{\Gamma(z+1)}{\Gamma(z+1-\beta)}(x-a)^{z-\beta} \qquad (2.14) $$

or, formed as

$$ {}_a^{C}D_x^{\beta}B_{r,N}(x) = \sum_{z=\lceil\beta\rceil}^{N} \frac{(-1)^{z-r}}{(b-a)^z}\binom{N}{z}\binom{z}{r}\ S_z(x;\beta) $$

Where $S_z(x;\beta) = \begin{cases} 0 & if \quad z\epsilon\{0,1,\dots,\lceil\beta\rceil-1\} \\ \frac{\Gamma(z+1)}{\Gamma(z+1-\beta)}(x-a)^{z-\beta} & if \quad z \in \mathbb{N} \end{cases}$.

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Lemma 2.6. At the boundary points the n^{th} derivative of the universal Bernstein polynomial of degree $r, y(x) = B_{r,N}(x)$, for all $r = 0,1, \dots, N(\in \mathbb{N})$. Formed [36]:

1. At $x = a$ is

$$y^{(n)}(a) = \sum_{r=0}^{n} \frac{(-1)^{n-r}\, n!}{(b-a)^n} \binom{N}{n} \binom{n}{r} \qquad (2.15)$$

2. At $x = b$ is

$$y^{(n)}(b) = \sum_{r=N-n}^{N} \frac{(-1)^{2n-r-N}\, n!}{(b-a)^n} \binom{N}{n} \binom{n}{N-r} \qquad (2.16)$$

Lemma 2.7. (New) The $\alpha-$fractional derivative for the N^{th} degree $\tau-$ Bernstein polynomials in the Caputo sense is supplied, if $\alpha \in \mathbb{R}^+ \backslash \mathbb{N}$ and $\tau \geq 0$.Then:

$$ {}_a^C D_x^{\alpha} B_{r,N}(x-\tau) = \sum_{i=r}^{N} \sum_{k=0}^{i} (-1)^{2i-k-r} \binom{N}{i} \binom{i}{r} \binom{i}{i-k} \frac{(\tau)^{i-k}}{(b-a)^i} M_k(x;\alpha) \qquad (2.17)$$

Where $M_k(x;\alpha) = \begin{cases} 0 & if \quad k \in \{0,1, \dots, \lceil \alpha \rceil - 1\} \\ \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} (x-a)^{k-\alpha} & if \quad k \in \mathbb{N} \ and \ k \geq \lceil \alpha \rceil \end{cases}$

Proof. The constant time delay general Bernstein polynomials of degree N defined on the closed bounded interval $[a, b]$, define:

$$B_{r,N}(x-\tau) = \frac{1}{(b-a)^N} \binom{N}{r} ((x-a)-\tau)^r \big((b-x)+(\tau)\big)^{N-r} \ for \ r = 0,1, \dots, N \qquad (2.18)$$

First way of prove: Then, using the binomial formula, we obtain:

$$\begin{aligned} \big((b-x)+(\tau)\big)^{N-r} &= \big((b-a)-((x-a)-\tau)\big)^{N-r} \\ &= \sum_{i=0}^{N-r} (-1)^{N-r+1} \binom{N-r}{i} (b-a)^{N-r-i} ((x-a)-\tau)^i \end{aligned} \qquad (2.19)$$

putting equation (2.19) in to equation (2.18) we get

$$\begin{aligned} B_{r,N}(x-\tau) &= \sum_{i=0}^{N-r} \frac{1}{(b-a)^N} (-1)^i \binom{N}{r} \binom{N-r}{i} ((x-a)-\tau)^i \ (b-a)^{N-r-i} \ ((x-a)-\tau)^r \\ &= \sum_{i=0}^{N-r} (-1)^i \binom{N}{r} \binom{N-r}{i} \frac{\big((x-a)-\tau\big)^{r+i}}{(b-a)^{r+i}} \end{aligned} \qquad (2.20)$$

yield,

$$B_{r,N}(x-\tau) = \sum_{i=r}^{N} (-1)^{i-r} \binom{N}{r} \binom{N-r}{i-r} \frac{\big((x-a)-\tau\big)^i}{(b-a)^i} \qquad (2.21)$$

apply the binomial theorem to $\big((x-a)-\tau\big)^i$ and putting in equation (2.21), obtain (2.22) after some simple manipulations:

$$B_{r,N}(x-\tau)=\sum_{i=r}^{N}\sum_{k=0}^{i}(-1)^{i+k-r}\binom{N}{i}\binom{i}{r}\binom{i}{k}\frac{1}{(b-a)^i}(\tau)^k(x-a)^{i-k} \qquad (2.22)$$

Apply the $\alpha-$Caputo fractional derivative for both sides of the equation (2.22) we obtain:

$${}_a^C D_x^\alpha\, B_{r,N}(x-\tau)=\sum_{i=r}^{N}\sum_{k=0}^{i}(-1)^{i+k-r}\binom{N}{i}\binom{i}{r}\binom{i}{k}\frac{1}{(b-a)^i}(\tau)^k\,{}_a^C D_x^\alpha\,(x-a)^{i-k}$$

Consequently, listed in:

$${}_a^C D_x^\alpha B_{r,N}(x-\tau)=\sum_{i=r}^{N}\sum_{k=0}^{i}(-1)^{i+k-r}\binom{N}{i}\binom{i}{r}\binom{i}{k}\frac{(\tau)^k}{(b-a)^i}M_{i-k}(x;\alpha)$$

Where $M_{i-k}(x;\alpha)=\begin{cases}0 & if \quad i-k\in\{0,1,\dots,\lceil\alpha\rceil-1\}\\ \frac{\Gamma(i-k+1)}{\Gamma(i-k+1-\alpha)}(x-a)^{i-k-\alpha} & if \quad i-k\in\mathbb{N}\ and\ i-k\geq\lceil\alpha\rceil\end{cases}$

Second way of prove: As k' moves from i to 0, k moves from 0 to i, therefore this may be restated by replacing $k'=i-k,\rightarrow k=i-k'$.

$$B_{r,N}(x-\tau)=\sum_{i=r}^{N}\sum_{k'=0}^{i}(-1)^{i+i-k'-r}\binom{N}{i}\binom{i}{r}\binom{i}{i-k'}\frac{(\tau)^{i-k'}}{(b-a)^i}(x-a)^{k'}$$

Perhaps we could revise

$$B_{r,N}(x-\tau)=\sum_{i=r}^{N}\sum_{k=0}^{i}(-1)^{2i-k-r}\binom{N}{i}\binom{i}{r}\binom{i}{i-k}\frac{(\tau)^{i-k}}{(b-a)^i}(x-a)^{k} \qquad (2.23)$$

We obtain the following by taking the Caputo fractional derivative on both sides of equation (2.23):

$${}_a^C D_x^\alpha\, B_{r,N}(x-\tau)=\sum_{i=r}^{N}\sum_{k=0}^{i}(-1)^{2i-k-r}\binom{N}{i}\binom{i}{r}\binom{i}{i-k}\frac{(\tau)^{i-k}}{(b-a)^i}\,{}_a^C D_x^\alpha\,(x-a)^{k}$$

Thus,

$${}_a^C D_x^\alpha B_{r,N}(x-\tau)=\sum_{i=r}^{N}\sum_{k=0}^{i}(-1)^{2i-k-r}\binom{N}{i}\binom{i}{r}\binom{i}{i-k}\frac{(\tau)^{i-k}}{(b-a)^i}M_k(x;\alpha)$$

Where $M_k(x;\alpha)=\begin{cases}0 & if \quad k\in\{0,1,\dots,\lceil\alpha\rceil-1\}\\ \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}(x-a)^{k-\alpha} & if \quad k\in\mathbb{N}\ and\ k\geq\lceil\alpha\rceil\end{cases}$

which completes the proof.

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2.2. The Clenshaw-Curtis Quadrature Formula:

This subsection defines the R-Cleanshaw-Curtis quadrature rule, which is based on the extreme Chebyshev zeros in the closed bounded interval $[a, b]$. The following relation will be used to demonstrate the expansion of the integrand using Chebyshev polynomials (see [36],[37]):

$$I = \int_a^b g(t)dt \cong \frac{b-a}{2} \sum_{d=0}^{R}{}'' \quad W_d^{(R)} \; g\left(t_d^{(R)}\right), \quad for\ all\ d = 0,1,..,R. \qquad (2.24)$$

The terms with the prefixes $d = 0$ and $d = R$ are to be halved, as indicated by the double prime sign $\sum''$ on the summation. The following is the definition of t_i, which are $R-$shifted Chebyshev collocation points:

$$t_d^{(R)} = \frac{b-a}{2}\xi_d^{(R)} + \frac{b+a}{2}, \qquad \xi_d^{(R)} = \cos(d\frac{\pi}{R})$$

also,

$$W_d^{(R)} = \frac{4}{R} \sum_{q=0}^{Q}{}'' \quad v_q \; \cos\left(\frac{qd\pi}{Q}\right), \quad and \quad v_q = \begin{cases} \frac{1}{1-q^2} & if \quad q-even \\ 0 & if \quad q-odd \end{cases}, \quad Q \in \mathbb{Z}^+, R \in \mathbb{Z}^+.$$

3.PROPOSED METHOD

In this section, we attempt to use generalized Bernstein polynomials to solve Fredholm integro-fractional differential equations of constant delay types with variable coefficients(FIFDEs-Delays). First, the form is the ideal method for estimating the solution of equation (1.1) under the boundary conditions (1.2) for this purpose.

$$y(x) \cong y_N(x) = \sum_{r=0}^{N} C_r \; B_{r,N}(x) \qquad (3.1)$$

For every r, the coordinate C_r are unknown constant coefficients, while the coordinate functions $B_{r,N}(x)$ are Bernstein polynomials on any closed bounded interval $[a, b]$.

Equation (3.1) and using historical functions in (1.2) with a fundamental understanding of the delays definition, then for any delays constant $\tau_* > 0$ obtain the following

$$y(x-\tau_*) \cong y_N(x-\tau_*) = \begin{cases} \varphi(x-\tau_*) & if \quad x-\tau_* \le a \\ \sum_{r=0}^{N} C_r \; B_{r,N}(x-\tau_*) & if \quad x-\tau_* > a \end{cases} \qquad (3.2)$$

Solve Fredholm integral part in equation (1.1) for any fixed points x and applying the Cleanshow-Curtis quadrature formula (2.24). yields

$${}_a^C D_x^{\beta} y(x) + \sum_{\ell=1}^{n} p_\ell(x) \; {}_a^C D_x^{\alpha_\ell} y(x-\tau_\ell) + p_0(x) y(x-\tau_0)$$

$$= f(x) + \sum_{j=0}^{m} \lambda_j \left[\frac{b-a}{2} \sum_{d=0}^{R}{}'' \quad k_j\left(x, t_d^{(R)}\right) y\left(t_d^{(R)} - \tau_j^*\right) \right]$$

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Once all y_N and y have been substituted as in equations (3.1 and 3.2), respectively, we obtain:

$$\sum_{r=0}^{N} C_r \, {}_a^C D_x^{\beta} B_{r,N}(x) + \sum_{\ell=1}^{n} p_\ell(x) \, {}_a^C D_x^{\alpha_\ell} * \begin{Bmatrix} \varphi(x-\tau_\ell) & if \quad x-\tau_\ell \le a \\ \sum_{r=0}^{N} C_r \, B_{r,N}(x-\tau_\ell) & if \quad x-\tau_\ell > a \end{Bmatrix}$$

$$+p_0(x) * \begin{Bmatrix} \varphi(x-\tau_0) & if \quad x-\tau_0 \le a \\ \sum_{r=0}^{N} C_r \, B_{r,N}(x-\tau_0) & if \quad x-\tau_0 > a \end{Bmatrix}$$

$$= f(x) + \frac{b-a}{2} \sum_{j=0}^{m} \lambda_j \left[\sum_{d=0}^{R}{}'' \quad k_j\left(x, t_d^{(R)}\right) * \begin{Bmatrix} \varphi\left(t_d^{(R)} - \tau_j^*\right) & if \quad t_d^{(R)} - \tau_j^* \le a \\ \sum_{r=0}^{N} C_r \, B_{r,N}\left(t_d^{(R)} - \tau_j^*\right) & if \quad t_d^{(R)} - \tau_j^* > a \end{Bmatrix} \right]$$

$$+ R_N(x; \bar{C}) \qquad (3.3)$$

Where $R_N(x;\bar{C})$ is error reminder function at any fixed points x. After applying the lemmas (2.5 and 2.7) to above equation with equation (2.22) we obtain:

$$\sum_{r=0}^{N} C_r \left\{ \sum_{z=\lceil\beta\rceil}^{N} \frac{(-1)^{z-r}}{(b-a)^z} \binom{N}{z} \binom{z}{r} S_z(x;\beta) \right\} + \sum_{\ell=1}^{n} p_\ell(x)$$

$$* \begin{Bmatrix} {}_a^C D_x^{\alpha_\ell} \varphi(x-\tau_\ell) & if \quad x-\tau_\ell \le a \\ \sum_{r=0}^{N} C_r \sum_{i=r}^{N} \sum_{k=0}^{i} \frac{(-1)^{2i-k-r} (\tau_\ell)^{i-k}}{(b-a)^i} \binom{N}{i} \binom{i}{r} \binom{i}{i-k} M_k(x;\alpha_\ell) & if \quad x-\tau_\ell > a \end{Bmatrix}$$

$$+p_0(x) * \begin{Bmatrix} \varphi(x-\tau_0) & if \quad x-\tau_0 \le a \\ \sum_{r=0}^{N} C_r \sum_{i=r}^{N} \sum_{k=0}^{i} \frac{(-1)^{i+k-r} (\tau_0)^k}{(b-a)^i} \binom{N}{i} \binom{i}{r} \binom{i}{k} (x-a)^{i-k} & if \quad x-\tau_0 > a \end{Bmatrix}$$

$$= f(x)$$

$$+ \frac{b-a}{2} \sum_{j=0}^{m} \lambda_j \left[\sum_{d=0}^{R}{}'' \quad k_j\left(x, t_d^{(R)}\right) \right.$$

$$\left. * \begin{Bmatrix} \varphi\left(t_d^{(R)} - \tau_j^*\right) & if \quad t_d^{(R)} - \tau_j^* \le a \\ \sum_{r=0}^{N} C_r \sum_{i=r}^{N} \sum_{k=0}^{i} \frac{(-1)^{i+k-r}}{(b-a)^i} \binom{N}{i} \binom{i}{r} \binom{i}{k} (\tau_j^*)^k (t_d^{(R)} - a)^{i-k} & if \quad t_d^{(R)} - \tau_j^* > a \end{Bmatrix} \right]$$

$$+ R_N(x;\bar{C}) \qquad (3.4)$$

The error function involved, $R_N(x;\bar{C})$ is dependent on the point x and the selection of the constant coefficients $\bar{C} = [C_0, C_1, \dots, C_N]$. Now the equation (3.4) can be written as:

$$R_N(x;\bar{C}) = \sum_{r=0}^{N} C_r\, \psi_r(x) - F(x) \qquad (3.5)$$

Where $\psi_r(x)$ and $F(x)$ are defined for fixed any point $x \in [a,b]$, here we have $n+1$ constant time delays $\tau = \tau_0, \tau_1, \dots, \tau_n$ or we have $m+1$ constant time delays $\tau =, \tau_0^*, \tau_1^*, \dots, \tau_m^*$ respectively. Thus, we have $n+2$ basic parts which construct $\psi_r(x)$ and $F(x)$ as follows:

$$\psi_r(x) = \begin{cases} \psi_r^{I_0}(x) & if\ all\ \tau\ 's\ satisfies\ (x-\tau) \le a \\ \psi_r^{I_1}(x) & if\ one\ \tau\ 's\ satisfies\ (x-\tau) > a\ and\ other\ satisfies\ \le a \\ \psi_r^{I_2}(x) & if\ tow\ \tau\ 's\ satisfies\ (x-\tau) > a\ and\ other\ satisfies\ \le a \\ \vdots & \\ \psi_r^{I_n}(x) & if\ n-\tau\ 's\ satisfies\ (x-\tau) > a\ and\ other\ satisfies\ \le a \\ \psi_r^{I_{n+1}}(x) & if\ all\ \tau\ 's\ satisfies\ (x-\tau) > a \end{cases} \qquad (3.6)$$

and

$$F(x) = \begin{cases} F^{I_0}(x) & if\ all\ \tau\ 's\ satisfies\ (x-\tau) \le a \\ F^{I_1}(x) & if\ one\ \tau\ 's\ satisfies\ (x-\tau) > a\ and\ other\ satisfies\ \le a \\ F^{2}(x) & if\ tow\ \tau\ 's\ satisfies\ (x-\tau) > a\ and\ other\ satisfies\ \le a \\ \vdots & \\ F^{I_n}(x) & if\ n-\tau\ 's\ satisfies\ (x-\tau) > a\ and\ other\ satisfies\ \le a \\ F^{I_{n+1}}(x) & if\ all\ \tau\ 's\ satisfies\ (x-\tau) > a \end{cases} \qquad (3.7)$$

On the other hand, let $\mathcal{L}$ be any non negative integer number which takes the values $\{0,1,\dots,n-1\}$ and $\{v_0, v_1, \dots, v_{\mathcal{L}}\}$ also takes the values from the positive integer set $\{0,1,\dots,n\}$. while, the set $\{d_j\}_{j=0}^{m}$ takes all values starting from 0 to R that the points $\left\{t_{d_j}^{(R)}\right\}_{j=0}^{m}$ satisfies the inequality $\wedge_{j=0}^{m}\left\{\left(t_{d_j}^{(R)} - \tau_j^*\right) \le a\right\}$, and also $\{d_j^*\}_{j=0}^{m}$ takes the all values starting from 0 to R that the points $\left\{t_{d_j^*}^{(R)}\right\}_{j=0}^{m}$ satisfies the inequality $\wedge_{j=0}^{m}\left\{\left(t_{d_j^*}^{(R)} - \tau_j^*\right) > a\right\}$. We observe that the following symbols indicate

$$\wedge_{j=0}^{m}\left\{(t_{d_j}^{(R)} - \tau_j^*) \le a\right\} = \left\{\left(t_{d_0}^{(R)} - \tau_0^*\right) \le a \wedge \left(t_{d_1}^{(R)} - \tau_1^*\right) \le a \wedge (t_{d_2}^{(R)} - \tau_2^*) \le a \wedge \dots \wedge (t_{d_m}^{(R)} - \tau_m^*) \le a\right\}$$

as well,

$$\wedge_{j=0}^{m}\left\{\left(t_{d_j^*}^{(R)} - \tau_j^*\right) > a\right\} = \left\{(t_{d_0^*}^{(R)} - \tau_0^*) > a \wedge (t_{d_1^*}^{(R)} - \tau_1^*) > a \wedge (t_{d_2^*}^{(R)} - \tau_2^*) > a \wedge \dots \wedge (t_{d_m^*}^{(R)} - \tau_m^*) > a\right\}.$$

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Additionally, define δ as follows:

$$\delta(\theta) = \begin{cases} \frac{1}{2} & if \quad \theta = 0 \; or \; R \\ 1 & if \quad o.w. \end{cases}.$$

Consequently, if all of constant delay time (τ) is $\leq a$. That is

$$\psi_r^{I_0}(x) = \sum_{z=\lceil\beta\rceil}^{N} \frac{(-1)^{z-r}}{(b-a)^z} \binom{N}{z}\binom{z}{r} S_z(x;\beta) - \left[\frac{b-a}{2}\sum_{j=0}^{m}\lambda_j \sum_{d_j^*}\delta(d_j^*) W_{d_j^*}^{(R)} k_j\left(x, t_{d_j^*}^{(R)}\right) * \sum_{i=r}^{N}\sum_{k=0}^{i}(-1)^{i+k-r}\binom{N}{i}\binom{i}{r}\binom{i}{k}\frac{(\tau_j^*)^k}{(b-a)^i}\left(t_{d_j^*}^{(R)} - a\right)^{i-k}\right] \quad (3.8)$$

with

$$F^{I_0}(x) = f(x) + \frac{b-a}{2}\sum_{j=0}^{m}\lambda_j\sum_{d_j}\delta(d_j) W_{d_j}^{(R)} k_j\left(x, t_{d_j}^{(R)}\right)\varphi\left(t_{d_j}^{(R)} - \tau_j^*\right) - p_0(x)\varphi(x-\tau_0) - \sum_{\ell=1}^{n} p_\ell(x)\, {}_a^C D_x^{\alpha_\ell}\varphi(x-\tau_\ell) \quad (3.9)$$

and

$$\psi_r^{I_{\mathcal{L}+1}^{[v_0,v_1,\ldots,v_{\mathcal{L}}]}}(x) = \sum_{z=\lceil\beta\rceil}^{N} \frac{(-1)^{z-r}}{(b-a)^z} \binom{N}{z}\binom{z}{r} S_z(x;\beta) - \left[\frac{b-a}{2}\sum_{j=0}^{m}\lambda_j \sum_{d_j^*}\delta(d_j^*) W_{d_j^*}^{(R)} k_j\left(x, t_{d_j^*}^{(R)}\right) * \sum_{i=r}^{N}\sum_{k=0}^{i}(-1)^{i+k-r}\binom{N}{i}\binom{i}{r}\binom{i}{k}\frac{(\tau_j^*)^k}{(b-a)^i}\left(t_{d_j^*}^{(R)} - a\right)^{i-k}\right]$$

$$+ \begin{cases} if \quad v_0 = 0; & \left[\begin{array}{l} p_0(x)\sum_{i=r}^{N}\sum_{k=0}^{i}(-1)^{i+k-r}\binom{N}{i}\binom{i}{r}\binom{i}{k}\frac{(\tau_0)^k}{(b-a)^i}(x-a)^{i-k} \\ + \sum_{\ell=\{v_1,v_2\ldots,v_{\mathcal{L}}\}} p_\ell(x)\sum_{i=r}^{N}\sum_{k=0}^{i}(-1)^{2i-k-r}\binom{N}{i}\binom{i}{r}\binom{i}{i-k}\frac{(\tau_\ell)^{i-k}}{(b-a)^i}M_k(x;\alpha_\ell) \end{array}\right] \\ if \quad v_0 \neq 0; & \sum_{\ell=\{v_0,v_1\ldots,v_{\mathcal{L}}\}} p_\ell(x)\sum_{i=r}^{N}\sum_{k=0}^{i}(-1)^{2i-k-r}\binom{N}{i}\binom{i}{r}\binom{i}{i-k}\frac{(\tau_\ell)^{i-k}}{(b-a)^i}M_k(x;\alpha_\ell) \end{cases} \quad (3.10)$$

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with

$$F^{I^{[v_0,v_1,\dots,v_{\mathcal{L}}]}_{\mathcal{L}+1}}(x) = f(x) + \frac{b-a}{2}\sum_{j=0}^{m}\lambda_j \sum_{d_j}\delta(d_j)W_{d_j}^{(R)}\,k_j\left(x,t_{d_j}^{(R)}\right)\,\varphi\left(t_{d_j}^{(R)}-\tau_j^*\right)$$

$$+\begin{cases} if\ \ v_0=0; & -\sum_{\ell=1,\ell\neq\{v_1,v_2\dots,v_{\mathcal{L}}\}}^{n} p_\ell(x)\,{}_a^C D_x^{\alpha_\ell}\varphi(x-\tau_\ell) \\ if\ \ v_0\neq 0; & -p_0(x)\varphi(x-\tau_0)-\sum_{\ell=1,\ell\neq\{v_0,v_1,v_2,\dots,v_{\mathcal{L}}\}}^{n} p_\ell(x)\,{}_a^C D_x^{\alpha_\ell}\varphi(x-\tau_\ell) \end{cases} \qquad (3.11)$$

and if all of constant delay time (τ) is $>\ a$. That is

$$\psi_r^{I_{n+1}}(x) = \sum_{z=\lceil\beta\rceil}^{N}\frac{(-1)^{z-r}}{(b-a)^z}\binom{N}{z}\binom{z}{r}\,S_z(x;\beta)$$

$$-\left[\frac{b-a}{2}\sum_{j=0}^{m}\lambda_j\sum_{d_j^*}\delta(d_j^*)W_{d_j^*}^{(R)}\,k_j\left(x,t_{d_j^*}^{(R)}\right)\right.$$

$$\left.*\sum_{i=r}^{N}\sum_{k=0}^{i}(-1)^{i+k-r}\binom{N}{i}\binom{i}{r}\binom{i}{k}\frac{(\tau_j^*)^k}{(b-a)^i}\left(t_{d_j^*}^{(R)}-a\right)^{i-k}\right]$$

$$+\sum_{\ell=1}^{n}p_\ell(x)\sum_{i=r}^{N}\sum_{k=0}^{i}(-1)^{2i-k-r}\binom{N}{i}\binom{i}{r}\binom{i}{i-k}\frac{(\tau_\ell)^{i-k}}{(b-a)^i}M_k(x;\alpha_\ell)$$

$$+p_0(x)\sum_{i=r}^{N}\sum_{k=0}^{i}(-1)^{i+k-r}\binom{N}{i}\binom{i}{r}\binom{i}{k}\frac{(\tau_0)^k}{(b-a)^i}(x-a)^{i-k} \qquad (3.12)$$

with

$$F^{I_{n+1}}(x) = f(x) + \frac{b-a}{2}\sum_{j=0}^{m}\lambda_j\sum_{d_j}\delta(d_j)W_{d_j}^{(R)}\,k_j\left(x,t_{d_j}^{(R)}\right)\,\varphi\left(t_{d_j}^{(R)}-\tau_j^*\right) \qquad (3.13)$$

The main points here are how to find the coefficients $C_r\ (r\ =\ 0,1,\dots,N)$ of $y_N(x)$ in equation (3.1), such that coefficients $C_r\ (r\ =\ 0,1,\dots,N)$ of) so $y_N(x)$ that $R_N(x;\ \bar{C})$ is the Residual Sum of Squares (RSS) is minimized. We now wish to minimize of residual, i.e, make $R_N(x;\ \bar{C})$ vanishes as possible as at selecting points, say $x\ =\ x_p$. From equation (3.5) we get :

$$R_N(x;\ \bar{C}) = \left\|\sum_{r=0}^{N}C_r\,\psi_r\,(x) - F(x)\right\|_2$$

by using Newten-cotes point [38], let $x\ =\ x_p$ where $x_p = \frac{2p-1}{2(\bar{N}+1)}, p = 1,2,\dots,\bar{N}+1$ so we get

$$R_N(x;\ \bar{C}) = \sum_{p=1}^{\bar{N}+1}\left(\sum_{r=0}^{N} C_r\ \psi_r\left(x_p\right) - F\left(x_p\right)\right)^2 \qquad (3.14)$$

After some simple manipulation, rewrite the equation (3.14) in the matrix form as :

$$\Phi C = F \qquad (3.15)$$

where

$$\Phi = \begin{bmatrix} \psi_0(x_1) & \psi_1(x_1) & \psi_2(x_1) & \cdots & \psi_N(x_1) \\ \psi_0(x_2) & \psi_1(x_2) & \psi_2(x_2) & \cdots & \psi_N(x_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_0(x_{\bar{N}+1}) & \psi_1(x_{\bar{N}+1}) & \psi_2(x_{\bar{N}+1}) & \cdots & \psi_N(x_{\bar{N}+1}) \end{bmatrix}_{(\bar{N}+1)\times(N+1)}, C = \begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_N \end{bmatrix}_{(N+1)\times 1},$$

$$F = \begin{bmatrix} F(x_1) \\ F(x_2) \\ \vdots \\ F(x_{\bar{N}+1}) \end{bmatrix}_{(\bar{N}+1)\times 1}$$

According to the boundary conditions, they can be expressed as simply as follows:

$$\sum_{\rho=0}^{\mu-1}\left\{g_{k\rho}y^{(\rho)}(a) + h_{k\rho}y^{(\rho)}(b)\right\} = \vartheta_k\ ;k = 0,1,\dots,\mu-1 \qquad (3.16)$$

by using lemma 2.6 can be substituted into equation (3.16) to produce the following expression:

$$\sum_{r=0}^{N} C_r\ [\Omega_{kr}] = \vartheta_k \qquad for\ each\ \ k = 0,1,\dots,\ \mu-1 \qquad (3.17)$$

where

$$\Omega_{kr} = \sum_{\rho=0}^{\mu-1}\left\{g_{k\rho}\frac{d^\rho}{dx^\rho}\ B_{r,N}(x)\Big|_{x=a} + h_{k\rho}\frac{d^\rho}{dx^\rho}\ B_{r,N}(x)\Big|_{x=b}\right\} \qquad (3.18)$$

in general, the matrix conditions form became :

$$\bar{\Omega}_{\mu-1} = \begin{bmatrix} \Omega_{00} & \Omega_{01} & \cdots & \Omega_{0N} \\ \Omega_{10} & \Omega_{11} & \cdots & \Omega_{1N} \\ \vdots & \vdots & \vdots & \vdots \\ \Omega_{(\mu-1)0} & \Omega_{(\mu-1)1} & \cdots & \Omega_{(\mu-1)N} \end{bmatrix}_{(\mu-1)\times(N+1)}, C = \begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_N \end{bmatrix}_{(N+1)\times 1},$$

$$V = \begin{bmatrix} \vartheta_0 \\ \vartheta_1 \\ \vdots \\ \vartheta_{(\mu-1)} \end{bmatrix}_{(\mu-1)\times 1}$$

where

$$\Omega_{kr} = \sum_{\rho=0}^{\mu-1}\left\{g_{k\rho}\frac{d^\rho}{dx^\rho}\ B_{r,N}(x)\Big|_{x=a} + h_{k\rho}\frac{d^\rho}{dx^\rho}\ B_{r,N}(x)\Big|_{x=b}\right\} \qquad (3.19)$$

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In matrix form, this gives

$$\bar{\Omega}_{\mu-1}\, C = V \qquad (3.20)$$

The result of adding the last rows (3.20) to (3.15) is a new matrix:

$$GC = W \qquad (3.21)$$

where

$$G = \begin{bmatrix} \Phi \\ \cdots \\ \bar{\Omega}_{\mu-1} \end{bmatrix} \qquad W = \begin{bmatrix} F \\ \cdots \\ V \end{bmatrix}$$

The constant coefficients C_r in equation (3.21) can be found by storing the matrix G, computing $G^T G$ and $G^T W$, and then using the LU-factorization approach or any numerical computations methods to solve: $[G^T G\,]C = [G^T W\,]$.

The values C_r 's in equation (3.1) are thus substituted to obtain the approximate solution for higher order fractional linear FIFDEs with constant multi-time Retarded Delay with variable coefficients equation (1.1).

The Algorithm (ADBM):

We summarize the procedures of the proposed Bernstein collocation method for approximate the solution of FIFDEs-Delays multi-time in the following stages:

Step 1. Set $\mu = \max\{\omega^{\beta}, \omega_{\ell}^{\alpha} : \ell = \overline{1:n}\}$ where $\omega^{\beta} = \lceil \beta \rceil$, $\omega_{\ell}^{\alpha} = \lceil \alpha_{\ell} \rceil$.

Step 2. Set the Newten-cotes points, $x_p = \frac{2p-1}{2(\bar{N}+1)}$, $where\ \ p = 1,2,\ldots,\bar{N}+1$.

Step 3. For each $x = x_p$:

a. For the Bernstein polynomial at fixed point, evaluate $\psi_r(x)$ by going over all the fundamental components as in equation (3.6), and then use equations (3.8, 3.10, and 3.12) for cases where all $r = 0,1,\ldots,N$.

b. After going over all the fundamental components in equation (3.7), compute $F(x)$ for each case using equations (3.9, 3.11, and 3.13).

Step 4. Equation 3.15 is used to construct the matrices Φ and F.

Step 5. Using equation (3.20), the boundary condition matrices $\bar{\Omega}_{\mu-1}$ and V are constructed.

Step 6. The matrix G and W, which describe the linear system (3.21), are constructed using steps (4 and 5).

Step 7. In order to determine constant coefficients $C_r (r = 0,1,\ldots,N)$, the LU-factorization technique is applied to construct the system in step 5 after multiplying both sides by G^T, as defined in equation (3.21).

Step 8. The approximate solution $y_N(x)$ of $y(x)$ can be obtained by substituting C_r's in equation (3.1), with $B_{r,N}\ (x)$.

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4. TESTING EXAMPLES

The numerical section uses the L_2 error norm to verify the correctness and efficacy of the proposed schemes. The suggested algorithm **ADBM** produces numerical results that are compared, and a Python program is used to generate both the numerical and graphical results.

Test Example 4.1. Consider a linear FIFDEs-Delays constant multi-time delay containing various fractional orders on the closed bounded interval $[0,1]$ with variable coefficients:

$$ {}_0^C D_x^{0.9} y(x) - \frac{1}{3} {}_0^C D_x^{0.6} y(x-0.1) + \cos(x)\, y(x-0.43) = f(x) $$
$$ + \int_0^1 \left[6(xt) y(t-0.5) + (x^2+t^2)\, y(t-0.62) - e^{-(x+t)} y(t-0.3) \right] dt \qquad (4.1) $$

Where

$$ f(x) = 2\frac{\Gamma(2)}{\Gamma(1.1)} x^{0.1} - \frac{\Gamma(4)}{\Gamma(3.1)} x^{2.1} + \frac{1}{3}\frac{\Gamma(4)}{\Gamma(3.4)} x^{2.4} - \frac{0.3}{3}\frac{\Gamma(3)}{\Gamma(2.4)} x^{1.4} - \frac{1.97}{3}\frac{\Gamma(2)}{\Gamma(1.4)} x^{0.4} - \cos(x)\, x^3 + $$
$$ \frac{129}{100}\cos(x)\, x^2 - \frac{49483}{62500} x^2 + \frac{14453}{10000} cox(x)x - \frac{157}{40} x + \frac{219507}{1000000}\cos(x) - \frac{104119}{250000} - \frac{2043}{1000} e^{-x} + $$
$$ \frac{7613}{1000} e^{-(x+1)} $$

with boundary condition $1y(0) - 0y(1) = 1$ and the historical function $\varphi(x) = 1 + 2x - x^3$. While the exact solution is $y(x) = 1 + x(2 - x^2)$.

Here, based on the given example:

Assume the approximate solution has the following form, while the numerical approximation numbers in the approach and general Clenshaw-Curtis formula, respectively, are $N = 3$ and $R = 18$:

$$ y(x) \cong y_3(x) = \sum_{r=0}^{3} C_r B_{r,3}(x) \qquad for\ all\ r = 0,1,2,3 $$

Then the Newten-cotes points x_p, such that $x_p = \frac{2p-1}{2(\bar{N}+1)}, p = 1,2,\dots,\bar{N}+1$ where here take $\bar{N} = 10$ so:

$x_1 = 0.045454545455$, $x_2 = 0.136363636364$, $x_3 0.227272727273$, $x_4 = 0.318181818182$,
$x_5 = 0.409090909091$, $x_6 = 0.5$, $x_7 = 0.590909090909$, $x_8 = 0.681818181818$,
$x_9 = 0.772727272727$, $x_{10} = 0.863636363636$, $x_{11} = 0.954545454545$

The fundamental matrix for equation (3.15) given the following manner :

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$$\Phi = \begin{bmatrix} -2.1356657438950 & 1.9322320513566 & 0.20043285036204 & 0.020263839863690 \\ -1.5723482025663 & 0.81358765858055 & 0.49892116512337 & 0.047053536120602 \\ -1.3588492175817 & 0.16863598860945 & 0.63106260253410 & 0.11294704784978 \\ -1.1705702605709 & -0.37801122294281 & 0.64971447171118 & 0.21563759204135 \\ -1.0169073310821 & -0.81808545627682 & 0.55618975709560 & 0.35472124126400 \\ -0.19517586762326 & -0.98982012776041 & 0.36345220950811 & 0.53016621950332 \\ -0.33376483780295 & -1.0885842960374 & 0.090513866832708 & 0.74422327191852 \\ -0.46221583145738 & -1.1547349187714 & -0.27765137473397 & 0.99952296508770 \\ -0.58705057578179 & -1.1683912017417 & -0.75553500885278 & 1.2974919168082 \\ -0.71527028673478 & -1.1104287723212 & -1.3548629517277 & 1.6380932302396 \\ -0.85402231411338 & -0.96349998528176 & -2.0837155374467 & 2.0196281725443 \end{bmatrix}_{11\times4}$$

$$F = \begin{bmatrix} 1.59249171228282 \\ 1.04136315846234 \\ 0.619967420820720 \\ 0.145976142753862 \\ -0.373969972201051 \\ 0.0626098659853248 \\ -0.449425444917605 \\ -1.03667881993776 \\ -1.70349753627017 \\ -2.45244308143859 \\ -3.28400160623916 \end{bmatrix}_{11\times1}$$, with Bernstein parameter terms: $C = \begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{bmatrix}_{4\times1}$

For the boundary conditions the fundamental matrices for equation (3.20) we get:

$$\bar{\Omega}_{\mu-1} = [1 \quad 0 \quad 0 \quad 0], \text{ and } V = [1].$$

solving the system above by any numerical methods (here using LU-factorization procedure), by procedure that $[G^T G; G^T W]$, the approximate solutions $y_3(x)$ are obtained.

$[G^T G; G^T W] =$

$$\begin{bmatrix} 14.232883626545 & -0.9673211364400 & -0.1768267194844 & -5.3559237889929 & ; & 1.50432 \\ -0.96732113644008 & 12.260734641454 & 4.4562372347466 & -8.0452009048009 & ; & 13.7817 \\ -0.17682671948448 & 4.4562372347466 & 8.3845493686193 & -6.9892799268755 & ; & 12.8394 \\ -5.3559237889929 & -8.0452009048009 & -6.9892799268755 & 10.467431741170 & ; & -14.147 \end{bmatrix}$$

Thus we obtaining the Bernstein parameters $C_r = [C_0, C_1, C_2, C_3]$, see table(1).

Table 1. Bernstein parameters for $(y_3(x))$ while $N = 3$ and $R = 18$

C_r	$R = 18$
C_0	1.000003476961
C_1	1.666532788504
C_2	2.332648626121
C_3	1.998546923461

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After putting all C_r's in the Bernstein approximation equation (3.1) for applying the Clenshaw-Curtis quadrature formula (2.24) for $R = 18$ we obtain:

$$y_3^{(R=18)}(x) \cong 1.000003476961(1-x)^3 + 1.666532788504(3x)(1-x)^2 + 2.332648626121(3x^2)(1-x) + 1.998546923461x^3$$

Using the same step as before, we can get approximation solution to the problem for $N = 3$, $\bar{N} = 10$ with $R = 19$ and $R = 20$, respectively. After that, we run the general python program created for this purpose:

Table 2. Bernstein parameters for $(y_3(x))$ while $N = 3$ and $R = 19, 20$ respectively

C_r	$R = 19$	$R = 20$
C_0	1.000000136263	0.999999998725
C_1	1 .666668704122	1.666666679188
C_2	2.333350474952	2.333333361985
C_3	2.000040353488	2.000000040551

So, the approximate solution for $R = 19$ become:

$$y_3^{(R=19)}(x) \cong 1.000000136263(1-x)^3 + 1.666668704122(3x)(1-x)^2 + 2.333350474952(3x^2)(1-x) + 2.000040353488x^3$$

The approximate solution for $R = 20$ become:

$$y_3^{(R=20)}(x) \cong 0.999999998725(1-x)^3 + 1.666666679188(3x)(1-x)^2 + 2.333333361985(3x^2)(1-x) + 2.000000040551x^3$$

Table (3) compares the exact solution $y(x)$ and the approximate solution $y_3(x)$. It also displays the running times, least square errors, and residual equation values $R_3(x;\bar{C})$. The formula (3.5) is applied for three distinct values $R = (18, 19, 20)$ respectively.

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Table 3. A comparison between the exact solution and the approximate solution

		The approximate solutions $y_3(x)$ with respective to integral parts R		
x	Exact	$R = 18$	$R = 19$	$R = 20$
0	1	1.00000347696	1.00000013626	0.999999998725
0.1	1.199	1.19895006214	1.19900109761	1.19900000293
0.2	1.392	1.39187301449	1.39200282057	1.39200000723
0.3	1.573	1.5727735096	1.57300527457	1.57300001159
0.4	1.736	1.73565272308	1.73600842902	1.73600001598
0.5	1.875	1.87451183054	1.87501225337	1.87500002035
0.6	1.984	1.98335200757	1.98401671704	1.98400002466
0.7	2.057	2.05617442977	2.05702178946	2.05700002888
0.8	2.088	2.08698027275	2.08802744005	2.08800003296
0.9	2.071	2.06977071212	2.07103363825	2.07100003686
1	2	1.99854692346	2.00004035349	2.00000004055
$L.S.E$		$6.192723962682916 \times 10^{-6}$	$4.525326179507544 \times 10^{-9}$	$6.3977047977615004 \times 10^{-15}$
$R_3 = L.S.E_y$		$1.28608853760722 \times 10^{-9}$	$6.36107340448893 \times 10^{-13}$	$1.52104183320445 \times 10^{-17}$
$R.Time /Sec$		0.5465397834777832	0.6033868789672852	0.656287431716919

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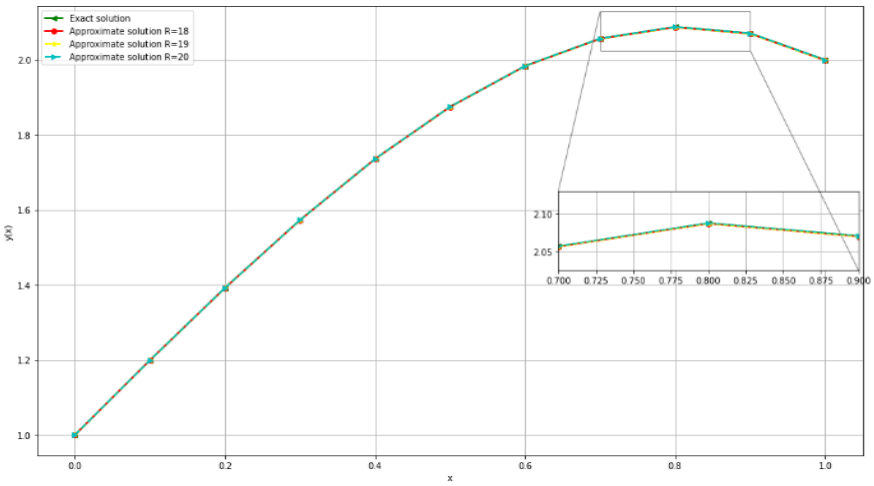

Figure 1. *Exact and approximate solution of Test Example 4.1*

The absolute error values for a numerical method used with various values of $R = (18, 19, 20)$ are displayed in the table(4). The absolute error typically drops as R rises, indicating that the approach gets more precise. This demonstrates the connection between the precision of the findings and the parameter R.

Table 4. Absolute error of the method with respective different input R

		Absolute error		
x	Exact	$R = 18$	$R = 19$	R = 20
0	1	3.47696×10^{-6}	1.36263×10^{-7}	1.275×10^{-9}
0.1	1.199	4.99379×10^{-5}	1.09761×10^{-6}	2.927×10^{-9}
0.2	1.392	1.2698×10^{-4}	2.82057×10^{-6}	7.230×10^{-9}
0.3	1.573	2.264×10^{-4}	5.27457×10^{-6}	1.1594×10^{-8}
0.4	1.736	3.4727×10^{-4}	8.42902×10^{-6}	1.5981×10^{-8}
0.5	1.875	4.8816×10^{-4}	1.22534×10^{-6}	2.0349×10^{-8}
0.6	1.984	6.4799×10^{-4}	1.6717×10^{-5}	2.4661×10^{-8}
0.7	2.057	8.2557×10^{-4}	2.17895×10^{-5}	2.8877×10^{-8}
0.8	2.088	1.01973×10^{-3}	2.74401×10^{-5}	3.2956×10^{-8}
0.9	2.071	1.22929×10^{-3}	3.36383×10^{-5}	3.6861×10^{-8}
1	2	1.45308×10^{-3}	4.0353×10^{-5}	4.0551×10^{-8}

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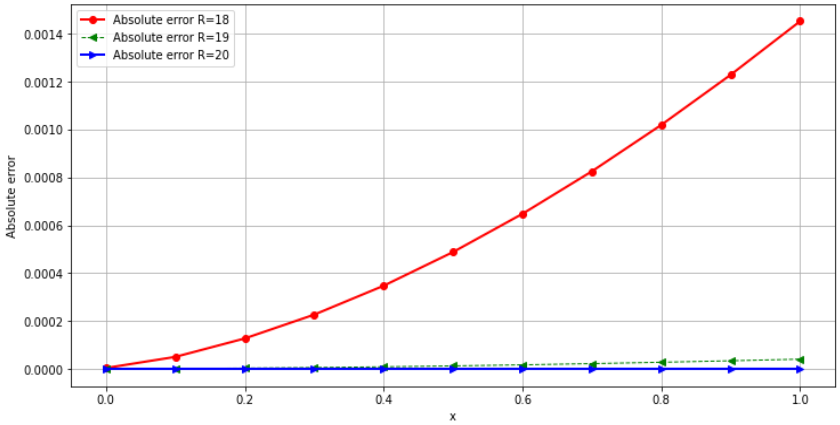

Figure 2. *Absolute error of Test Example 4.1*

Apply the same stages as in the algorithm **ADBM** and running the Python program which are written for this purpose, for $N = 3$ and $N = 6$ and $N = 8$ with $R = 20$ we obtain the Bernstein approximate solution for $y_N(x)$ with respective to N. we listed all values with error in table (5).

Table 5. A comparison between the exact solution and the approximate solution

		The approximate solutions and absolute error for $R = 20$ with respective different input N					
x	Exact	$y_{3(x)}$	absolute error $y_{3(x)}$	$y_{6(x)}$	absolute error $y_{6(x)}$	$y_{8(x)}$	absolute error $y_{8(x)}$
0	1	0.9999999987	1.275×10^{-9}	0.9999999999	7.5×10^{-11}	0.9999999999	7.2×10^{-11}
0.1	1.19	1.1990000029	2.927×10^{-9}	1.1990000040	4.004×10^{-9}	1.1990000039	3.967×10^{-9}
0.2	1.39	1.3920000072	7.23×10^{-9}	1.3920000083	8.391×10^{-9}	1.3920000082	8.246×10^{-9}
0.3	1.57	1.5730000115	1.1594×10^{-8}	1.5730000129	1.2945×10^{-8}	1.5730000126	1.2623×10^{-8}
0.4	1.736	1.7360000159	1.5981×10^{-8}	1.7360000175	1.7566×10^{-8}	1.7360000171	1.7139×10^{-8}
0.5	1.875	1.8750000203	2.0349×10^{-8}	1.8750000221	2.2116×10^{-8}	1.8750000216	2.1642×10^{-8}
0.6	1.984	1.9840000246	2.4661×10^{-8}	1.9840000265	2.6524×10^{-8}	1.9840000259	2.5955×10^{-8}
3.0.7	2.057	2.0570000288	2.8877×10^{-8}	2.0570000307	3.0755×10^{-8}	2.0570000300	3.0053×10^{-8}
0.8	2.088	2.0880000329	3.2956×10^{-8}	2.0880000348	3.4841×10^{-8}	2.0880000340	3.4048×10^{-8}
0.9	2.071	2.0710000368	3.6861×10^{-8}	2.0710000388	3.8895×10^{-8}	2.0710000380	3.8042×10^{-8}

1	2	2.0000000405	4.0551×10^{-8}	2.0000000431	4.3135×10^{-8}	2.0000000421	4.2197×10^{-8}
$L.S.E$		$6.3977047977 \times 10^{-15}$		$7.2886388018 \times 10^{-15}$		$6.9690903926 \times 10^{-15}$	
$R_N = L.S.E_y$		$1.5210418332 \times 10^{-17}$		$1.6959268826 \times 10^{-18}$		$1.6005283543 \times 10^{-19}$	
$R.Time /sec$		0.6562874317		1.7596397399		3.1995050907	

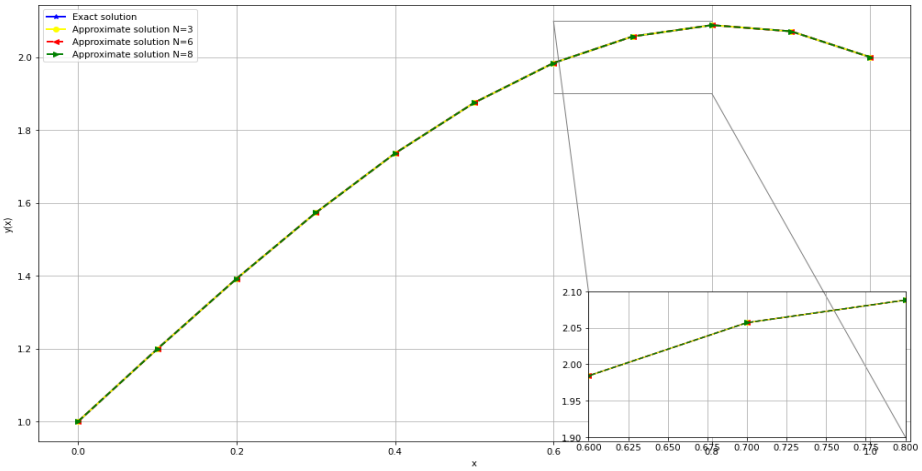

Figure 3. *Exact and approximate solution of Test Example 4.1*

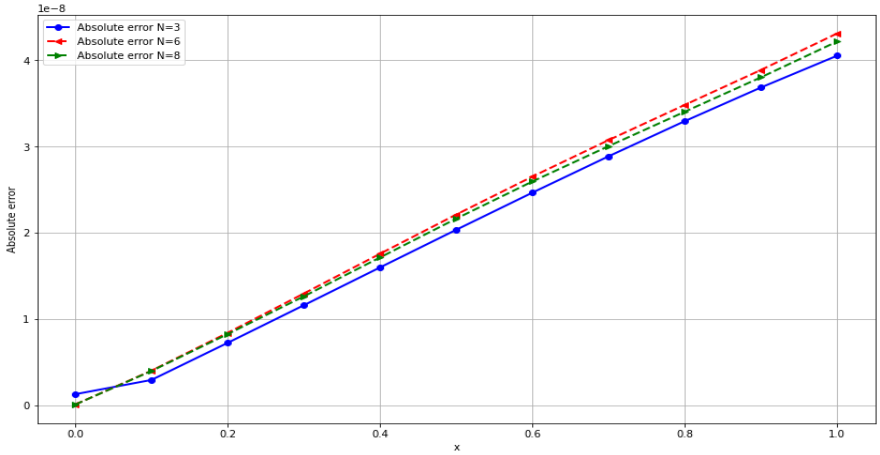

Figure 4. *Absolute error of Test Example 4.1*

Test Example 4.2. Consider the following (FIFDEs-Delays) in which $0 < \alpha, \beta \leq 1$ that is:

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$$ {}_0^C D_x^{0.7} y(x) + \cosh(x)\, {}_0^C D_x^{0.6} y(x-0.1) - \frac{x}{3}\, {}_0^C D_x^{0.5} y(x-0.2) + \frac{1}{5}\, {}_0^C D_x^{0.4} y(x-0.3) + \frac{e^x}{6} y(x-0.5) = f(x) + \int_0^1 (-t\sin(x)) y(t-0.7)dt + \int_0^1 (t+x^2) y(t-0.9)dt \qquad (4.2) $$

Where

$$ f(x) = \frac{\Gamma(5)}{\Gamma(4.3)}x^{3.3} - 2\frac{\Gamma(2)}{\Gamma(1.3)}x^{0.3} + \cos(x)\left[\frac{\Gamma(5)}{\Gamma(4.4)}x^{3.4} - 0.4\frac{\Gamma(4)}{\Gamma(3.4)}x^{2.4} + 0.06\frac{\Gamma(3)}{\Gamma(2.4)}x^{1.4} - 2.004\frac{\Gamma(2)}{\Gamma(1.4)}x^{0.4}\right] - \frac{x}{3}\left[\frac{\Gamma(5)}{\Gamma(4.5)}x^{3.5} - 0.8\frac{\Gamma(4)}{\Gamma(3.5)}x^{2.5} + 0.24\frac{\Gamma(3)}{\Gamma(2.5)}x^{1.5} - 2.032\frac{\Gamma(2)}{\Gamma(1.5)}x^{0.5}\right] + \frac{1}{5}\left[\frac{\Gamma(5)}{\Gamma(4.6)}x^{3.6} - 1.2\frac{\Gamma(4)}{\Gamma(3.6)}x^{2.6} + 0.54\frac{\Gamma(3)}{\Gamma(2.6)}x^{1.6} - 2.108\frac{\Gamma(2)}{\Gamma(1.6)}x^{0.6}\right] + \frac{e^x x^4}{6} - \frac{e^x x^3}{3} + \frac{e^x x^2}{4} - \frac{9181}{10000}x^2 - \frac{5}{12}e^x x + \frac{17}{96}e^x + \frac{94291666666667}{2500000000000000}\sin(x) - \frac{5012}{20000} $$

With boundary conditions $1y(0) + 0y(1) = 0$, while the historical function is $\varphi(x) = x^4 - 2x$. where $y(x) = x(x^3 - 2)$ is the exact solution.

Take $N = 4$ and $\bar{N} = 7$ and $(R = 10, 15, 25)$ respectively . Assume the approximate solution has the following form (numerical approximation number in approach, general Clenshaw-Curtis formula):

$$ y(x) \cong y_4(x) = \sum_{r=0}^{4} C_r B_{r,4}(x) \qquad for\ all\ r = 0,1,\dots,4 $$

After executing a Python program to get the constant coefficients C_r, we can obtain an approximate solution to our problem by using the same procedure as in Test Example 4.1.

Table 6. Bernstein parameters for $(y_4(x))$ while $N = 4$ and $R = 10,15,25$ respectively

C_r	$R = 10$	$R = 15$	$R = 25$
C_0	$-1.525656939160 \times 10^{-5}$	$-5.597377912818 \times 10^{-6}$	$1.123689858556 \times 10^{-16}$
C_1	$-4.998453337048 \times 10^{-1}$	$-4.999340339434 \times 10^{-1}$	$-5.000000000000 \times 10^{-1}$
C_2	-1.000240669159	-1.000104953236	-1.000000000000
C_3	-1.500733498773	-1.500326259683	-1.500000000000
C_4	-1.001679607642	-1.000743519947	-1.00000000000

So the approximate solution for $R = (10, 15, 25)$ respectively we get :

$$ y_4^{(R=10)}(x) = -1.525656939160 \times 10^{-5}(1-x)^4 - 4.998453337048 \times 10^{-1}(4x)(1-x)^3 - 1.000240669159(6x^2)(1-x)^2 - 1.500733498773(4x^3)(1-x) - 1.001679607642\, x^4 $$

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$$y_4^{(R=15)}(x) = -5.597377912818 \times 10^{-6}(1-x)^4 - 4.999340339434 \times 10^{-1}(4x)(1-x)^3 - 1.000104953236(6x^2)(1-x)^2 - 1.500326259683(4x^3)(1-x) - 1.000743519947x^4$$

$$y_4^{(R=25)}(x) = 1.123689858556 \times 10^{-16}(1-x)^4 - 5.0 \times 10^{-1}(4x)(1-x)^3 - 1.0(6x^2)(1-x)^2 - 1.5(4x^3)(1-x) - 1.0x^4$$

Furthermore, after running the programs, table (7) and the approximation expression, the table(7) compares the exact solution $y(x)$ and the approximate solution $y_4(x)$. It also displays the running times, least square errors, and residual equation values $R_4(x;\bar{C})$. The formula (3.5) is applied for three distinct values of R. the following approximate formulas are obtaine

Table 7. A comparison between the exact solution and the approximate solution

		The approximate solutions $y_4(x)$ with respective to integral parts R		
x	Exact	$R=10$	R=15	$R=25$
0	0	$-1.52565693916 \times 10^{-5}$	$-5.59737791282 \times 10^{-6}$	$1.12368985856 \times 10^{-16}$
0.0526316	−0.105255484534	−0.105244096275	−0.105249936675	−0.105255484534
0.105263	−0.210403542023	−0.210382750426	−0.210394273417	−0.210403542023
0.157895	−0.315167931492	−0.315153568813	−0.315161787751	−0.315167931492
0.210526	−0.419088251318	−0.419094892247	−0.419091561363	−0.419088251318
0.263158	−0.521519939227	−0.521561052551	−0.521538577471	−0.521519939227
0.315789	−0.621634272297	−0.621722372566	−0.621673720828	−0.621634272297
0.368421	−0.718418366955	−0.718565166147	−0.718483777722	−0.718418366955
0.421053	−0.810675178981	−0.810891738167	−0.810771435977	−0.810675178981
0.473684	−0.897023503503	−0.897320384513	−0.897155284948	−0.897023503503
0.526316	−0.975897975	−0.97628539209	−0.976069815529	−0.975897975
0.578947	−1.045549067303	−1.04603703882	−1.04576542014	−1.0455490673
0.631579	−1.104043093592	−1.10464159363	−1.10430839276	−1.10404309359
0.684211	−1.149262206398	−1.14998131648	−1.14958092886	−1.1492622064
0.736842	−1.178904397603	−1.17975445833	−1.17928112549	−1.1789043976
0.789474	−1.190483498438	−1.19147526117	−1.1909229812	−1.19048349844
0.842105	−1.181329179488	−1.182473958	−1.1818363961	−1.18132917949
0.894737	−1.148586950683	−1.14989677283	−1.14916717183	−1.14858695068
0.947368	−1.089218161309	−1.0907059207	−1.08987701154	−1.08921816131
1	−1	−1.00167960764	−1.00074351995	−1
	$L.S.E$	$1.1197471434910678 \times 10^{-5}$	$2.1972158044305715 \times 10^{-6}$	$3.085062829263179 \times 10^{-30}$
	$R_4 = L.E.S_y$	$8.22667352497636 \times 10^{-8}$	$1.47764708280376 \times 10^{-8}$	$2.97085518356102 \times 10^{-30}$
	$R.Time/Sec$	0.6163537502288818	0.618492841720581	0.7572336196899414

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The absolute error values for a numerical method used with various values of $R = 10, 15, 25$ are displayed in the table(8). The absolute error typically drops as R rises, indicating that the approach becomes more precise. This demonstrates the connection between the precision of the findings and the parameter R.

Table 8. Absolute error of the method with respective different input R

		Absolute error		
x	Exact	$R = 10$	$R = 15$	$R = 25$
0	0	1.52566×10^{-5}	5.59737×10^{-6}	1.12369×10^{-16}
0.0526316	-0.105255484534	1.13883×10^{-5}	5.54785×10^{-6}	4.21191×10^{-14}
0.105263	-0.210403542023	2.07916×10^{-5}	9.26860×10^{-6}	4.17999×10^{-14}
0.157895	-0.315167931492	1.43627×10^{-5}	6.14374×10^{-6}	3.83582×10^{-14}
0.210526	-0.419088251318	6.64093×10^{-6}	3.31004×10^{-6}	3.88578×10^{-16}
0.263158	-0.521519939227	4.11133×10^{-5}	1.86382×10^{-5}	1.43219×10^{-14}
0.315789	-0.621634272297	8.81003×10^{-5}	3.94485×10^{-5}	3.30846×10^{-14}
0.368421	-0.718418366955	1.46799×10^{-4}	6.54107×10^{-5}	4.08562×10^{-14}
0.421053	-0.810675178981	2.16559×10^{-4}	9.62569×10^{-5}	3.11973×10^{-14}
0.473684	-0.897023503503	2.96881×10^{-4}	1.31781×10^{-4}	1.11022×10^{-14}
0.526316	-0.975897975	3.87417×10^{-4}	1.71840×10^{-4}	8.21565×10^{-15}
0.578947	-1.045549067303	4.87972×10^{-4}	2.16352×10^{-4}	4.41869×10^{-14}
0.631579	-1.104043093592	5.98500×10^{-4}	2.65299×10^{-4}	2.33147×10^{-14}
0.684211	-1.149262206398	7.19110×10^{-4}	3.18722×10^{-4}	4.77396×10^{-14}
0.736842	-1.178904397603	8.50061×10^{-4}	3.76727×10^{-4}	1.57874×10^{-13}
0.789474	-1.190483498438	9.91763×10^{-4}	4.39482×10^{-4}	4.71179×10^{-13}
0.842105	-1.181329179488	1.14478×10^{-3}	5.07216×10^{-4}	4.26992×10^{-13}
0.894737	-1.148586950683	1.30982×10^{-3}	5.80221×10^{-4}	3.12861×10^{-13}
0.947368	-1.089218161309	1.48776×10^{-3}	6.58850×10^{-4}	3.82361×10^{-13}
1	-1	1.67961×10^{-3}	7.43519×10^{-4}	0

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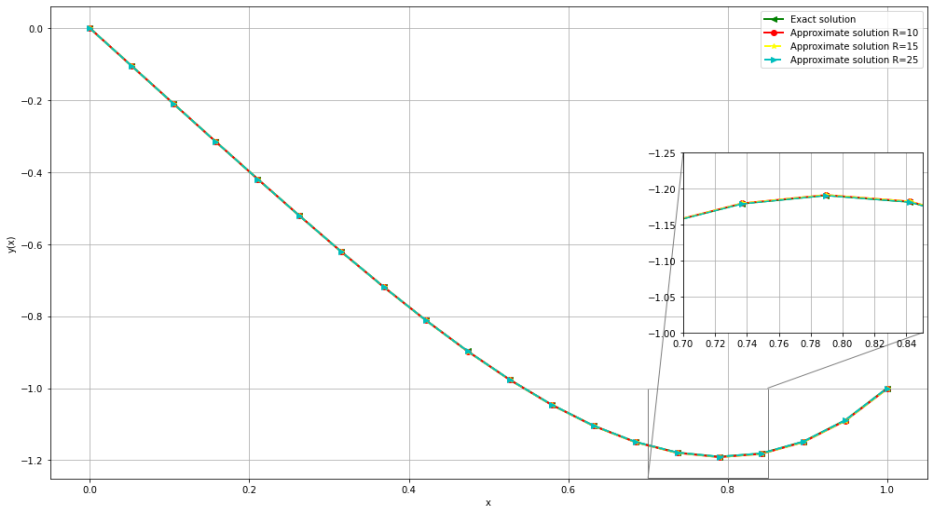

Figure 5. *Exact and approximate solution of Test Example 4.2*

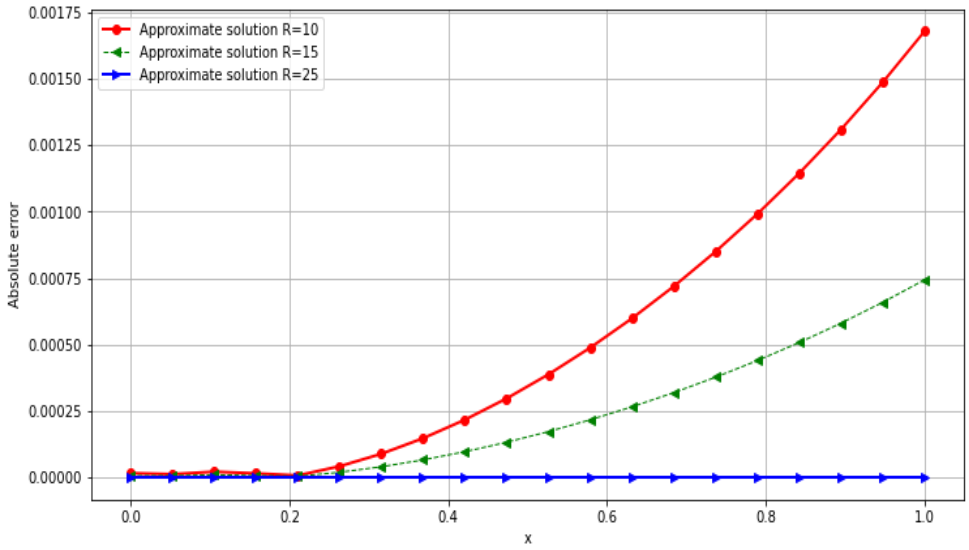

Figure 6. *Absolute error of Test Example 4.2*

Test Example 4.3. Consider the following linear fractional (FIFDEs-Delays) in which $0 < \alpha, \beta \leq 1$

$$ {}_0^C D_x^{0.5} y(x) + {}_0^C D_x^{0.4}\, y(x-0.2) \; + \sinh(x) y(x-0.4) $$
$$ = f(x) + \int_0^1 \cos(x)\, y(t-0.6)dt + \int_0^1 2(x-t) y(t-0.8)dt \qquad (4.3) $$

where

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$$f(x)=\sum_{h=0}^{\infty}\left[\frac{-x^{h+0.5}}{\Gamma(h+1.5)}-e^{-0.2}\frac{x^{h+0.6}}{\Gamma(h+1.6)}\right]+\sin(x)\left[1-e^{(x-0.4)}\right]-\cos(x)\left[1-e^{0.4}+e^{-0.6}\right]$$
$$-2x[1-e^{0.2}+e^{-0.8}]-2e^{-0.8}+1$$

Using a boundary condition $1y(0)+0\,y(1)=0$ and the function of history $\varphi(x)=1-e^{x}$, where $y(x)=1-e^{x}$ is the exact solution.

Suppose that, we take $\overline{M}$ terms from part $f(x)$:

$$f(x)=\sum_{h=0}^{\overline{M}}\left[\frac{-x^{h+0.5}}{\Gamma(h+1.5)}-e^{-0.2}\frac{x^{h+0.6}}{\Gamma(h+1.6)}\right]+\sin(x)\left[1-e^{(x-0.4)}\right]-\cos(x)\left[1-e^{0.4}+e^{-0.6}\right]$$
$$-2x[1-e^{0.2}+e^{-0.8}]-2e^{-0.8}+1$$

Take $N=8$ and $\overline{N}=8$ and $R=22$. Assume the approximate solution has the following form (numerical approximation number in approach, general Clenshaw-Curtis formula):

$$y(x)\cong y_8(x)=\sum_{r=0}^{8}C_rB_{r,8}(x)\qquad for\ all\ r=0,1,\dots,8$$

After executing a Python program to get the constant coefficients C_r,

Table 9. Bernstein parameters for $(y_8(x))$ while $N=8$ and $\overline{M}=4,8,10$ respectively

C_r	$\overline{M}=4$	$\overline{M}=8$	$\overline{M}=10$
C_0	$-4.425507899959\times10^{-10}$	$1.304776482284\times10^{-14}$	$3.795959002425\times10^{-12}$
C_1	$-1.249999850265\times10^{-1}$	$-1.250000000971\times10^{-1}$	$-1.250000003792\times10^{-1}$
C_2	$-2.678571188803\times10^{-1}$	$-2.678571424636\times10^{-1}$	$-2.678571417154\times10^{-1}$
C_3	$-4.315476242192\times10^{-1}$	$-4.315476185230\times10^{-1}$	$-4.315476177733\times10^{-1}$
C_4	$-6.196428068734\times10^{-1}$	$-6.196428624525\times10^{-1}$	$-6.196428735721\times10^{-1}$
C_5	$-8.364585055317\times10^{-1}$	$-8.364583216315\times10^{-1}$	$-8.364582870416\times10^{-1}$
C_6	-1.087252202546	-1.087251993991	-1.087252101780
C_7	-1.378450634366	-1.378497110930	-1.378496618155
C_8	-1.717962516236	-1.718277272478	-1.718281786193

We can obtain an approximate solution to our problem by using the same procedure as in Test Example 4.1. The approximate solution for $\overline{M}=(4,8,10)$ respectively we get:

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$$y_8^{(\overline{M}=4)} = -4.425507899959 \times 10^{-10}(1-x)^8 - 1.249999850265 \times 10^{-1}(8x)(1-x)^7 - 2.678571188803 \times 10^{-1}(28x^2)(1-x)^6 - 4.315476242192 \times 10^{-1}(56x^3)(1-x)^5 - 6.196428068734 \times 10^{-1}(70x^4)(1-x)^4 - 8.364585055317 \times 10^{-1}(56x^5)(1-x)^3 - 1.087252202546(28x^6)(1-x)^2 - 1.378450634366(8x^7)(1-x) - 1.717962516236x^8$$

$$y_8^{(\overline{M}=8)} = 1.304776482284 \times 10^{-14}(1-x)^8 - 1.250000000971 \times 10^{-1}(8x)(1-x)^7 - 2.678571424636 \times 10^{-1}(28x^2)(1-x)^6 - 4.315476185230 \times 10^{-1}(56x^3)(1-x)^5 - 6.196428624525 \times 10^{-1}(70x^4)(1-x)^4 - 8.364583216315 \times 10^{-1}(56x^5)(1-x)^3 - 1.087251993991(28x^6)(1-x)^2 - 1.378497110930(8x^7)(1-x) - 1.718277272478x^8$$

$$y_8^{(\overline{M}=10)} = 3.795959002425 \times 10^{-12}(1-x)^8 - 1.250000003792 \times 10^{-1}(8x)(1-x)^7 - 2.678571417154 \times 10^{-1}(28x^2)(1-x)^6 - 4.315476177733 \times 10^{-1}(56x^3)(1-x)^5 - 6.196428735721 \times 10^{-1}(70x^4)(1-x)^4 - 8.364582870416 \times 10^{-1}(56x^5)(1-x)^3 - 1.087252101780(28x^6)(1-x)^2 - 1.378496618155(8x^7)(1-x) - 1.718281786193x^8$$

Furthermore, after running the programs, table (10) shows a comparison between the exact solution $y(x)$ and the approximate solution $y_8(x)$ and shows the least square errors, running times and residual equations of the values $R_8(x;\bar{C})$are also included by applying formula (3.5) for three different values $\overline{M}$ and the approximation expression, the following approximate formulas are obtained for $\overline{M} = (4, 8, 10)$:

Table 10. A comparison between the exact solution and the approximate solution

x	Exact	$\overline{M} = 4$	$\overline{M} = 8$	$\overline{M} = 10$
0	0	$-4.42550789996 \times 10^{-10}$	1.30477648228×10	3.79595900243×10
0.1	−0.105170918	-0.105170908948	-0.105170918057	-0.105170918066
0.2	−0.221402758	-0.221402741843	-0.22140275815	-0.221402758201
0.3	−0.349858807	-0.349858724607	-0.349858807573	-0.349858807886
0.4	−0.491824697	-0.491824138576	-0.491824696564	-0.491824698055
0.5	−0.648721270	-0.648718625855	-0.648721261726	-0.648721270984
0.	−0.822118800	-0.822109384969	-0.822118753465	-0.822118800486

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0.6	-1.013752707	-1.01372528933	-1.0137525195	-1.01375270695
0.7	-1.225540928	-1.22547198959	-1.22554030616	-1.22554092516
0.8	-1.459603111	-1.45944808977	-1.45960132882	-1.45960309822
0.9	-1.718281828	1.71796251624	-1.71827727248	-1.71828178619
	$L.S.E$	$1.31592219584245 \times 10^{-7}$	$2.43585838154564 \times 10^{-11}$	$1.96545936985566 \times 10^{-15}$
	$R_8 = L.S.E_y$	$7.70802276676131 \times 10^{-18}$	$3.57011757183415 \times 10^{-27}$	$5.66771539696522 \times 10^{-22}$
	$R.Time/Sec$	1.76134705543518	1.768518209457397	1.798195838928222

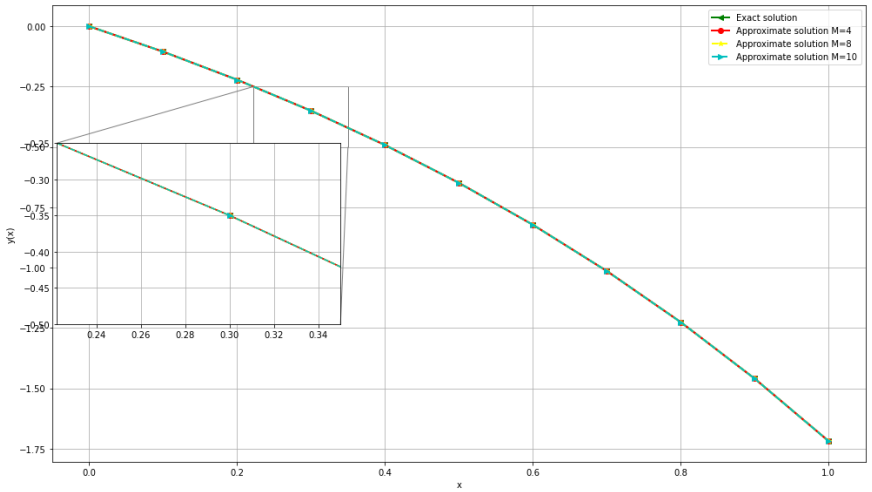

Figure 7. *Exact and approximate solution of Test Example 4.3*

Table (11) shows the absolute error values for a numerical method applied with different values of $\bar{M} = 4, 8, 10$. As $\bar{M}$ increases, the absolute error generally decreases, suggesting that the method becomes more accurate. This highlights the relationship between the parameter $\bar{M}$ and the precision of the results.

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Table 11. Absolute error of the method with respective different values $\bar{M}$

x	Exact	Absolute error		
		$\bar{M}=4$	$\bar{M}=8$	$\bar{M}=10$
0	0	4.42551×10^{-10}	1.30478×10^{-14}	3.79596×10^{-12}
0.1	-0.1051709181	9.12795×10^{-9}	1.88477×10^{-11}	9.64771×10^{-12}
0.2	-0.2214027581	1.63175×10^{-8}	1.05699×10^{-11}	4.12302×10^{-11}
0.3	-0.3498588076	8.29692×10^{-8}	3.00321×10^{-12}	3.10297×10^{-10}
0.4	-0.4918246976	5.59066×10^{-7}	1.07737×10^{-9}	4.1373×10^{-10}
0.5	-0.6487212707	2.64484×10^{-6}	8.97463×10^{-9}	2.84272×10^{-10}
0.6	-0.8221188004	9.41542×10^{-6}	4.6926×10^{-8}	9.55909×10^{-11}
0.7	-1.013752707	2.74181×10^{-5}	1.87969×10^{-7}	5.21477×10^{-10}
0.8	-1.225540928	6.89389×10^{-5}	6.22329×10^{-7}	3.32947×10^{-9}
0.9	-1.459603111	1.55021×10^{-4}	1.78233×10^{-6}	1.29349×10^{-8}
1	-1.718281828	3.19312×10^{-4}	4.55598×10^{-6}	4.2266×10^{-8}

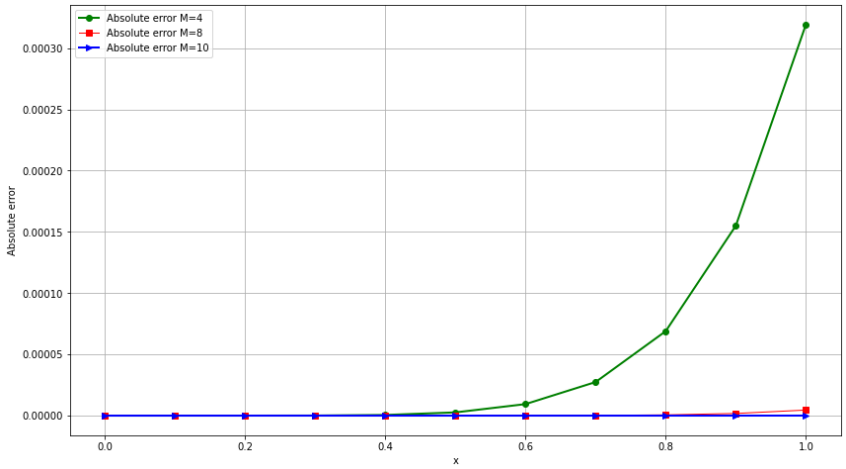

Figure 8. *Absolute error of Test Example 4.3*

5. CONCLUSIONS

In this study, under practical circumstances, Fredholm integro-fractional differential equations (FIFDEs-Delays) of delay types with variable coefficients are solved using the Bernstein polynomial approximation. Good results were obtained by relying solely on the computer program that was built and included multiple examples for illustration. Additionally, tabular forms of the running time, least square error, and least square error function $y(x)$ were provided. thus the following points have been identified.

1. The good results are determined by the number of approximate parts of the integral R (the exact solution of $y_N(x)$ if R is a large number) and the number of polynomials N that are obtained with a suitable number of $\overline{N}$.

2. In a situation with Mittag-Leffler terms (see Test Example 4.3), the number of Mittag-Leffler words and the selection of the aforementioned factors determine how accurate the findings are. In order to save time, we converted the infinite Mittag-Leffler terms into (4,8, and 10) terms.

Conflict of interests.

There is no conflict of interest, according to the authors.

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الخلاصة

المقدمة:

قدم هذه الدراسة إطارًا جديدًا مفيدًا يستخدم متعددات حدود برنشتاين لتحسين تقنية التجميع الطيفي لحل معادلات فريدهولم التكاملية التفاضلية ذات الرتب الكسرية ذات المعاملات المتغيرة وتأخير ثابت متعدد الأوقات (FIFDEs–Delays) عدديًا في ظل الظروف الحدودية

طرق العمل:

يُفترض أن تكون الحلول التقريبية على شكل متسلسلة حدود برنشتاين المقطوعة. يعتمد هذا النهج الجديد على استخدام تقنية المصفوفة لتحويل معادلة العرض مع الشروط إلى نظام معادلات خطي جبري ذي معاملات برنشتاين مجهولة.

الإستنتاجات:

يُحسّن هذا النهج دقة الحلول المُتوصل إليها، ويُبسّط المسألة في الوقت نفسه. يُحدد حل هذا النظام معاملات الحل المُفترض. إضافةً إلى ذلك، قُيّمت مُعاملات التكامل المُستخدمة في هذه التقنية كمّيًا باستخدام صيغة كلينشو–كيرتس

الكلمات المفتاحية: معادلات فريدهولم التكاملية، المشتقة الكسرية، أنواع التأخير الثابت، متعددة حدود برنشتاين، مشتقة كابوتو، تقنية المصفوفة، نقاط التجميع القياسية.