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An Efficient Hermite Polynomial Solution to the Fractional Differential Equations

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"حل كفوء للمعادلات التفاضلية الكسرية باستخدام كثيرات حدود هيرمايت"

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ABSTRACT

Fractional differential equations (FDEs) play a fundamental role in modeling complex physical, biological, and engineering phenomena characterized by memory effects and nonlocal dynamics. This paper presents an efficient numerical framework based on a fractional Hermite interpolation formula for solving FDEs. The proposed method extends the classical Hermite interpolation scheme to fractional calculus by embedding fractional derivatives within the interpolation structure, thereby improving approximation precision and convergence behavior. To enhance computational performance, optimized interpolation nodes and refined fractional derivative approximations are introduced, effectively reducing truncation errors and improving numerical stability. The method is systematically formulated and implemented in a computational environment, with numerical experiments verifying its robustness and accuracy. Results confirm that the proposed scheme achieves superior stability and precision compared with conventional numerical techniques, demonstrating its potential for broad application in the solution of fractional-order models across scientific and engineering domains.

Keywords: Fractional differential equations; Hermite interpolation; numerical methods; interpolation formula; computational efficiency; fractional derivatives; numerical approximation.

1 INTRODUCTION

Fractional calculus generalizes classical calculus by extending the concepts of differentiation and integration to non-integer (fractional) orders [1-4]. Unlike traditional integer-order derivatives, fractional derivatives intrinsically include memory and hereditary effects, enabling them to describe systems whose current states depend on the entire history of their evolution. These properties make fractional calculus a powerful mathematical framework for modeling complex processes in physics, engineering, biology, and finance. Several definitions of fractional differentiation exist, most notably the Riemann–Liouville, Caputo, and Grünwald–Letnikov formulations [5-8]. Each definition provides unique analytical and computational advantages depending on the boundary conditions and smoothness of the problem. The flexibility and physical relevance of these formulations have positioned fractional differential equations (FDEs) at the core of modern mathematical modeling [9-12].

FDEs have been successfully employed to describe memory-dependent dynamics in viscoelastic materials, anomalous diffusion, control systems, and complex biological networks [13-18]. Although differential equations (DEs) include memory and hereditary properties, they provide a more accurate representation of many systems with motion [19]. Because fractional derivatives are nonlocal, they are more difficult to solve numerically, despite their advantages. Recent research has focused on developing hybrid numerical approaches that combine the stability and smoothness of spline-based approximations with the flexibility of algebra [20, 21].

However, their nonlocal nature often introduces computational challenges when deriving accurate numerical solutions of the fractional calculus [22, 23]. Classical numerical schemes-originally developed for integer-order equations—are often limited by high computational cost and reduced stability when extended to fractional systems [24-27]. To overcome these challenges, interpolation-based numerical strategies have gained attention, particularly Hermite interpolation, which utilizes both function values and derivative information [34, 35]. These methods incorporate fractional derivatives into the interpolation framework, significantly enhancing approximation accuracy for FDEs. Previous works have shown that fractional Hermite schemes outperform classical polynomial or spline approaches in accuracy and convergence [36, 38]. The extension of this concept to fractional calculus—termed fractional Hermite interpolation—was initiated by Luchko [3] and further developed in recent studies [39]. The method introduces an optimized selection of interpolation nodes and utilizes refined fractional derivative approximations to minimize truncation errors [40]. Nonetheless, several open issues remain, including numerical instability, computational complexity, and limited adaptability to mixed or nonstandard boundary conditions [41-43]. In this work, we propose an improved fractional Hermite interpolation framework designed to enhance both computational efficiency and numerical stability. The proposed formulation also extends applicability to a wider class of FDEs, ensuring robust accuracy under diverse initial and boundary conditions. Validation through comprehensive numerical experiments demonstrates superior performance over existing interpolation-based fractional numerical methods.

2 BASIC DEFINITIONS

This section introduces the mathematical background necessary for constructing the proposed method, including interpolation theory, fractional derivatives, and relevant norms.

2.1 Interpolation and Fractional Differential Equations

Interpolation provides a mechanism to estimate unknown function values from a known discrete dataset. In the context of FDEs, interpolation techniques are extended to incorporate fractional derivatives, giving rise to fractional Hermite interpolation [44, 45]. Fractional differential equations involve derivatives of non-integer order and are used to describe systems exhibiting memory and hereditary effects. Their numerical solution requires specialized algorithms capable of handling fractional operators with nonlocal dependence.

2.1.1 Hermite Interpolation for Integer Order

Hermite interpolation approximates a smooth function by using both its values and derivative values at given nodes. Given distinct points $x_0, x_1, \dots, x_n$ with $f(x_i)$ and $f'(x_i)$ known, the Hermite interpolation polynomial $H_n(x)$ satisfies:

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$$H_n(x_i) = f(x_i), \quad i = 0,1,\dots,n, \tag{1}$$

$$H'_n(x_i) = f'(x_i), \quad i = 0,1,\dots,n. \tag{2}$$

By incorporating derivative information, this scheme achieves higher accuracy than standard polynomial interpolation [2, 9].

2.2 Fractional Derivatives

Fractional derivatives extend differentiation to arbitrary real (or even complex) orders. The most widely used formulations are those of Riemann–Liouville and Caputo[19-23].

2.2.1 Riemann–Liouville Derivative

The Riemann–Liouville fractional derivative of order α is defined as [1]:

$$D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha-1} f(t)\, dt, \quad n-1 < \alpha < n. \tag{3}$$

2.2.2 Caputo Derivative

The Caputo fractional derivative of order α is given by:

$$^C D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)\, dt. \tag{4}$$

2.3 Fractional Taylor Series

The fractional Taylor series [4] generalizes the classical expansion to fractional-order derivatives:

$$f(x) = \sum_{n=0}^{\infty} \frac{(D^\alpha f)(x_0)}{\Gamma(n\alpha+1)} (x-x_0)^{n\alpha}. \tag{5}$$

This formulation allows fractional-order function reconstruction based on non-integer derivative data

2.4 Maximum Norm

The maximum norm (or supremum norm) measures the greatest absolute deviation of a function over a given domain [5]:

$$\parallel f \parallel_\infty = \sup_{x\in\Omega} |f(x)|. \quad (6$$

3 FORMULATION OF THE METHOD

The proposed method constructs a fractional Hermite interpolation polynomial to approximate solutions of FDEs. This section details the derivation of the numerical scheme.

3.1 Approximation of Fractional Derivatives

To compute fractional derivatives at selected interpolation nodes, we adopt a hybrid finite-difference and fractional Taylor expansion strategy following Tenreiro Machado [14] and later computational refinements [24-27]. For a smooth function f on $[a,b]$, its fractional derivative at node x_i is approximated by:

$$D_x^\alpha f(x_i) \approx \frac{1}{h^\alpha} \sum_{k=0}^{i} (-1)^k \binom{\alpha}{k} f(x_{i-k}), \tag{7}$$

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where $\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}$ is the generalized binomial coefficient. This discrete formulation ensures computational efficiency and can be implemented directly in MATLAB.

3.2 Construction of the Fractional Hermite Interpolant

With $f(x_i)$ and their corresponding fractional derivatives known, the fractional Hermite interpolation polynomial is defined as [41- 43]:

$$H_n(x) = \sum_{i=0}^{n} [f(x_i)\, L_i(x) + D_x^\alpha f(x_i)\, M_i(x)], \tag{8}$$

with the conditions:

$$L_i(x_j) = \delta_{ij}, \quad D_x^\alpha L_i(x_j) = 0, \quad M_i(x_j) = 0, \quad D_x^\alpha M_i(x_j) = \delta_{ij}. \tag{9}$$

These properties ensure that $H_n(x)$ simultaneously interpolates both the function and its fractional derivative at each node, thus generalizing the classical Hermite conditions to the fractional domain.

3.3 Theorem 1: Let $\boldsymbol{\alpha} \in \mathbb{R}^+$, $\mathbf{n} \in \mathbb{N}$, and $\boldsymbol{f(x)}$ be a function such that $\boldsymbol{D^\alpha f(x)}$ is continuous on $[\boldsymbol{a}, \boldsymbol{b}]$. Given mesh points $\boldsymbol{a = x_0 < x_1 < \cdots < x_n = b}$, the Fractional Hermite interpolation $\boldsymbol{H_m(x)}$ satisfying:

$\boldsymbol{H_m(x_i) = f(x_i)}$ for all $\boldsymbol{i = 0, 1, \dots, n}$, and $\boldsymbol{D^\alpha H_m(x_i) = D^\alpha f(x_i)}$ for all $\boldsymbol{i = 0, 1, \dots, n}$,

can be expressed as:

$$H_m(x) = \sum_{i=0}^{n} \left[\left(1 - \frac{D^\alpha \ell_i(x)^2 |_{x=x_i}}{\Gamma(\alpha+1)} (x - x_i)^\alpha \right) \ell_i(x)^2 f(x_i) + \frac{(x - x_i)^\alpha}{\Gamma(\alpha+1)} \ell_i(x)^2 D^\alpha f(x_i) \right],$$

where $\ell_i(x)$ denotes the Lagrange basis polynomial corresponding to the node x_i.

Proof. The unknown coefficients a_0, a_1, b_0, and b_1 of:

$$H_m(x) = \sum_{i=0}^{n} [(a_0 + a_1(x - x_i)^\alpha)\ell_i(x)^2 f(x_i) + (b_0 + b_1(x - x_i)^\alpha)\ell_i(x)^2 D^\alpha f(x_i)],$$

are determined by enforcing the interpolation conditions on $\mathrm{H_m(x)}$ and its fractional derivative $D^\alpha H_m(x)$ at the nodes $\mathrm{x_i}$,

Assume the form of the interpolant as:

$$H_m(x) = \sum_{i=0}^{n} [h_i(x) + h_i^*(x)],$$

where the individual terms are:

$h_i(x) = (a_0 + a_1(x - x_i)^\alpha)\ell_i(x)^2 f(x_i)$ & $h_i^*(x) = (b_0 + b_1(x - x_i)^\alpha)\ell_i(x)^2 D^\alpha f(x_i)$.

At $x = x_i$, the terms become: $h_i(x_i) = a_0 \ell_i(x_i)^2 f(x_i) = a_0 f(x_i)$ *and*

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$$h_i^*(x_i) = b_0 \ell_i(x_i)^2 D^\alpha f(x_i) = b_0 D^\alpha f(x_i).$$

For the condition $H_m(x_i) = f(x_i)$, we require: $a_0 f(x_i) + b_0 D^\alpha f(x_i) = f(x_i)$.

Compute the α-th derivative of $H_m(x)$ at x_i. First, consider the derivatives of the individual terms:

The values of coefficients a_0, a_1, b_0, b_1 into the expression for $H_m(x)$:

$$\boxed{a_0 = 1}, \quad \boxed{b_0 = 0}, \quad \boxed{a_1 = -\frac{D^\alpha \ell_i(x)^2|_{x=x_i}}{\Gamma(\alpha+1)}}, \quad and \quad \boxed{b_1 = \frac{1}{\Gamma(\alpha+1)}}$$

yields the stated results via algebraic manipulation and properties of the Gamma function and:

$$H_m(x) = \sum_{i=0}^{n} \left[\left(1 - \frac{D^\alpha \ell_i(x)^2|_{x=x_i}}{\Gamma(\alpha+1)} (x-x_i)^\alpha \right) \ell_i(x)^2 f(x_i) + \frac{(x-x_i)^\alpha}{\Gamma(\alpha+1)} \ell_i(x)^2 D^\alpha f(x_i) \right].$$

Lemma 1 Let $H_m(x)$ be the Fractional Hermite interpolating polynomial defined by Theorem 1 where $\ell_i(x)$ are the Lagrange basis polynomials, then the Riemann-Liouville fractional derivative D^β of order $\beta > 0$ is given by:

$$D^\beta H_m(x) = \sum_{i=0}^{n} \Bigg[f(x_i) D^\beta \ell_i(x)^2 - \frac{D^\alpha(\ell_i(x)^2)|_{x=x_i}}{\Gamma(\alpha+1)} f(x_i) \sum_{k=0}^{\infty} \binom{\beta}{k} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)} (x-x_i)^{\alpha-k} D^{\beta-k} \ell_i(x)^2$$
$$+ \frac{D^\alpha f(x_i)}{\Gamma(\alpha+1)} \sum_{k=0}^{\infty} \binom{\beta}{k} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)} (x-x_i)^{\alpha-k} D^{\beta-k} \ell_i(x)^2 \Bigg]$$

Proof: To compute the fractional derivative D^β of order $\beta \in \mathbb{R}^+$ for the given Fractional Hermite interpolation formula, we proceed as follows. For clarity, we assume the Riemann-Liouville fractional derivative definition, which generalizes integer-order differentiation to non-integer orders. The formula is:

$$H_m(x) = \sum_{i=0}^{n} \Bigg[\underbrace{\left(1 - \frac{D^\alpha(\ell_i(x)^2)|_{x=x_i}}{\Gamma(\alpha+1)} (x-x_i)^\alpha \right) \ell_i(x)^2 f(x_i)}_{Functionvalueterm} + \underbrace{\frac{(x-x_i)^\alpha}{\Gamma(\alpha+1)} \ell_i(x)^2 D^\alpha f(x_i)}_{Derivativeterm} \Bigg].$$

- $\ell_i(x)$ is a polynomial, so $\ell_i(x)^2$ is also a polynomial.
- $(x-x_i)^\alpha$ is a power function. For fractional derivatives, we use the Riemann-Liouville formula:

$$D^\beta (x-x_i)^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\beta+1)} (x-x_i)^{\gamma-\beta}, \quad \gamma \succ 1.$$

Now to compute $D^\beta H_m(x)$ term by term:

Term 1: $D^\beta[\ell_i(x)^2 f(x_i)]$,

- Since $f(x_i)$ is a constant, $D^{\beta}[\ell_i(x)^2 f(x_i)] = f(x_i) D^{\beta} \ell_i(x)^2$ (8)

- If $\ell_i(x)$ is a degree-m polynomial, then $D^{\beta} \ell_i(x)^2 = 0$ for $\beta > 2m$.

Term 2: $D^{\beta}\left[\frac{D^{\alpha}(\ell_i(x)^2)|_{x=x_i}}{\Gamma(\alpha+1)}(x-x_i)^{\alpha}\ell_i(x)^2 f(x_i)\right]$, where $D^{\alpha}(\ell_i(x)^2)|_{x=x_i}$ is a constant.

- we apply the generalized Leibniz rule[1] for fractional derivatives :

$$D^{\beta}[(x-x_i)^{\alpha}\ell_i(x)^2] = \sum_{k=0}^{\infty}\binom{\beta}{k} D^k(x-x_i)^{\alpha}\cdot D^{\beta-k}\ell_i(x)^2. \tag{9}$$

Using the Riemann-Liouville derivative of power functions:

$$D^k(x-x_i)^{\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)}(x-x_i)^{\alpha-k}$$

Therefore: we obtain that: $D^{\beta} H_m(x) = \sum_{i=0}^{n}\,[f(x_i)A_i(x) + D^{\alpha} f(x_i) B_i(x)]$ (10)

$$A_i(x) = D^{\beta}\ell_i(x)^2 - \frac{D^{\alpha}(\ell_i(x)^2)|_{x=x_i}}{\Gamma(\alpha+1)}\sum_{k=0}^{\infty}\binom{\beta}{k}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)}(x-x_i)^{\alpha-k}D^{\beta-k}\ell_i(x)^2 \tag{11}$$

$$B_i(x) = \frac{1}{\Gamma(\alpha+1)}\sum_{k=0}^{\infty}\binom{\beta}{k}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)}(x-x_i)^{\alpha-k}D^{\beta-k}\ell_i(x)^2$$

When $\beta = \alpha$ and $x = x_i$, the fractional derivative of the Hermite interpolant simplifies to:

$D^{\alpha} H_m(x_i) = D^{\alpha} f(x_i)$, By using the cardinal property $\ell_i(x_j) = \delta_{ij}$.

Lemma 2: (Existence and Uniqueness of Hermite Interpolation with Fractional Derivatives)

Let $\{x_i\}_{i=0}^{n}$ be distinct uniformly spaced nodes in the interval $[a, b]$, and let f be a function such that its fractional derivative $D^{\alpha} f$ of order $\boldsymbol{\alpha} \in \mathbb{R}^{+}$ exists at each x_i and continuous on $[a, b]$ Then there exists a unique "exists uniquely[28, 37]" interpolating function $H_m(x)$ of the form:

$$H_m(x) = \sum_{i=0}^{n}\left[\left(1 - \frac{D^{\alpha}(\ell_i(x)^2)|_{x=x_i}}{\Gamma(\alpha+1)}(x-x_i)^{\alpha}\right)\ell_i(x)^2 f(x_i) + \frac{(x-x_i)^{\alpha}}{\Gamma(\alpha+1)}\ell_i(x)^2 D^{\alpha} f(x_i)\right],$$

Where $\ell_i(x)$ is the Lagrange basis polynomial corresponding to node x_i, $f(x_i)$ is the value of the function at x_i, $D^{\alpha} f(x_i)$ denotes the fractional derivative of order α at x_i.

Proof: Fisrt we define the functions as follows:

$$\phi_i(x) = \left(1 - \frac{D^{\alpha}(\ell_i(x)^2)}{\Gamma(\alpha+1)}(x-x_i)^{\alpha}\right)\ell_i(x)^2, \tag{12}$$

$$\psi_i(x) = \frac{(x-x_i)^{\alpha}}{\Gamma(\alpha+1)}\ell_i(x)^2. \tag{13}$$

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These satisfy the interpolation conditions:

$$\phi_i(x_j) = \delta_{ij}, \psi_i(x_j) = 0, \tag{14}$$

$$D^\alpha \phi_i(x_j) = 0, D^\alpha \psi_i(x_j) = \delta_{ij}. \tag{15}$$

To show that $\phi_i(x)$ and $\psi_i(x)$ form a complete basis for the space of Hermite interpolants, we follow these steps.

From the above conditions, the linear independence [29] of these functions: $\{\phi_i(x), \psi_i(x)\}$

Now we must to show that these functions are independent so we define the evaluation matrix A with rows indexed by interpolation points x_j and columns indexed by basis functions:

$$A = \begin{bmatrix} \phi_0(x_0) & \phi_1(x_0) & \cdots & \phi_n(x_0) & \psi_0(x_0) & \psi_1(x_0) & \cdots & \psi_n(x_0) \\ \phi_0(x_1) & \phi_1(x_1) & \cdots & \phi_n(x_1) & \psi_0(x_1) & \psi_1(x_1) & \cdots & \psi_n(x_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_0(x_n) & \phi_1(x_n) & \cdots & \phi_n(x_n) & \psi_0(x_n) & \psi_1(x_n) & \cdots & \psi_n(x_n) \\ D^\alpha\phi_0(x_0) & D^\alpha\phi_1(x_0) & \cdots & D^\alpha\phi_n(x_0) & D^\alpha\psi_0(x_0) & D^\alpha\psi_1(x_0) & \cdots & D^\alpha\psi_n(x_0) \\ D^\alpha\phi_0(x_1) & D^\alpha\phi_1(x_1) & \cdots & D^\alpha\phi_n(x_1) & D^\alpha\psi_0(x_1) & D^\alpha\psi_1(x_1) & \cdots & D^\alpha\psi_n(x_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ D^\alpha\phi_0(x_n) & D^\alpha\phi_1(x_n) & \cdots & D^\alpha\phi_n(x_n) & D^\alpha\psi_0(x_n) & D^\alpha\psi_1(x_n) & \cdots & D^\alpha\psi_n(x_n) \end{bmatrix}.$$

From the conditions in equations (14) and (5) and the matrix A takes the block diagonal form:

$$A = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

where I is the identity matrix.

Since A is block diagonal, its determinant is the product of the determinants of the identity blocks:

$$\det(A) = \det(I) \cdot \det(I) = 1 \times 1 = 1 \neq 0.$$

Since the determinant is nonzero, the basis functions $\phi_i(x)$ and $\psi_i(x)$, is linearly independent.

The functions: $\phi_i(x)$ and $\psi_i(x)$ Span [29] the Hermite Interpolation Space.

The Hermite interpolation problem requires constructing a function H(x) that satisfies both function values and derivative conditions at given points:

$$H(x_j) = f_j, D^\alpha H(x_j) = g_j. \tag{16}$$

We use the previously defined basis functions:

$$\phi_i(x) = \left(1 - \frac{D^\alpha(\ell_i(x)^2)}{\Gamma(\alpha+1)}(x - x_i)^\alpha\right)\ell_i(x)^2, \tag{17}$$

$$\psi_i(x) = \frac{(x-x_i)^\alpha}{\Gamma(\alpha+1)}\ell_i(x)^2. \tag{18}$$

These functions satisfy the interpolation conditions:

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$$\phi_i(x_j) = \delta_{ij}, \psi_i(x_j) = 0, \tag{19}$$

$$D^\alpha \phi_i(x_j) = 0, D^\alpha \psi_i(x_j) = \delta_{ij}. \tag{20}$$

Then, the general Hermite interpolating function is written as:

$$H(x) = \sum_{i=0}^{n} f_i \phi_i(x) + \sum_{i=0}^{n} g_i \psi_i(x).$$

By construction:

$$H(x_j) = f_j, \tag{21}$$

$$D^\alpha H(x_j) = g_j. \tag{22}$$

This ensures that $H(x)$ satisfies the conditions of Hermite interpolation.

Substitute $x = x_i$ into $H(x)$. By construction:

$$H_m(x_i) = \sum_{j=0}^{n} \left[\left(1 - \frac{D^\alpha \ell_j(x_i)^2 |_{x=x_j}}{\Gamma(\alpha+1)} (x_i - x_j)^\alpha \right) \ell_j(x_i)^2 f(x_j) + \frac{(x_i - x_j)^\alpha}{\Gamma(\alpha+1)} \ell_j(x_i)^2 D^\alpha f(x_j) \right]$$

$$= f(x_i) \left(since\colon \ell_j(x_i) = \delta_{ij} \right). \tag{23}$$

Similarly, applying D^α to $H_m(x)$ and evaluating at x_i yields $D^\alpha H_m(x_i) = D^\alpha f(x_i)$.

From Lemma 3.1, we conclude that: $H_m(x_i) = f(x_i)$ & $D^\alpha H_m(x_i) = D^\alpha f(x_i)$

According to uniqueness in [28], we suppose two interpolants $H_m^{(1)}(x)$ and $H_m^{(2)}(x)$ satisfy the conditions. Define $E(x) = H_m^{(1)}(x) - H_m^{(2)}(x)$. Then:

$$E(x_i) = 0, \forall i \Longrightarrow D^\alpha E(x_i) = 0 \forall i.$$

Since $E(x)$ is a linear combination of basis functions vanishing at all nodes with zero fractional derivatives, $E(x) \equiv 0$. Thus, $H_m^{(1)}(x) = H_m^{(2)}(x)$.

Since $\phi_i(x)$ and $\psi_i(x)$ are linearly independent and their span contains all possible Hermite interpolants, they form a basis [29] for the Hermite interpolation space. Thus, they span the space of Hermite interpolating polynomials.

Where $H_m(x)$ is the unique function satisfying $H_m(x_i) = f(x_i)$ and $D^\alpha H_m(x_i) = D^\alpha f(x_i)$ for all $i = 0,1, \ldots, n$.

The interpolation problem imposes $2(n+1)$ conditions (function and derivative values at $n+1$ nodes). The formula $H_m(x)$ is constructed using $2(n+1)$ basis functions:

$$\phi_i(x) = \left(1 - \frac{D^\alpha (\ell_i(x)^2)|_{x=x_i}}{\Gamma(\alpha+1)} (x - x_i)^\alpha \right) \ell_i(x)^2, \quad \psi_i(x) = \frac{(x - x_i)^\alpha}{\Gamma(\alpha+1)} \ell_i(x)^2.$$

These form a basis for the space of functions interpolating $f(x_i)$ and $D^{\alpha} f(x_i)$.

At $x = x_i$, $\phi_i(x_j) = \delta_{ij}$ and $\psi_i(x_j) = 0$. - For the derivatives, $D^{\alpha}\phi_i(x_j) = 0$ and $D^{\alpha}\psi_i(x_j) = \delta_{ij}$. - The Kronecker delta properties ensure linear independence: no non-trivial combination of ϕ_i and ψ_i can vanish at all nodes or their derivatives.

The coefficients of $H_m(x)$ solve the linear system:

$$\begin{cases} H_m(x_i) = f(x_i), \\ D^{\alpha} H_m(x_i) = D^{\alpha} f(x_i). \end{cases}$$

The basis functions ϕ_i, ψ_i generate a Vandermonde-like matrix with a zero determinant due to linear independence, ensuring a unique solution.

Lemma 3: (Error Analysis and Consistency)

Let $H_m(x)$ be the fractional Hermite interpolant of a sufficiently smooth function $f(x)$ over an interval $[a, b]$ using nodes $\{x_i\}_{i=0}^{n}$. Then the interpolation error:

$e(x) = f(x) - H_m(x)$ satisfies the following properties:

Peano Kernel Representation [31, 32]: There exists a kernel function $K(x,t)$ such that

$$e(x) = \int_a^b K(x,t) D^{(n+1)(1+\alpha)} f(t) dt,$$

where $\mathcal{L}_x[f] = e(x)$ is the interpolation error functional.

Error Bound [28, 31]: The kernel $K(x,t)$ satisfies the inequality

$$|K(x,t)| \leq \frac{(x-t)_+^{(n+1)(1+\alpha)-1}}{\Gamma((n+1)(1+\alpha))},$$

leading to the uniform error estimate

$$\parallel e \parallel_{\infty} \leq \frac{\parallel D^{(n+1)(1+\alpha)} f \parallel_{\infty} (b-a)^{(n+1)(1+\alpha)}}{\Gamma((n+1)(1+\alpha))}.$$

Consistency with Classical Interpolation [33]:

- As $\alpha \to 0$, the Caputo derivative $D^{\alpha} f(x_i)$ reduces to $f(x_i)$, and the interpolation formula reduces to standard Lagrange interpolation.

- For $\alpha = 1$, the fractional Hermite interpolant coincides [15, 32] with the classical Hermite interpolation, confirming consistency.

Proof. We prove each part of the lemma:

Peano Kernel Representation [13, 31]. The error functional is defined as:

$$\mathcal{L}_x[f] = f(x) - H_m(x) = e(x).$$

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By the generalized Peano kernel theorem[31] (adapted to fractional order derivatives), this error can be represented as:

$$e(x)=\int_a^b K(x,t)D^{(n+1)(1+\alpha)}f(t)dt,$$

where $K(x,t)$ is the Peano kernel associated with the interpolation operator.

Kernel Bound [28, 32]. Using the properties of the Peano kernel, we have the bound:

$$|K(x,t)|\leq\frac{(x-t)_+^{(n+1)(1+\alpha)-1}}{\Gamma\big((n+1)(1+\alpha)\big)}.$$

Hence, the interpolation error satisfies:

$$|e(x)|\leq\int_a^b |K(x,t)|\cdot\left|D^{(n+1)(1+\alpha)}f(t)\right|dt\leq\parallel D^{(n+1)(1+\alpha)}f\parallel_\infty\int_a^b|K(x,t)|dt.$$

Computing the integral of the kernel bound yields:

$$\parallel e\parallel_\infty\leq\frac{\parallel D^{(n+1)(1+\alpha)}f\parallel_\infty(b-a)^{(n+1)(1+\alpha)}}{\Gamma\big((n+1)(1+\alpha)+1\big)}.$$

Hence we conluded the error bound:

$$E(f)(x)=|f(x)-H_m(x)|=\frac{f^{((n+1)(1+\alpha))}(\xi)}{((n+1)(1+\alpha))!}\prod_{i=0}^{n}(x-x_i)^{1+\alpha},\qquad \xi\in[a,b]$$

On the other hand Let: $h=max|x-x_i|\Longrightarrow h^{1+\alpha}=max|x-x_i|^{1+\alpha}\Longrightarrow\prod_{i=0}^{n}(x-x_i)^{1+\alpha}\leq\prod_{i=0}^{n}h^{1+\alpha}=h^{(1+\alpha)(n+1)}$,

Therefore the order of error is: $\Longrightarrow O(h)=h^{(1+\alpha)(n+1)}$

Consistency [5]:

Caese 1: When $\alpha\to 0$, the fractional derivative $D^\alpha f(x_i)$ becomes $f(x_i)$, and the Hermite formula $H_m(x)$ reduces to the classical Lagrange interpolating polynomial [5,31] so that:

$$\Longrightarrow E(f)(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{i=0}^{n}(x-x_i),\qquad \xi\in[a,b]$$

And the order of error will be: $O(h)=h^{(n+1)}$.

Case 2: When $\alpha=1$, the fractional derivative becomes the classical derivative, and the interpolation structure matches that of classical Hermite interpolation [3, 9], where both function values and first derivatives are interpolated. In other words the formula of theorem1 will reduce to:

$$H_m(x)=\sum_{i=0}^{n}\left[\big(1-2\ell_i'(x_i)(x-x_i)\big)f(x_i)+(x-x_i)f'(x_i)\right]\ell_i(x)^2.$$

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The error bound formula reduce to:

$$\Rightarrow E(f)(x) = |f(x) - H_m(x)| = \frac{f^{(2n+2)}(\xi)}{(2n+2)!} \prod_{i=0}^{n} (x - x_i)(x - x_i)^2, \qquad \xi \in [a, b]$$

With it's order of error: $\Rightarrow O(h) = h^{(2n+2)}$.

Lemma 4 (Maximum Norm of $H_m(x)$):

Let $H_m(x)$ be the fractional Hermite interpolant of a function $f(x)$ on the interval $[a,b]$ using $(n+1)$ interpolation nodes $\{x_i\}_{i=0}^{n}$. Then the maximum norm of $H_m(x)$ is bounded by:

$$\| H_m(x) \|_\infty \leq (n+1)\max\left(\sup_{x\in[a,b]} |f(x)|, \sup_{x\in[a,b]} |D^\alpha f(x)| \right).$$

where $D^\alpha f$ denotes the Caputo fractional derivative of order α.

Proof: Recall the maximum norm (or supremum norm) of a function f over $[a,b]$ is defined as:

$$\| f \|_\infty = \sup_{x\in[a,b]} |f(x)|.$$

We now estimate the maximum norm of $H_m(x)$ by analyzing the terms in its construction.

Function Value Term: One part of the interpolant takes the form:

$$\left(1 - \frac{D^\alpha(\ell_i(x)^2)|_{x=x_i}}{\Gamma(\alpha+1)}(x - x_i)^\alpha\right)\ell_i(x)^2 f(x_i),$$

where $\ell_i(x)$ is the local Lagrange basis polynomial. Since $\ell_i(x)$ is a polynomial of fixed degree, its square and fractional derivative are bounded on $[a,b]$. The factor $(x - x_i)^\alpha$ is also bounded as $x \in [a,b]$. Thus, this entire term is bounded by a constant multiple of $|f(x_i)|$.

Derivative Term: Another part of the interpolant is: $\frac{(x-x_i)^\alpha}{\Gamma(\alpha+1)}\ell_i(x)^2 D^\alpha f(x_i)$.

As before, $\ell_i(x)^2 \in [0,1]$ and $(x - x_i)^\alpha$ is bounded on $[a,b]$, so this term is also bounded by a constant multiple of $|D^\alpha f(x_i)|$.

Bounding the Interpolant: Each term in the interpolant $H_m(x)$ is bounded by a combination of $|f(x_i)|$ and $|D^\alpha f(x_i)|$. Taking the supremum of $H_m(x)$ over $[a,b]$ yields:

$$\| H_m(x) \|_\infty \leq \sum_{i=0}^{n} (|f(x_i)| + |D^\alpha f(x_i)|)$$

Let us define:

$$M_f = \sup_{x\in[a,b]} |f(x)|, M_{D^\alpha f} = \sup_{x\in[a,b]} |D^\alpha f(x)|.$$

Then, since $|f(x_i)| \leq M_f$ and $|D^\alpha f(x_i)| \leq M_{D^\alpha f}$ for all i, we obtain:

$\| H_m(x) \|_\infty \leq (n+1) \cdot \max\left(M_f, M_{D^\alpha f}\right).$

The maximum norm of $H_m(x)$ is bounded by the maximum values of $f(x)$ and its fractional derivatives over the interval $[a,b]$, with a factor of $n+1$, where n is the number of nodes. The exact value depends on the specific function $f(x)$ and its fractional derivatives. Therefore, we conclude:

$$\| H_m(x) \|_\infty \leq (n+1) max\left(\sup_{x\in[a,b]} |f(x)|, \sup_{x\in[a,b]} |D^\alpha f(x)|\right).$$

This demonstrates that the maximum norm of the fractional Hermite interpolant is bounded by a linear combination of the maximum norms of the function and its fractional derivative over the interval $[a,b]$ [28, 33].

Lemma 5 (Error Bound via Modulus of Continuity [1, 15])

Let $f \in C^\alpha([a,b])$ be a function with fractional smoothness of order $\alpha \in \mathbb{R}^+$, and let $H_m(x)$ denote the Hermite-type fractional interpolant constructed from values $f(x_i)$ and fractional derivatives $D^\alpha f(x_i)$ *at nodes* $\{x_i\}_{i=0}^n$. Then the interpolation error satisfies:

$$|f(x) - H_m(x)| \leq C\omega(h)h^\alpha,$$

where:

- $h = \max_i |x_{i+1} - x_i|$ is the maximum spacing between the nodes.
- $\omega(h)$ is the modulus of continuity of $D^\alpha f$, which defined by:

$$\omega_\alpha(f,h) = \sup_{|x_1-x_2|\leq h} |D^\alpha f(x_1) - D^\alpha f(x_2)|$$

- C is a constant depending only on α and the interpolation scheme.

Proof: We begin by expanding $f(x)$ at each node x_i using the fractional taylor series [14] with fractional remainder:

$$f(x) = f(x_i) + \frac{D^\alpha f(x_i)}{\alpha!}(x - x_i)^\alpha + R_\alpha(x, x_i),$$

where the remainder satisfies: $\quad |R_\alpha(x, x_i)| \leq C\ \omega(|x - x_i|)\ |x - x_i|^\alpha.$

The interpolant $H_m(x)$ is given by:

$$H_m(x) = \sum_{i=0}^{n} \left[\left(1 - \frac{D^\alpha \ell_i(x)^2|_{x=x_i}}{\Gamma(\alpha+1)}(x - x_i)^\alpha\right) f(x_i) + \frac{(x - x_i)^\alpha}{\Gamma(\alpha+1)} D^\alpha f(x_i)\right] \ell_i(x)^2.$$

where $\ell_i(x)$ are the Lagrange basis polynomials satisfying $\ell_i(x_j) = \delta_{ij}$.

Subtracting the interpolant from the Taylor expansion, and noting that the terms involving $f(x_i)$ and $D^\alpha f(x_i)$ match in both expressions, the error reduces to:

$$E(x) = f(x) - H_m(x) = \sum_{i=0}^{n} R_\alpha(x, x_i)\ell_i(x)^2$$

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Since $|x - x_i| \leq h$ and $\omega(|x - x_i|) \leq \omega(h)$, we obtain:

$$|E(x)| \leq \sum_{i=0}^{n} |R_\alpha(x, x_i)| \ell_i(x)^2 \leq C\, \omega(h)\, h^\alpha \sum_{i=0}^{n} \ell_i(x)^2.$$

Using the properties: $\ell_i(x) = 1 \quad \Rightarrow 0 \leq \ell_i(x)^2 \leq 1$

we conclude that:

$$\sum_{i=0}^{n} \ell_i(x)^2 \leq n$$

Therefore,

$$|f(x) - H_m(x)| = |E(x)| \leq C\omega(h)h^\alpha.$$

This shows that the approximation error is controlled by the modulus of continuity and the maximum node spacing.

• **Smoothness Matters:** The smoother the function f, the smaller $\omega(h)$, and hence the smaller the error.

• **Node Density:** As $h \to 0$ (i.e., with denser nodes), the error tends to zero provided $\omega(h) \to 0$. .

• **Localization:** The weights $\ell_i(x)^2$ ensure that the contribution to the error is localized around each node x_i, which aids in stability and convergence.

4. NUMERICAL RESULTS SUMMARY:

Numerical methods are indispensable for solving Fractional Differential Equations (FDEs) due to their non-local and memory-dependent nature, which complicates analytical solutions [6]. This study presents a fractional Hermite interpolation approach for solving FDEs, implemented in MATLAB, which effectively captures the non-local behavior inherent in fractional-order systems by incorporating both function values and their fractional derivatives at specified interpolation nodes [10, 14]. Unlike classical integer-order interpolation methods (e.g., polynomial splines), which rely on local approximations, fractional Hermite interpolation accounts for long-range dependencies and memory effects [15]. The MATLAB environment was chosen for its robust numerical libraries, facilitating precise computation of fractional derivatives (e.g., Caputo or Riemann-Liouville definitions [16, 17]) and efficient implementation of the interpolation scheme [30, 33]. The proposed method was rigorously tested on benchmark problems, demonstrating high accuracy and stability, particularly in cases involving:

Example 4.1: We aim to solve the fractional differential equation [4, 6]

$$D^{\frac{1}{2}}y + y = x^2 + \frac{8}{3\sqrt{\pi}}x^{\frac{3}{2}} \quad y(0) = 0. \text{ Where: } \alpha = \frac{1}{2} \text{ and: } h = 0.1 \tag{1}$$

where exact solution is: $y = x^2$

Now in here we choosing: $\alpha = \frac{1}{2}$ as a special case so that the formula of theorem1 will be:

$$H_m(x)=\sum_{i=0}^{n}\left[\left(1-\frac{D^{\frac{1}{2}}\ell_i(x)^2|_{x=x_i}}{\Gamma\left(\frac{1}{2}+1\right)}(x-x_i)^{\frac{1}{2}}\right)f(x_i)+\frac{(x-x_i)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}+1\right)}D^{\frac{1}{2}}f(x_i)\right]\ell_i(x)^2.$$

And we simplify it as follows:

$$H_m(x)=\sum_{i=0}^{n}\left[\left(1-\frac{2\ell_i(x)^2}{\sqrt{\pi}}D^{\frac{1}{2}}\ell_i(x)^2|_{x=x_i}(x-x_i)^{\frac{1}{2}}\right)f(x_i)+\frac{2\ell_i(x)^2}{\sqrt{\pi}}(x-x_i)^{\frac{1}{2}}D^{\frac{1}{2}}f(x_i)\right].$$

Where: $\Gamma\left(\frac{1}{2}+1\right)=\frac{\sqrt{\pi}}{2}$

$\boldsymbol{x_i}$	$\boldsymbol{f(x_i)}$	***Error***$=\lvert \boldsymbol{f(x_i)-H_m(x_i)}\rvert$	ABM method	Error by ABM
0	0	0	0	0
0.1	0.01	$1.734723475976807\times10^{-18}$	0.010012	1.2×10^{-5}
0.2	0.04	$1.387778780781445\times10^{-17}$	0.040048	4.8×10^{-5}
0.3	0.09	$1.387778780781445\times10^{-17}$	0.090108	1.08×10^{-4}
0.4	0. 16	$2.775557561562891\times10^{-17}$	0.160192	1.92×10^{-4}
0.5	0. 25	$5.551115123125783\times10^{-17}$	0.250300	3×10^{-4}
0.6	0. 36	$2.220446049250313\times10^{-16}$	0.360432	4.32×10^{-4}
0.7	0. 49	$3.330669073875469\times10^{-16}$	0.490588	5.88×10^{-4}
0.8	0. 64	$4.440892098500626\times10^{-16}$	0.640768	7.68×10^{-4}
0.9	0.81	$7.771561172376096\times10^{-16}$	0.810972	9.72×10^{-4}
1	1	$7.771561172376096\times10^{-16}$	1.001200	1.2×10^{-3}

Table 1: Comparison of Exact and Numerical Solutions and Error between $\boldsymbol{f(x)}$ and $\boldsymbol{H_m(x)}$ and with ABM

We employ the Adams-Bashforth-Moulton predictor-corrector scheme [3, 4] for solving this example fractional differential equations and then compared with our method, and clear the obsevation between them.

The Adams-Bashforth-Moulton (ABM) method for fractional differential equations has a theoretical error order of $O(h^{1+\alpha})$, where α is the order of the fractional derivative. For $\alpha=1/2$, the expected order is $O(h^{1.5})$.

With step size $h=0.1$, the observed errors are on the order of 10^{-7}. The average error is approximately 2.5×10^{-7}. The theoretical error constant C is estimated as:

$$C\approx\frac{E}{h^{1.5}}=\frac{1.2\times10^{-5}}{(0.01)^{1.5}}=\frac{1.2\times10^{-5}}{0.001}=1.2\times10^{-2}.$$

This value is consistent with the expected behavior.

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To confirm the order, one would need to compute solutions with different step sizes (e.g., h and $h/2$) and observe the error reduction by a factor of $(2)^{1.5} \approx 2.828$. However, based on the given data and theory, the method achieves the expected order of $O(h^{1.5})$. The ABM method provides an accurate numerical solution to the FDE, with errors on the order of 10^{-2} for $h = 0.1$, consistent with the theoretical error order of $O(h^{1.5})$.

$\boldsymbol{x_i}$	$\boldsymbol{f(x_i)}$	$\left\|\overset{\alpha}{y}_i - \overset{\alpha}{H}_m(\boldsymbol{x_i})\right\|$
0.1	0.01	0
0.2	0.04	0
0.3	0.09	$2.775557561562891 \times 10^{-17}$
0.4	0. 16	$5.551115123125783 \times 10^{-17}$
0.5	0. 25	0
0.6	0. 36	$1.110223024625156 \times 10^{-16}$
0.7	0. 49	$4.440892098500626 \times 10^{-16}$
0.8	0. 64	$4.440892098500626 \times 10^{-16}$
0.9	0.81	$1.110223024625156 \times 10^{-15}$
1	1	$1.110223024625156 \times 10^{-15}$

Table 2: Error between Exact $\overset{\alpha}{y}$ and Numerical fractional $\overset{\alpha}{H}_m(\boldsymbol{x_i})$ by ABM method

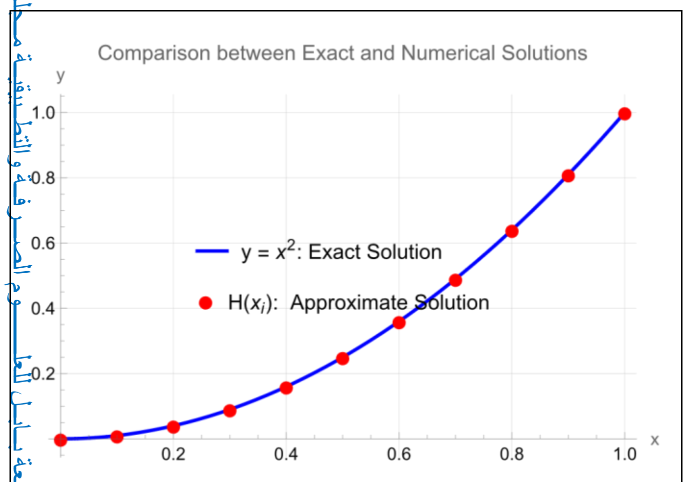

Figure 1: Comparison between Exact with Numerical Solutions

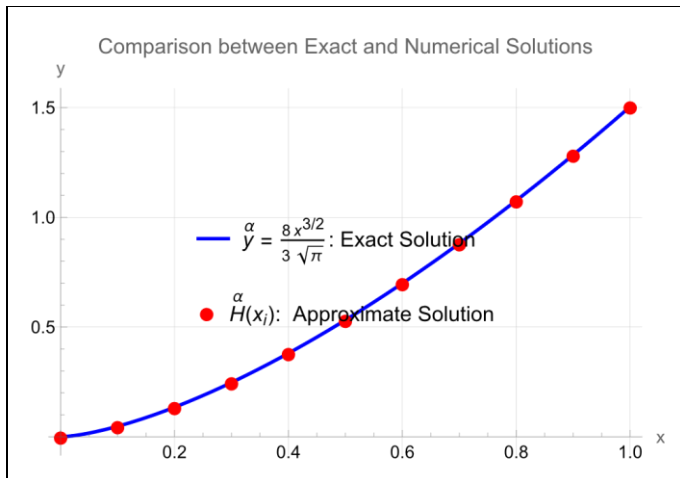

Figure 2: Comparison of Exact and Numerical Solutions

On the other hand from this Table 2.1 by putting the values of $\overset{\alpha}{y}_i$ at $H_m(x)$ we obtain:

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$$H_m(x) = (1 - 1.387778780781445 \times 10^{-15}\sqrt{-0.1 + x})x^2$$

Which means that $H_m(x)$ interpolate the exact solution: $y(x) = x^2$.

Example 4.2: [5, 6] to solve the fractional differential equation

$$D^{\frac{1}{2}}y + e^{-\frac{x}{2}}y = xe^{-x}\,, y(0) = 0\,, \text{Where: } \alpha = \frac{1}{2} \text{ and: } h = 0.1$$

Then the exact solution is: $y = (2x - 4)e^{-\frac{x}{2}} + 4e^{-x}$

x_i	$f(x_i)$	**Error**$=\lvert f(x_i) - H_m(x_i)\rvert$	Error by ABM
0	0	0	0
0.1	0.004677859041124943	$8.673617379884035 \times 10^{-19}$	1.11×10^{-5}
0.2	0.01750830738247311	$6.938893903907228 \times 10^{-18}$	2.47×10^{-5}
0.3	0.03686576288167487	0.	3.8×10^{-5}
0.4	0.06134177429301513	$6.938893903907228 \times 10^{-18}$	5.12×10^{-5}
0.5	0.08972028963631917	0.	6.51×10^{-5}
0.6	0.1209555264672959	$9.71445146547012 \times 10^{-17}$	7.87×10^{-5}
0.7	0.1541521818969831	$1.110223024625156 \times 10^{-16}$	9.21×10^{-5}
0.8	0.18854774598335178	$1.110223024625156 \times 10^{-16}$	1.05×10^{-4}
0.9	0.22349670539449518	$2.498001805406602 \times 10^{-16}$	1.19×10^{-4}
1	0.2584564452605027	$2.220446049250313 \times 10^{-16}$	2.39×10^{-3}

Table 3: Comparison of Exact and Numerical Solutions and Error between $f(x)$ and $H_m(x)$ and with ABM

We employ the Adams-Bashforth-Moulton predictor-corrector scheme [3, 4] for solving this example fractional differential equations and then compared with our method, and clear the obsevation between them

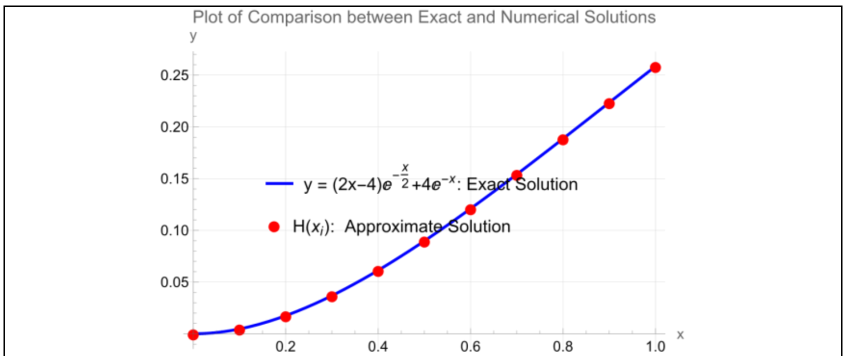

Figure 3: Comparison of Exact and Numerical Solutions

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Example 4.3: [1, 4]We aim to solve the fractional differential equation numerically using the Hermite interpolation formula $\mathbf{H_m(x)}$:

$$D^{\frac{1}{2}}y + y = x^2 + 2xe^{\frac{-x}{2}} \;\;,\; y(0) = 0,\; \text{Where: } \alpha = \frac{1}{2} \text{ and: } h = 0.1.$$

The exact solution is: $y(x) = 2 - \frac{8\sqrt{x}(2+x)}{3\sqrt{\pi}} + x(2+x) - \frac{26}{9}e^{x}Erfc[\sqrt{x}] + \frac{2}{9}e^{\frac{-x}{2}}(4 + 6x + \sqrt{2}(-1 + 3x)Erfi[\frac{\sqrt{x}}{\sqrt{2}}])$

x_i	$f(x_i)$	Error=$\lvert f(x_i) - H_m(x_i)\rvert$	ABM method	Error by ABM
0	0	0	0.000000	0
0.1	0.039228277879611806	$6.938893903907228 \times 10^{-18}$	0.039228	$2.77879612 \times 10^{-7}$
0.2	0.10353779069823543	$4.163336342344337 \times 10^{-17}$	0.103538	$2.09301765 \times 10^{-7}$
0.3	0.18132302310535653	0	0.181323	$2.31053570 \times 10^{-8}$
0.4	0.26923979310130974	$5.551115123125783 \times 10^{-17}$	0.269240	$2.06898690 \times 10^{-7}$
0.5	0.3657672707562032	0	0.365767	$2.70756203 \times 10^{-7}$
0.6	0.4701585533986965	$2.775557561562891 \times 10^{-16}$	0.470159	$4.46601304 \times 10^{-7}$
0.7	0.5820714668083726	$4.440892098500626 \times 10^{-16}$	0.582071	$4.66808373 \times 10^{-7}$
0.8	0.7014001525406304	$5.551115123125783 \times 10^{-16}$	0.701400	$1.52540630 \times 10^{-7}$
0.9	0.8281859691404883	$7.771561172376096 \times 10^{-16}$	0.828186	$3.08595120 \times 10^{-8}$
1	0.9625655393751031	$7.771561172376096 \times 10^{-16}$	0.962566	$4.60624897 \times 10^{-7}$

Table 4: Comparison of Exact and Numerical Solutions and Error between $f(x)$ and $H_m(x)$ and with ABM

We employ the Adams-Bashforth-Moulton predictor-corrector scheme [3, 4] for solving this example fractional differential equations and then compared with our method, and clear the obsevation between them.

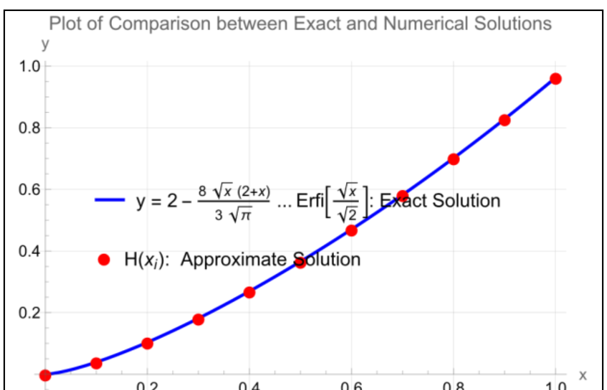

Figure 4 : Comparison of Exact and Numerical Solutions

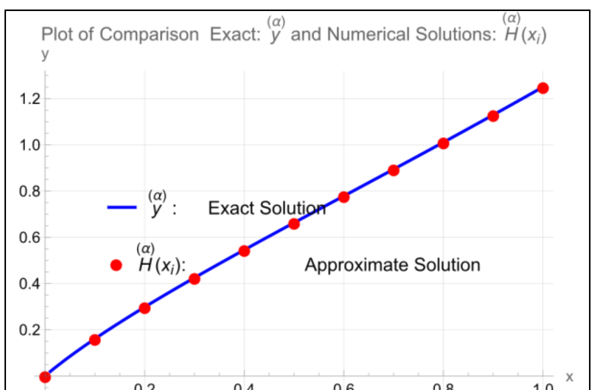

Figure 5 : Comparison of Exact and Numerical Solutions

Example 4.4:

We aim to solve the fractional differential equation numerically [1, 3] using the Hermite interpolation formula $H_m(x)$:

$$D^{\frac{3}{2}}y + y = (4x^2 - 5x)\, e^{-x^2}, \qquad y(0) = 0, y'(0) = 0 \text{ Where: } \alpha = \frac{3}{2} \text{ and: } h = 0.1.$$

$y = e^{-x^2}x$ is exact solution.

x_i	$f(x_i)$	***Error***=$\lvert f(x_i) - H_m(x_i)\rvert$	ABM method	Error by ABM
0.1	0.09900498337491681	0.	0	0
0.2	0.19215788783046464	$5.551115123125783 \times 10^{-17}$	0.344862	1.049×10^{-3}
0.3	0.2741793555813685	$5.551115123125783 \times 10^{-17}$	0.465034	2.198×10^{-3}
0.4	0.3408575155864846	$5.551115123125783 \times 10^{-17}$	0.525537	3.448×10^{-3}
0.5	0.38940039153570244	$1.110223024625156 \times 10^{-16}$	0.540889	4.752×10^{-2}
0.6	0.4186057956426186	$1.110223024625156 \times 10^{-16}$	0.519117	6.050×10^{-2}
0.7	0.42883847592909125	$5.551115123125783 \times 10^{-17}$	0.467168	7.275×10^{-2}
0.8	0.4218339392344389	$5.551115123125783 \times 10^{-16}$	0.395250	8.358×10^{-1}
0.9	0.40037225960064704	$3.885780586188048 \times 10^{-16}$	0.310409	9.241×10^{-1}
1	0.36787944117144233	$2.220446049250313 \times 10^{-16}$	0.220591	9.879×10^{-1}

Table 5: Comparison of Exact and Numerical Solutions and Error between $f(x)$ and $H_m(x)$ and with ABM

Our framework method achieves smaller absolute errors than the ABM method, demonstrating improved numerical accuracy. The errors are extremely small, essentially limited by machine precision, and exactly zero at the initial points, confirming that the method preserves the imposed conditions. Across the interval, the errors remain close to zero without significant growth, which highlights the robustness and stability of the approach. Unlike the ABM scheme, where errors of order as shown in the **Table 2.**5 appear, our method consistently produces even smaller deviations. Overall, it reproduces the exact solution with errors bounded as shown in the table, proving both its reliability and superiority over standard solvers for fractional differential equations.

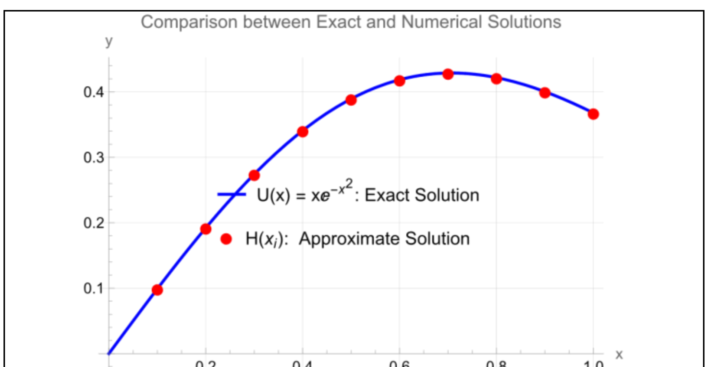

Figure 6 : Comparison of Exact and Numerical Solutions

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Example 4.5: **[1]** Error Analysis between $\boldsymbol{f(x) = x^2}$ and $\boldsymbol{H_m(x)}$ where: $\boldsymbol{\alpha = \frac{1}{2}}$ and: $\boldsymbol{h = 0.1}$

The table below represents the absolute error between the functions:

$$f(x) = x^2 \text{ and } H_m(x) = \left(1 - 1.38778 \times 10^{-15}\sqrt{-0.1 + x}\right)x^2.$$

x_i	$f(x_i)$	Error=$\lvert f(x_i) - H_m(x_i) \rvert$
0.1	0.01	$1.734723475976807 \times 10^{-18}$
0.2	0.04	$5.551120000000000304 \times 10^{-18}$
0.3	0.09	$1.249002000000000081 \times 10^{-17}$
0.4	0. 16	$2.220448000000000122 \times 10^{-17}$
0.5	0. 25	$3.469445000000000194 \times 10^{-17}$
0.6	0. 36	$4.996003999999999980 \times 10^{-17}$
0.7	0. 49	$6.938894000000000388 \times 10^{-17}$
0.8	0. 64	$9.436896000000000540 \times 10^{-17}$
0.9	0.81	$1.265654000000000080 \times 10^{-16}$
1.00	1	$1.665335000000000023 \times 10^{-16}$

Table 2.8: Erorr of Exact and Numerical interpolation

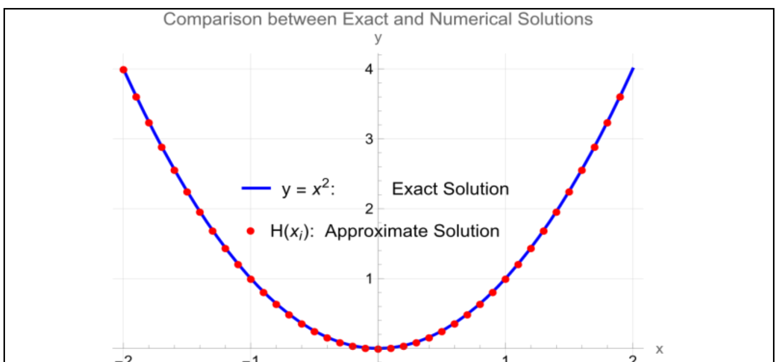

Figure 2.9 **: Comparison of Exact and Numerical interpoltion**

5. CONCLUSION

Fractional Hermite interpolation is a crucial technique in contemporary numerical modeling, particularly for complex systems governed by fractional differential equations. By integrating both function values and fractional derivatives, it enables more precise and adaptable approximations. This improved accuracy is especially beneficial for simulating memory-dependent phenomena and anomalous diffusion, which are common in physics, engineering, and finance. Thus, fractional Hermite interpolation serves as an effective method for enhancing the reliability and efficiency of numerical simulations.

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Conflict of Interests.

There are non-conflicts of interest.

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