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A New Subclass of Univalent Functions Using Chebyshev Polynomials

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فئة جزئية جديدة من الدوال أحادية التكافؤ باستخدام متعددات الحدود تشيبيشيف

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ABSTRACT

Background:

Geometric Function Theory (GFT) is a vital area in complex analysis, offering powerful techniques for solving physical and engineering problems. Particularly in two-dimensional problems, isomorphic mappings play a crucial role in simplifying complex regions, enabling more tractable analysis.

Materials and Methods:

This study investigates key topics within GFT, focusing on a q-generalized subclass of univalent functions $K(\delta, t, q)$. Chebyshev polynomials of the second kind are employed to estimate bounds for the coefficients of functions in this subclass. Theoretical tools from mathematical analysis and quantum calculus are integrated into the study to support this exploration.

Results:

We derive estimate coefficient bounds for functions in the $K(\delta, t, q)$ class and provide estimations for two important functionals: the Fekete–Szegö functional and the second-order Hankel determinant. These findings offer deeper insight into the geometric properties of the considered function class.

Conclusion:

The work strengthens the connection between geometric function theory and quantum calculus by introducing and analyzing a new subclass of univalent functions. The methods and estimations developed here enhance the understanding of analytic function behavior and open pathways for future work in applied mathematics and engineering models.

Keywords:

Univalent Function, Fekete-Szegö functional, Chebyshev Polynomials, Second Hankel Determinant and Coefficient Bounds.

1. INTRODUCTION

Assume that $\boldsymbol{U} = \{t \in \mathbb{C};\ |t| < 1\}$ is an open unit disk, and $\mathcal{A}$ represents the class of analytical functions h, described on $\boldsymbol{U}$ and satisfying the standard normalization terms:

$$h\,(0) = 0 \text{ and } h'(0) = 1. \tag{1}$$

Thus, any function $h \in \mathcal{A}$ can be expressed by the following Taylor - Maclaurin series expansion:

$$h(t) = t + \sum_{i=2}^{\infty} b_i\, t^i. \tag{2}$$

We say that the analytic function h, defined in the domain $\boldsymbol{U}$, is univalent while it is injective in $\boldsymbol{U}$. The category $\mathcal{S}$ is a subset of $\mathcal{A}$ that contains all univalent functions inside $\mathcal{A}$. The Koebe function, defined by

$$K(t) = t(1-t)^{-2} = \frac{1}{4}\left[\left(\frac{1+t}{1-t}\right)^2 - 1\right], \quad (t \in \boldsymbol{U})$$

is the well-known univalent function in $\mathcal{S}$. Considering that h and k are two analytical functions in $\boldsymbol{U}$, in this case the function h is called subordinate to the function k (in symbol $h \prec k$) if there exists an analytical function ω in $\boldsymbol{U}$ such that $\omega(0) = 0$, $|\omega(t)| < 1$ and $h(t) = k(\omega(t))$ for each $t \in \boldsymbol{U}$ (ω is named a Schwarz function). Especially, while the function k is a univalent function in $\boldsymbol{U}$, in this case we obtain the following result:

$$h \prec k \text{ if and only if } h\,(0) = k\,(0) \text{ and } h(\boldsymbol{U}) \subset k(\boldsymbol{U}).$$

In 1916, Bieberbach [1] established that if $h \in \mathcal{S}$ and is expressed as in Eq. (2), then $|b_2| \leq 2$. Equality holds if and only if h corresponds to one of the rotations of the Koebe function or the Koebe function itself. This result served as the foundation for the renowned Bieberbach conjecture which is discussed below.

Bieberbach's Conjecture. [2] For any $h \in \mathcal{S}$ described by Eq. (2), $|b_i| \leq i$ for all integers $i \geq 2$. Equality occurs if and only if h is the Koebe function or one of its rotations.

The difficulties of proving Bieberbach's conjecture led many mathematicians to study subclasses of $\mathcal{S}$, such as convex functions, close -to- convex functions, and starlike functions for which sharp coefficient bounds can be derived. The Bieberbach conjecture remained unproven until de Branges discovered a proof in 1984 [2]. The subclasses

$$\mathcal{S}^* = \left\{ h \in \mathcal{A};\ Re\left(\frac{t h'(t)}{h(t)}\right) > 0; \quad t \in \boldsymbol{U} \right\}$$

is the standard classes of starlike functions in $\boldsymbol{U}$ and

$$C = \left\{ h \in \mathcal{A};\ Re\left(1 + \frac{t h''(t)}{h'(t)}\right) > 0;\ \ t \in \boldsymbol{U} \right\}$$

represents the standard classes of convex functions which is defined in $\boldsymbol{U}$. At first, these subclasses were introduced and presented by Shanmugam et al. [3] in a comprehensive method, which utilized properties of the convolution operator and subordination. Subsequently, Ma and Minda [4] derived coefficient problems for these general classes.

Inspired by the work of Ma and Minda, many researchers presented new ideas in the form of articles in this field, some of which we will mention. Bulut et al. [5] obtained the first two coefficient bounds of the functions in a novel category of univalent functions using the famous Chebyshev polynomials. Finding an upper bound for the Fekete-Szegö functional was another goal of their work. In 2022, Hameed Mohammed et al. [6] introduced a special subclass of normalized analytic functions using a differential inequality, and they studied some geometrical properties of it. Hameed Mohammed [7] obtained the upper bounds for the logarithmic coefficients of a certain category of univalent functions.

Since quantum calculus is used in a lot of fields of mathematics [8], study of it has attracted many mathematicians. If we combine quantum calculus with complex analysis methods, we can easily study and investigate many structures of geometric function theory, which makes $q-$calculus useful of the most other aspects in mathematics. Therefore, new subclasses of univalent functions have recently been introduced and investigated by some researchers using $q-$calculus; for example, see [8-11]. For additional literature on this topic, readers may consult the references cited in these works.

Assume that $0 < q < 1$, the q-number $[i]_q$ is defined following the work of [9]

$$[i]_q = \begin{cases} \frac{1-q^i}{1-q}; & i \in \mathbb{C}, \\ \sum_{j=0}^{i-1} q^j; & i \in \mathbb{N}, \\ 0; & i = 0. \end{cases}$$

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Also, for a function h belongs to $\mathcal{A}$ its q-derivative is defined by

$$D_q h(t) = \begin{cases} \frac{h(qt)-h(t)}{(q-1)z}; & t \neq 0, \\ h'(0); & t = 0, \end{cases}$$

for $t \in \boldsymbol{U}$. We note that $\lim\limits_{q\to 1^-} \mathrm{D}_q h\,(t) = h'(t)$, if h is differentiable in $\boldsymbol{U}$.

For $h(t)$ given by Eq. (2), the q-derivative can be extracted as

$$D_q h(t) = 1 + \sum_{i=2}^{\infty} [i]_q b_i t^{i-1}, \quad (t \in \boldsymbol{U}).$$

If h and k belong to the class $\mathcal{A}$, the following identities apply to the q-difference operator D_q [10].

a) $D_q\big(h\,(t)k(t)\big) = h\,(qt)D_q k(t) + k(t)D_q\,h\,(t),$

b) $D_q\left(\gamma h\,(t) \pm \sigma\,k(t)\right) = \gamma D_q\,h\,(t) \pm \sigma\,D_q k(t),\ for\ \gamma, \sigma\ \in \mathbb{C} \setminus \{0\}.$

Another interesting subject worth mentioning is the Henkel determinant. For $h \in \mathcal{A}$ Noonan and Thomas in 1976 introduced the m^{th} determinant for $m > 1$ and $i \geq 0$ as [11]

$$H_m(i) = \begin{vmatrix} b_i & b_{i+1} & \cdots & b_{i+m-1} \\ b_{i+1} & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \vdots \\ b_{i+m-1} & \cdots & \cdots & b_{i+2(m-1)} \end{vmatrix}.$$

This determinant has been explored by a lot of studies. For instance, Noor [12] determined the growth rate of $H_m(i)$ in the case $i \to \infty$ for the function h which is given by Eq. (2) Researcher Ehrenborg investigated Hankel determinant for the functions of the form of exponential polynomials [13]. As everyone knows for $h \in \mathcal{S}$ that is given by Eq. (2) the inequality $|b_3 - b_2^2| \leq 1$ is sharp [1]. This accords with the Hankel determinant with $m = 2$ and $i = 1$. It has been clear that

$$H_2(1) = \begin{vmatrix} b_1 & b_2 \\ b_2 & b_3 \end{vmatrix}$$

and

$$H_2(2) = \begin{vmatrix} b_2 & b_3 \\ b_3 & b_4 \end{vmatrix},$$

where the Hankel determinant $H_2(1) = b_3 - b_2^2$ is called Fekete-Szegö and $H_2(2) = b_2 b_4 - b_3^2$ is said to be the second Hankel determinant. In 1933, Fekete and Szegö refuted Littlewood and Paley the conjecture, which claimed that the coefficients for odd univalent functions are bounded by one. Additionally, the Fekete-Szegö functional $|b_3 - \psi b_2^2|$ for normalized univalent functions $h \in \mathcal{A}$ is illustrious for its extensive report in the field of geometric function theory [14]. For more about the Fekete-Szegö functional see [15, 16]. The estimating of $|H_2(2)|$ has been the focus of new Hankel determinant researches for instance, the sharp bounds of $H_2(2)$ were given by many authors [17-19] for various subclasses of univalent functions. Lee et al. [20] have presented a summary overview of the Hankel determinants of univalent functions. Also, they have gotten bounds for $H_2(2)$ for the functions which belong to some categories defined by using subordination.

Chebyshev polynomials have gained significant importance in the numerical analysis, because of their theoretical insights and also practical applications. They are categorized into four kinds. The first kind, $T_i(x)$ and the second kind $\mathfrak{U}_i(x)$, and the applications of them are the main subject of many books and research articles in the field of orthogonal polynomials, Doha [21]and Mason [22]. For any real variable x in the open interval $(-1, 1)$, the above two types of Chebyshev polynomials are defined as follows:

$$T_i(x) = \cos i\theta,$$

$$\mathfrak{U}_i(x) = \frac{\sin(i+1)\theta}{\sin\theta},$$

where $x = \cos\theta$ and i denotes the degree of the polynomial.

Definition 1.1. The function $h \in \mathcal{A}$ is said to belong to the class $K(\delta, t, q)$, such that $\delta \geq 0$, $t \in (\frac{1}{2}, 1]$ and $0 < q < 1$, if the following subordination condition holds:

$$(1-\delta)\left(\frac{\mathscr{t} D_q h(\mathscr{t})}{h(\mathscr{t})}\right) + \delta\left(1 + \frac{\mathscr{t} D_q\big(D_q h(\mathscr{t})\big)}{D_q h(\mathscr{t})}\right) \prec H(\mathscr{t}, t) := \frac{1}{1 - 2t\mathscr{t} + \mathscr{t}^2} \quad (\mathscr{t} \in \boldsymbol{U}). \tag{3}$$

We observe that if $t = cos\beta,\ \beta \in \left(\frac{-\pi}{3}, \frac{\pi}{3}\right)$, then

$$H(\mathscr{t}, t) = \frac{1}{1-2t\mathscr{t}+\mathscr{t}^2} = 1 + \sum_{i=1}^{\infty} \frac{\sin\big((i+1)\beta\big)}{sin\beta} \mathscr{t}^i, \quad (\mathscr{t} \in \boldsymbol{U}).$$

Therefore,

$$H(\mathscr{t},t) = 1 + 2\cos\beta\,\mathscr{t} + (3\,cos^2\beta - sin^2\beta)\mathscr{t}^2 + (4\,cos^3\beta - 4sin^2\beta\cos\beta)\mathscr{t}^3 + \cdots (\mathscr{t} \in \boldsymbol{U}).$$

Following [15], we can write

$$H(\mathscr{t},t) = 1 + \mathfrak{U}_1(t)\mathscr{t} + \mathfrak{U}_2(t)\mathscr{t}^2 + \mathfrak{U}_3(t)\mathscr{t}^3 + \cdots \qquad \big(\mathscr{t} \in \boldsymbol{U},\ \ t \in (-1,1)\big),$$

where, $\mathfrak{U}_{i-1} = \frac{\sin(i\arccos t)}{\sqrt{1-t^2}}$ $(\mathrm{i} \in \mathbb{N})$ are the Chebyshev polynomials of the second-kind. Furthermore, it is noted that

$$\mathfrak{U}_i(\mathrm{t}) = 2\mathrm{t}\,\mathfrak{U}_{i-1}(\mathrm{t}) - \mathfrak{U}_{i-2}(\mathrm{t}),$$

and

$$\mathfrak{U}_1(\mathrm{t}) = 2\mathrm{t},$$

$$\mathfrak{U}_2(\mathrm{t}) = 4t^2 - 1,$$

$$\mathfrak{U}_3(\mathrm{t}) = 8t^3 - 4t, \ldots . \tag{4}$$

The first kind of Chebyshev polynomials, $T_i(t)$, have the following generating function

$$\sum_{i=0}^{\infty} T_i(t)\mathscr{t}^i = \frac{1-t\mathscr{t}}{1-2t\mathscr{t}+\mathscr{t}^2}, \quad (\mathscr{t} \in \boldsymbol{U}).$$

Furthermore, the above mentioned two kinds of Chebyshev polynomials are closely related through the following relationships:

$$\frac{dT_i(t)}{dt} = \mathrm{i}\mathfrak{U}_{i-1}(\mathrm{t}),$$

$$T_i(t) = \mathfrak{U}_i(\mathrm{t}) - \mathrm{t}\mathfrak{U}_{i-1}(\mathrm{t}),$$

$$2T_i(t) = \mathfrak{U}_i(\mathrm{t}) - \mathfrak{U}_{i-2}(\mathrm{t}).$$

In this article, inspired by the papers of Dziok et al. [23] and Şahsene et al. [15], we employ Chebyshev polynomial expansions to estimate the bounds for the coefficients and for the Fekete - Szegö functional of univalent functions belonging to $K(\delta,t,q)$. It should also be mentioned that obtaining an interesting upper bound for second Hankel determinant associated with this subclass was inspired by the techniques of Lee et al. [20].

Lemma 1.2. [24]. Let $\omega(t) = d_1 t + d_2 t^2 + d_3 t^3 + \cdots$ be any Schwarz function that is defined in the unit disk $\boldsymbol{U}$. Then

$$|d_j| \leq 1, \ for\ all\ j \in \mathbb{N}, \text{and} \tag{5}$$

$$|d_2 - \psi\, d_1^2| \leq \max\{1, |\psi|\}, \ \text{for all } \psi \in \mathbb{R}. \tag{6}$$

Lemma 1.3. [25]. If $p(t) = \sum_{i=1}^{\infty} d_i t^i$, $t \in \boldsymbol{U}$ is any Schwarz function with $d_1 \in \mathbb{R}$, then

$$d_2 = x(1 - d_1^2),$$

$$d_3 = (1 - d_1^2)(1 - |x|^2)s - (1 - d_1^2)d_1 x^2,$$

for some s, x, such that $|s| \leq 1$ and $|x| \leq 1$.

2. MAIN RESULTS

2.1. The Bounds of the Coefficients for the Function Class $K(\delta, t, q)$

Theorem 2.1. If the function $h(t)$ which is given by Eq. (2) belongs to the class $K(\delta, t, q)$, then

$$|b_2| \leq \frac{2t}{A},\ |b_3| \leq \frac{2t}{B} + \mu \text{ ,and } |b_4| \leq \frac{2}{C}\left(4t^3\left(1 - \frac{E}{A^3}\right) + 2t^2\left(2 + \frac{D}{AB}\right) + t\left(\frac{\mu D}{A} - 1\right) - 1\right),$$

where,

$$\mathrm{A} = [2]_q - 1 + \delta,$$

$$\mathrm{B} = \big([3]_q - 1\big)(1 - \delta) + [2]_q[3]_q\delta,$$

$$C = \big([4]_q - 1\big)(1 - \delta) + [4]_q[3]_q\delta,$$

$$D = \big([2]_q + [3]_q - 2\big)(1 - \delta) + [2]_q[3]_q\delta(1 + [2]_q),$$

$$E = \big([2]_q - 1\big)(1 - \delta) + [2]_q^3\delta$$

and

$$\mu = \frac{4t^2\Big([2]_q^2(1+\delta) + (\delta - 1)\big(\delta + [2]_q\big)\Big)}{A^2 B} - \frac{1}{B}.$$

We note that the value of μ depends on both A and B.

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Proof. Since $h \in K(\delta, t, q)$ then from Eq. (3) this research concluded that there is an analytic function ω given by

$$\omega(t) = d_1 t + d_2 t^2 + d_3 t^3 + \cdots, \quad (t \in \boldsymbol{U})$$

where, $\omega(0) = 0$, $|\omega(t)| < 1$ for all $t \in \boldsymbol{U}$ and

$$(1-\delta)\left(\frac{tD_q h(t)}{h(t)}\right) + \delta\left(1 + \frac{tD_q\big(D_q h(t)\big)}{D_q h(t)}\right) = 1 + \mathfrak{A}_1(t)\omega(t) + \mathfrak{A}_2(t)\omega^2(t) + \mathfrak{A}_3(t)\omega^3(t) + \cdots \tag{7}$$

Eq. (7) allows us to derive that

$$(1 - \delta)\left(\frac{tD_q h(t)}{h(t)}\right) + \delta\left(1 + \frac{tD_q\big(D_q h(t)\big)}{D_q h(t)}\right) = 1 + \mathfrak{A}_1(t)d_1 t + [\mathfrak{A}_1(t)d_2 + \mathfrak{A}_2(t)d_1^2]t^2 + [\mathfrak{A}_1(t)d_3 + 2\mathfrak{A}_2(t)d_1d_2 + \mathfrak{A}_3(t)d_1^3]t^3 + \cdots. \tag{8}$$

By comparing both sides of Eq. (8), it can be concluded that

$$\big([2]_q - 1 + \delta\big)b_2 = \mathfrak{A}_1(t)d_1, \tag{9}$$

$$\Big(\big([3]_q - 1\big)(1-\delta) + [2]_q\,[3]_q\,\delta\Big)\, b_3 + \Big(\big(1-[2]_q\big)(1-\delta) - [2]_q^2\,\delta\Big)\, b_2^2 = \mathfrak{A}_1(t)d_2 + \mathfrak{A}_2(t)d_1^2, \tag{10}$$

$$\Big(\big([4]_q - 1\big)(1-\delta) + [4]_q[3]_q\delta\Big)\, b_4 - \Big(\big([2]_q + [3]_q - 2\big)(1-\delta) + [2]_q[3]_q\delta\big(1 + [2]_q\big)\Big)\, b_2b_3 + \Big(\big([2]_q - 1\big)(1-\delta) + [2]_q^3\,\delta\Big)\, b_2^3 = \mathfrak{A}_1(t)d_3 + 2\mathfrak{A}_2(t)d_1d_2 + \mathfrak{A}_3(t)d_1^3. \tag{11}$$

By using Eq. (4) in Eq. (9) and then by applying Lemma 1.2, the following result is obtained

$$|b_2| \le \frac{2t}{A} \tag{12}$$

where,

$$A = [2]_q - 1 + \delta.$$

To determine the upper bound of $|b_3|$, we utilize Eq. (9) in Eq. (10) and then we get

$$Bb_3 = \mathfrak{A}_1(t)d_2 + \left(\mathfrak{A}_2(t) - \Big(\big(1-[2]_q\big)(1-\delta) - \big([2]_q\big)^2\delta\Big)\frac{(\mathfrak{A}_1(t))^2}{\mathrm{A}^2}\right) d_1^2. \tag{13}$$

Now, by using Eq. (4) in Eq. (13) and by applying Lemma 1.2 we have

$$|b_3| \leq \frac{2t}{B} + \mu$$

where,

$$\mathrm{B} = ([3]_q - 1)(1-\delta) + [2]_q[3]_q\delta$$

and

$$\mu = \frac{4t^2\left([2]_q^2(1+\delta)+(\delta-1)\left(\delta+[2]_q\right)\right)}{A^2B} - \frac{1}{B}.$$

At this step we want to find a bound for $|b_4|$. We utilize Eq. (9) and Eq. (10) in Eq. (11) and we obtain

$$Cb_4 = \mathfrak{A}_1(t)d_3 + 2\mathfrak{A}_2(t)d_1d_2 + \mathfrak{A}_3(t)d_1^3 + (Db_3 - Eb_2^2)b_2 \tag{14}$$

where,

$$C = \left([4]_q - 1\right)(1-\delta) + [4]_q[3]_q\delta,$$

$$D = \left([2]_q + [3]_q - 2\right)(1-\delta) + [2]_q[3]_q\delta(1+[2]_q) \text{ and}$$

$$E = \left([2]_q - 1\right)(1-\delta) + [2]_q^3\,\delta.$$

Then, by using Eq. (4) in Eq. (14) and applying Lemma 1.2 we have

$$|b_4| \leq \frac{2}{C}\left(4t^3\left(1-\frac{E}{A^3}\right) + 2t^2\left(2+\frac{D}{AB}\right) + t\left(\frac{\mu D}{A} - 1\right) - 1\right).$$

The theorem's proof comes to end.

Now, if we take $\delta = 1$ in the above theorem, then the following result is obtained.

Corollary 2.2. If $h \in K(t,q)$, then

$$|b_2| \leq \frac{2t}{[2]_q},$$

$$|b_3| \leq \frac{8t^2+2t-1}{[2]_q[3]_q},$$

and

$$|b_4| \leq \frac{2}{[2]_q[3]_q}\left(8t^3 + 6t^2 - 2t - 1 + \frac{8t^3+2t^2-t}{[2]_q}\right).$$

For δ = 0 we obtain another result.

Corollary 2.3. If $h \in K(\delta, t, q)$, then

$$|b_2| \leq \frac{2t}{[2]_q-1},$$

$$|b_3| \leq \frac{[2]_q(4t^2+2t-1)-2t+1}{([2]_q-1)([3]_q-1)},$$

and

$$|b_4| \leq \frac{2}{[4]_q-1}\left(4t^3\left(1-\frac{1}{([2]_q-1)^2}\right) + 2t^2\left(2+\frac{[2]_q+[3]_q-2}{([3]_q-1)([2]_q-1)}\right) + t\left(\frac{([2]_q+[3]_q-2)([2]_q(4t^2-1)+1)}{([3]_q-1)([2]_q-1)^2} - 1\right) - 1\right).$$

2.2. Fekete - Szegö Inequalities Related the Function Class $K(\delta, t, q)$

Theorem 2.2. Suppose that $h(t)$ belongs to the class $K(\delta, t, q)$, then

$$|b_3 - \psi b_2^2| \leq \begin{cases} \frac{2t}{B}; & \psi \in [\psi_1, \psi_2], \\ \frac{2t}{B}\left|\frac{4t^2-1}{2t} - \frac{\left((1-[2]_q)(1-\delta)-([2]_q)^2\delta\right)2t}{A^2} - \psi\frac{2tB}{A^2}\right|; & \psi \notin [\psi_1, \psi_2], \end{cases}$$

where,

$$\psi_1 = \frac{4t^2\left([2]_q\left([2]_q(1+\delta)-(1-\delta)\right)-(1-\delta)\delta\right)-(1+2t)A^2}{4t^2B},$$

$$\psi_2 = \frac{4t^2\left([2]_q\left([2]_q(1+\delta)-(1-\delta)\right)-(1-\delta)\delta\right)-(1-2t)A^2}{4t^2B},$$

and A and B are defined in Theorem 2.1.

Proof. Since $h \in K(\delta, t, q)$, then from Eq. (9) and Eq. (13) we have

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$$|b_3-\psi b_2^2| = \frac{\mathfrak{A}_1(t)}{B}\left|d_2+\left(\frac{\mathfrak{A}_2(t)}{\mathfrak{A}_1(t)}-\frac{\left((1-[2]_q)(1-\delta)-([2]_q)^2\delta\right)\mathfrak{A}_1(t)}{A^2}-\psi\frac{\mathfrak{A}_1(t)B}{A^2}\right)d_1^2\right|.$$

Considering that $\omega(t)$ is a Schwarz function then by applying Lemma 1.2 we conclude that

$$|b_3-\psi b_2^2| \le \frac{\mathfrak{A}_1(t)}{B}\max\left(1,\left|\frac{\mathfrak{A}_2(t)}{\mathfrak{A}_1(t)}-\frac{\left((1-[2]_q)(1-\delta)-([2]_q)^2\delta\right)\mathfrak{A}_1(t)}{A^2}-\psi\frac{\mathfrak{A}_1(t)B}{A^2}\right|\right). \tag{15}$$

Finally, using Eq. (4) in Eq. (15)

$$|b_3-\psi b_2^2| \le \frac{2t}{B}\max\left(1,\left|\frac{4t^2-1}{2t}-\frac{\left((1-[2]_q)(1-\delta)-([2]_q)^2\delta\right)2t}{A^2}-\psi\frac{2tB}{A^2}\right|\right).$$

Since $t > 0$, we have

$$\left|\frac{4t^2-1}{2t}-\frac{\left((1-[2]_q)(1-\delta)-([2]_q)^2\delta\right)2t}{A^2}-\psi\frac{2tB}{A^2}\right| \le 1$$

$$\Leftrightarrow \frac{4t^2\left([2]_q\left([2]_q(1+\delta)-(1-\delta)\right)-(1-\delta)\delta\right)-(1+2t)A^2}{4t^2B} \le \psi \le \frac{4t^2\left([2]_q\left([2]_q(1+\delta)-(1-\delta)\right)-(1-\delta)\delta\right)-(1-2t)A^2}{4t^2B}$$

$$\Leftrightarrow \psi_1 \le \psi \le \psi_2.$$

Now, if we take $\delta = 1$ in the previous theorem, then the following result is derived.

Corollary 2.4. If $h \in K(\delta,t,q)$, then

$$|b_3-\psi b_2^2| \le \begin{cases} \dfrac{2t}{[2]_q[3]_q}; & \psi \in [\psi_1,\psi_2], \\ \left|\dfrac{[2]_q(8t^2-1)-\psi 4t^2[3]_q}{[2]_q^2[3]_q}\right|; & \psi \notin [\psi_1,\psi_2], \end{cases}$$

If $\delta = 0$, then the following corollary is obtained.

Corollary 2.5. If $h \in K(\delta,t,q)$, then

$$|b_3-\psi b_2^2| \le \begin{cases} \dfrac{2t}{[3]_q-1}; & \psi \in [\psi_1,\psi_2], \\ \dfrac{2t}{[3]_q-1}\left|\dfrac{4t^2-1}{2t}+\dfrac{2t}{[2]_q-1}\left(1-\psi\dfrac{[3]_q-1}{[2]_q-1}\right)\right|; & \psi \notin [\psi_1,\psi_2] \end{cases}$$

For $\psi = 1$, in Theorem 2.2, we have

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Corollary 2.6. If $h \in K(\delta, t, q)$, then

$$|b_3 - b_2^2| \leq \frac{2t}{B}\left|\frac{4t^2-1}{2t} - \frac{\left((1-[2]_q)(1-\delta)-([2]_q)^2\delta\right)2t}{A^2} - \frac{2tB}{A^2}\right|$$

where, A and B are defined in Theorem 2.1.

2.3. Second-Hankel Determinant Inequalities for the Function Class $K(\delta, t, q)$.

Theorem 2.3. Assume that $h(t)$ belongs to the class $K(\delta, t, q)$, which was defined by Eq. (2).

(i) If $\frac{4t}{AC}\left(2t^2\left(2+\frac{D}{AB}\right)-1+t\right)-\frac{4t}{B}\left(\mu+\frac{2t}{B}\right) \leq 0$ and

$$\frac{4t}{AC}\left(4t^3\left(1-\frac{E}{A^3}\right)-2t^2\left(\frac{D}{AB}+2\right)-t\left(3-\frac{D}{A}\mu\right)+1\right)+\frac{4t^2}{B^2}-\mu\left(\mu-\frac{4t}{B}\right) \leq -\frac{t}{AC}\left(2t^2\left(2+\frac{D}{AB}\right)-1+t\right)+\frac{4t}{B}\left(\mu+\frac{2t}{B}\right),$$

then $|b_2 b_4 - b_3^2| \leq \frac{4t^2}{B^2}$.

(ii) If $\frac{4t}{AC}\left(2t^2\left(2+\frac{D}{AB}\right)-1+t\right)-\frac{4t}{B}\left(\mu+\frac{2t}{B}\right) \geq 0$ and $\frac{4t}{AC}\left(4t^3\left(1-\frac{E}{A^3}\right)-2t^2\left(\frac{D}{AB}+2\right)-t\left(3-\frac{D}{A}\mu\right)+1\right)+\frac{4t^2}{B^2}-\mu\left(\mu-\frac{4t}{B}\right) \geq -\frac{t}{2AC}\left(2t^2\left(2+\frac{D}{AB}\right)-1+t\right)+\frac{t}{2B}\left(\mu+\frac{2t}{B}\right)$

or

$$\frac{4t}{AC}\left(2t^2\left(2+\frac{D}{AB}\right)-1+t\right)-\frac{4t}{B}\left(\mu+\frac{2t}{B}\right) \leq 0 \quad \text{and} \quad \frac{4t}{AC}\left(4t^3\left(1-\frac{E}{A^3}\right)-2t^2\left(\frac{D}{AB}+2\right)-t\left(3-\frac{D}{A}\mu\right)+1\right)+\frac{4t^2}{B^2}-\mu\left(\mu-\frac{4t}{B}\right) \geq -\frac{t}{AC}\left(2t^2\left(2+\frac{D}{AB}\right)-1+t\right)+\frac{t}{B}\left(\mu+\frac{2t}{B}\right)$$

then

$$|b_2 b_4 - b_3^2| \leq \frac{16t}{AC}\left[16t^3\left(1-\frac{E}{A^3}\right)-6t^2\left(\frac{D}{AB}+2\right)-t\left(11-\frac{4D}{A}\mu\right)+3\right]+\frac{36t^2}{B^2}+16\mu\left(\frac{3t}{B}-\mu\right).$$

(iii) If $\frac{4t}{AC}\left(2t^2\left(2+\frac{D}{AB}\right)-1+t\right)-\frac{4t}{B}\left(\mu+\frac{2t}{B}\right) > 0$ and $\frac{4t}{AC}\left(4t^3\left(1-\frac{E}{A^3}\right)-2t^2\left(\frac{D}{AB}+2\right)-t\left(3-\frac{D}{A}\mu\right)+1\right)+\frac{4t^2}{B^2}-\mu\left(\mu-\frac{4t}{B}\right) \leq -\frac{t}{2AC}\left(2t^2\left(2+\frac{D}{AB}\right)-1+t\right)+\frac{t}{2B}\left(\mu+\frac{2t}{B}\right)$

then

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$$|b_2b_4 - b_3^2| \le \frac{4\,J_1J_3 - J_2^2}{4J_1} = \frac{4t^2}{B^2} - \frac{\left(\frac{4t}{AC}\left(2t^2\left(2+\frac{D}{AB}\right)-1+t\right)-\frac{4t}{B}\left(\mu+\frac{2t}{B}\right)\right)^2}{4\left(\frac{4t}{AC}\left(4t^3\left(1-\frac{E}{A^3}\right)-2t^2\left(\frac{D}{AB}+2\right)-t\left(3-\frac{D}{A}\mu\right)+1\right)+\frac{4t^2}{B^2}-\mu\left(\mu-\frac{4t}{B}\right)\right)}.$$

Proof. By considering Eq. (9), Eq. (13) and Eq. (14) we get

$$b_2b_4 - b_3^2 = \left(\frac{4t^2}{AC}\left(4t^2 - 2 + \frac{D}{A}\mu - \frac{4t^2}{A^3}E\right) - \mu^2\right)d_1^4 + \frac{4t^2}{AC}d_1d_3 + \left(\frac{4t}{AC}\left(4t^2 - 1 + \frac{2t^2}{AB}D\right) - \frac{4t}{B}\mu\right)d_1^2d_2 - \frac{4t^2}{B^2}d_2^2, \tag{16}$$

and then

$$|b_2b_4 - b_3^2| = \left|\left(\frac{4t^2}{AC}\left(4t^2 - 2 + \frac{D}{A}\mu - \frac{4t^2}{A^3}E\right) - \mu^2\right)d_1^4 + \frac{4t^2}{AC}d_1d_3 + \left(\frac{4t}{AC}\left(4t^2 - 1 + \frac{2t^2}{AB}D\right) - \frac{4t}{B}\mu\right)d_1^2d_2 - \frac{4t^2}{B^2}d_2^2\right|. \tag{17}$$

Now by using Lemma 1.3 and doing some calculations we get

$$|b_2b_4 - b_3^2| = \left|\left(\frac{4t^2}{AC}\left(4t^2 - 2 + \frac{D}{A}\mu - \frac{4t^2}{A^3}E\right) - \mu^2\right)d_1^4 + \frac{4t^2}{AC}d_1(1-d_1^2)(1-|x|^2)s + \left(\frac{4t}{AC}\left(4t^2 - 1 + \frac{2t^2}{AB}D\right) - \frac{4t}{B}\mu\right)d_1^2x(1-d_1^2) - 4t^2(1-d_1^2)\left(\frac{d_1^2}{AC} + \frac{(1-d_1^2)}{B^2}\right)x^2\right|.$$

where, x and s are such that $|x| \le 1$, $|s| \le 1$. In this step we have

$$|b_2b_4 - b_3^2| \le \left(\frac{4t^2}{AC}\left(4t^2 - 2 + \frac{D}{A}\mu - \frac{4t^2}{A^3}E\right) - \mu^2\right)d_1^4 - \frac{4t^2d_1(1-d_1^2)(1-d_1)}{AC}|x|^2 + \frac{4t^2(1-d_1^2)^2}{B^2}|x|^2 + \left(\frac{4t}{AC}\left(4t^2 - 1 + \frac{2t^2}{AB}D\right) - \frac{4t}{B}\mu\right)d_1^2(1-d_1^2)|x| + \frac{4t^2}{AC}d_1(1-d_1^2).$$

By Eq. (5) in Lemma 1.2 $|d_1| \le 1$, and without loss of generality, may be assumed to be $|d_1| = d \in [0,1]$ and then

$$|b_2b_4 - b_3^2| \le \left(\frac{4t^2}{AC}\left(4t^2 - 2 + \frac{D}{A}\mu - \frac{4t^2}{A^3}E\right) - \mu^2\right)d^4 + 4t^2(1-d^2)\left(\frac{(1-d^2)}{B^2} - \frac{d(1-d)}{AC}\right)\rho^2 + \left(\frac{4t}{AC}\left(4t^2 - 1 + \frac{2t^2}{AB}D\right) - \frac{4t}{B}\mu\right)d^2(1-d^2)\rho + \frac{4t^2}{AC}d(1-d^2) = F(d,\rho), \tag{18}$$

where $\rho = |x| \le 1$.

To obtain an estimate of $|b_2b_4 - b_3^2|$, we must maximize $F(d,\rho)$ in $[0,1] \times [0,1]$. So,

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$$\frac{\partial F(d,\rho)}{\partial \rho} = 8t^2(1-d^2)\left(\frac{(1-d^2)}{B^2} - \frac{d(1-d)}{AC}\right)\rho + \left(\frac{4t}{AC}\left(4t^2 - 1 + \frac{2t^2}{AB}D\right) - \frac{4t}{B}\mu\right)d^2(1-d^2). \quad (19)$$

For any fixed d with $0 < d < 1$ and for $0 < \rho < 1$ and, from Eq. (19), we notice that $\frac{\partial F}{\partial \rho} > 0$. As a consequence $F(d,\rho)$ is an increasing function of ρ and therefore it cannot have a maximum value at any point in the interior of the closed square $[0,1] \times [0,1]$. Furthermore, for fixed $d \in [0,1]$, we have

$$\max_{0 \le \rho \le 1} F(d,\rho) = F(d,1) = G(d) \quad (20)$$

$$G(d) = \left(\frac{4t}{AC}\left(4t^3\left(1 - \frac{E}{A^3}\right) - 2t^2\left(\frac{D}{AB} + 2\right) - t\left(3 - \frac{D}{A}\mu\right) + 1\right) + \frac{4t^2}{B^2} - \mu\left(\mu - \frac{4t}{B}\right)\right)d^4 +$$
$$\left(\frac{4t}{AC}\left(2t^2\left(2 + \frac{D}{AB}\right) - 1 + t\right) - \frac{4t}{B}\left(\mu + \frac{2t}{B}\right)\right)d^2 + \frac{4t^2}{B^2} . \quad (21)$$

Let

$$J_1 = \frac{4t}{AC}\left(4t^3\left(1 - \frac{E}{A^3}\right) - 2t^2\left(\frac{D}{AB} + 2\right) - t\left(3 - \frac{D}{A}\mu\right) + 1\right) + \frac{4t^2}{B^2} - \mu\left(\mu - \frac{4t}{B}\right),$$
$$J_2 = \frac{4t}{AC}\left(2t^2\left(2 + \frac{D}{AB}\right) - 1 + t\right) - \frac{4t}{B}\left(\mu + \frac{2t}{B}\right),\ J_3 = \frac{4t^2}{B^2}\ \ and\ \ \nu = d^2. \quad (22)$$

So, $J_1\nu^2 + J_2\nu + J_3$ is a quadratic function and then it is a parabola whose orientation is determined by the sign of J_1. Now, since

$$\max_{0 \le \nu \le 1}(J_1\nu^2 + J_2\nu + J_3) = \begin{cases} J_3; & J_2 \le 0,\ J_1 \le -\frac{J_2}{4}\ , \\ 16\,J_1 + 4J_2 + J_3; & J_2 \ge 0,\ J_1 \ge -\frac{J_2}{8}\ \ or\ J_2 \le 0, J_1 \ge -\frac{J_2}{4}\ , \\ \frac{4\,J_1J_3 - J_2^2}{4J_1}; & J_2 > 0,\ J_1 \le -\frac{J_2}{8}\ , \end{cases}$$

and therefore, we have

$$|b_2b_4 - b_3^2| \le \begin{cases} J_3; & J_2 \le 0,\ J_1 \le -\frac{J_2}{4}\ , \\ 16\,J_1 + 4J_2 + J_3; & J_2 \ge 0,\ J_1 \ge -\frac{J_2}{8}\ \ or\ J_2 \le 0, J_1 \ge -\frac{J_2}{4}\ , \\ \frac{4\,J_1J_3 - J_2^2}{4J_1}; & J_2 > 0,\ J_1 \le -\frac{J_2}{8}\ . \end{cases}$$

where, $J_1,\ J_2,\ J_3$ are given by Eq. (22).

The following corollary is obtained by assuming $\delta = 0$ in the above theorem.

Corollary 2.7. If $h \in K(\delta, t, q)$, then

$$|b_2b_4 - b_3^2| \le \frac{4t^2}{([3]_q - 1)^2}; \qquad J_2 \le 0,\ J_1 \le -\frac{J_2}{4},$$

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$$|b_2b_4-b_3^2|\leq\frac{16t}{([2]_q-1)([4]_q-1)}\left[16t^3\left(1-\frac{1}{([2]_q-1)^2}\right)-6t^2\left(\frac{[2]_q+[3]_q-2}{([2]_q-1)([3]_q-1)}+2\right)-t\left(11-\frac{4([2]_q+[3]_q-2)([2]_q(4t^2-1)+1)}{([2]_q-1)^2([3]_q-1)}\right)+3\right]+\frac{36t^2}{([3]_q-1)^2}+\frac{16([2]_q(4t^2-1)+1)}{([2]_q-1)([3]_q-1)}\left(\frac{3t}{([3]_q-1)}-\frac{[2]_q(4t^2-1)+1}{([2]_q-1)([3]_q-1)}\right);\qquad J_2\geq 0,\ J_1\geq-\frac{J_2}{8}\ \ or\ J_2\leq 0, J_1\geq-\frac{J_2}{4},$$

$$|b_2b_4-b_3^2|\leq\frac{4t^2}{([3]_q-1)^2}-\frac{\left(\frac{4t}{([2]q-1)([4]q-1)}\left(2t^2\left(2+\frac{[2]q+[3]q-2}{([2]q-1)([3]q-1)}\right)-1+t\right)-\frac{4t}{[3]q-1}\left(\frac{[2]q(4t^2-1)+1}{([2]q-1)([3]q-1)}+\frac{2t}{[3]q-1}\right)\right)^2}{4\left(\frac{4t}{([2]q-1)([4]q-1)}\left(4t^3\left(1-\frac{1}{([2]q-1)^2}\right)-2t^2\left(\frac{[2]q+[3]q-2}{([2]q-1)([3]q-1)}+2\right)-t\left(3-\frac{([2]q+[3]q-2)([2]q(4t^2-1)+1)}{([2]q-1)^2([3]q-1)}\right)+1\right)+\frac{4t^2}{([2]q-1)^2}-\frac{[2]q(4t^2-1)+1}{([2]q-1)([3]q-1)}\left(\frac{[2]q(4t^2-1)+1}{([2]q-1)([3]q-1)}-\frac{4t}{[3]q-1}\right)\right)};\quad J_2>0,\ J_1\leq-\frac{J_2}{8}.$$

The following corollary is obtained, by taking $\delta=1$ in the above theorem.

Corollary 2.8. If $h\in K(\delta,t,q)$, then

$$|b_2b_4-b_3^2|\leq\frac{4t^2}{[2]_q^2[3]_q^2};\qquad J_2\leq 0,\ J_1\leq-\frac{J_2}{4},$$

$$|b_2b_4-b_3^2|\leq\frac{16t^2}{[2]_q[3]_q[4]_q}\left[-6t^2\left(\frac{1+3[2]_q}{[2]_q}\right)-t\left(11-\frac{4(1+[2]_q)(8t^2-1)}{[2]_q}\right)+3\right]+\frac{36t^2}{[2]_q^2[3]_q^2}+\frac{16(8t^2-1)(3t-8t^2+1)}{[2]_q^2[3]_q^2};\qquad J_2\geq 0,\ J_1\geq-\frac{J_2}{8}\ \ or\ J_2\leq 0, J_1\geq-\frac{J_2}{4},$$

$$|b_2b_4-b_3^2|\leq\frac{4t^2}{[2]_q^2[3]_q^2}-\frac{\left(\frac{4t}{[2]_q[3]_q[4]_q}\left(2t^2\left(\frac{3[2]_q+1}{[2]_q}\right)-1+t\right)-\frac{4t(8t^2+2t-1)}{[2]_q^2[3]_q^2}\right)^2}{4\left(\frac{4t}{[2]_q[3]_q[4]_q}\left(-2t^2\left(\frac{1+3[2]_q}{[2]_q}\right)-t\left(3-\frac{([2]_q+1)(8t^2-1)}{[2]_q}\right)+1\right)+\frac{4t^2-(8t^2-1)(8t^2-4t-1)}{[2]_q^2[3]_q^2}\right)};\quad J_2>0,\ J_1\leq-\frac{J_2}{8}.$$

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3. CONCLUSION

In this article, we were able to define a new subclass of univalent functions and obtain bounds for the coefficients of functions in this subclass. Also, considering the importance of GFT and its wide application, especially in physical and engineering issues, we have studied other issues in this field, including the Fekete - Szegö and Second Hankel Determinant problems. The bounds obtained in this article are useful in practical problems such as electromagnetism and electrostatics, fluid mechanics, control theory and electrical engineering.

Author Contributions

The main idea of this work belongs to both authors, and they equally contributed to the proof of the theorems, analysis of the results, and writing of this article. They both also read and approved the final version of the article.

Conflict of interests.

There are non-conflicts of interest.

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الخلاصة

المقدمة

تُعد نظرية الدوال الهندسية (GFT) مجالًا حيويًا في التحليل العقدى، إذ تُقدم تقنيات فعّالة لحل المسائل الفيزيائية والهندسية. خاصةً في المسائل ثنائية الأبعاد تلعب التحويلات التماثلية دورا حاسمًا في تبسيط المناطق المعقدة، مما يُتيح تحليلًا أكثر سهولة.

طرق العمل:

تبحث هذه الدراسة في مواضيع رئيسية ضمن GFT، مع التركيز على فئة جزئية معممة من الدوال أحادية التكافؤ $K(\delta,t,q)$. وتُستخدم متعددات حدود تشيبيشيف من النوع الثاني لتقدير حدود معاملات الدوال في هذه الفئة الجزئية. وتُدمج في هذه الدراسة أدوات نظرية من التحليل الرياضي وحساب الكم لدعم هذا الاستكشاف.

النتائج:

نستنتج حدود معاملات الدوال في فئة $K(\delta,t,q)$ونقدم تقديرات لدالتين مهمتين: دالة فيكيتي–شيغو ومحدد هانكل من الدرجة الثانية. تُقدم هذه النتائج فهمًا أعمق للخصائص الهندسية لفئة الدوال المعنية.

الاستنتاجات:

يُعزز هذا العمل الصلة بين نظرية الدوال الهندسية وحساب الكم من خلال طرح وتحليل فئة جزئية جديدة من الدوال أحادية التكافؤ. تُعزز الأساليب والتقديرات المُطوّرة هنا فهم سلوك الدوال التحليلية، وتفتح آفاقًا جديدة للأعمال المستقبلية في الرياضيات التطبيقية والنماذج الهندسية.

الكلمات المفتاحية:

الدالة أحادية التكافؤ، مسائل فيكيتي–شيغو، متعددات الحدود تشيبيشيف، محدد هانكل الثاني و حدود المعاملات.