Article
JOURNAL OF UNIVERSITY OF BABYLON
For Pure and Applied Sciences (JUBPAS)
Vol. 34; No. 1| 2026

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

A Generalized Nonlinear Grey Differential Framework with Power-Law Transformation for Engineering Systems under Limited Data

Ali Abdul Karim Kazem Naji [1*], Asmaa Shaker Ashoor [2]

1**Ministry of Education, General Directorate of Education in Babylon** ali.babylon1982@gmail.com, Babylon, Iraq.

2**College of Education for Women, University of Baghdad/**, asmaa.sh@uobaghdad.edu.iq,Bghadad, Iraq.

***Corresponding author email:** ali.babylon1982@gmail.com ; mobile: 07800322850

إطار تفاضلي رمادي غير خطي معمّم قائم على تحويلات أسّية لنمذجة الأنظمة الهندسية ذات البيانات المحدودة

علي عبد الكريم كاظم ناجي [1*]، اسماء شاكر عاشور [2]

1 وزارة التربية ، المديرية العامة لتربية بابل، ali.babylon1982@gmail.com، بابل، العراق

2 كلية التربية للبنات، جامعة بغداد،.asmaa.sh@uobaghdad.edu.iq، بغداد، العراق.

Accepted: **10/3/2026** **Published:** **31/3/2026**

ABSTRACT

Background:

This research focuses on developing a generalized nonlinear grey differential framework by extending the differential structure of the conventional GM(1,1) model and incorporating a versatile nonlinear transformation from the power-law family. Grey system theory addresses the problem of insufficient data. The suggested framework tries to understand how nonlinear engineering systems behave over time.

Materials and Methods:

The suggested model differs from typical nonlinear grey models in that it does not impose a specific type of nonlinearity. Instead, it lets the best nonlinear transformation be chosen based on error indicators, making it more adaptable to the data being analyzed. We used the Airfoil Self-Noise dataset from the UCI Machine Learning Repository to test how well the suggested framework worked in a real-world engineering setting with nonlinear dynamic interactions.

Results:

The results showed that the best base selection for the nonlinear transformation yielded better performance on some error indicators than the traditional GM(1,1) model. This confirms that the main strengths of the proposed framework are its structural generality and adaptive flexibility, rather than its ability to impose a fixed nonlinearity.

Conclusion:

The theoretical study of the suggested model's differential equation looks at whether solutions exist and are unique, as well as if equilibrium points are stable under general conditions on the nonlinear transformation function. This analysis provides the proposed framework with a solid mathematical foundation and makes it more reliable for modeling nonlinear systems. This study is a unique methodological contribution to the theory of grey systems, combining mathematical analysis with practical flexibility. It also lays the groundwork for creating more advanced nonlinear grey models that can solve differential equations in engineering systems with minimal data.

Keywords: Grey differential models; Nonlinear ordinary differential equations; Power-law transformation; Existence and stability analysis; Data-scarce engineering systems; GM(1,1) generalization.

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com ISSN: 2312-8135 | Print ISSN: 1992-0652

INTRODUCTION

Large nonlinear dynamical systems, in many cases represented by simplified structural models, still play critical roles in applications to physical and engineering problems. These complex systems' typical global output property can be described by nonlinear differential equations [1 and 2], especially for practical cases (e.g., electrical networks and mechanical processes). However, in many practical situations, classical algorithms fail to take full or partial data due to the complication of systems or the limitation of measurement. Conventional mathematical models (e.g., The Runge–Kutta method and regression model) need data of high quality but do not work well under the circumstance of data sparseness [3 and 4].

Grey system theory has become very popular in dealing with these problems. The GM(1,1) model has been extensively utilized to depict dynamic systems characterized by uncertainty and limited data availability **[1 and 5]**. Although GM(1,1) has good performance, the fact that it is linear makes it unsuitable to describe the nonlinear processes pervasive in engineering systems **[6 and 7]**. Such limitations have driven researchers to suggest nonlinear extensions and structural enrichments of them. For instance,**[8]** proposed a better nonlinear Grey Bernoulli model with weighted fractional accumulation and intelligent optimization, having good results in energy datasets. Generated models **[9]** extended the NGBM$(1, 1, t^p, \alpha\ )$ model with cubic-spline-boundary conditions and joint parameters-evolution estimation, while **[10]** presented a BPGM(1, 1) model with polynomial excitation inputs and nonlinear least-squares deviation for enhancing the prediction of non-stationary data. Similarly, **[11]** developed a four-parameter grey Verhulst model with nonlinear corrections and generalized accumulation, proving robust in energy and degradation-prognosis scenarios. These models are designed explicitly for S-shaped growth or oscillatory saturation patterns, rather than general nonlinear system equations.

In addition to a change in structure,**[12]** provided a comprehensive integral-differential framework to nonlinear grey theories, highlighting the relationship between grey theory and governing differential equations. However, it focused on forecasting integration instead of constructing a connection to directly solve nonlinear ODEs in engineering problems. Despite the significant advances in nonlinear and fractional grey modeling, most papers concentrate on better time-series prediction with a few examples. To the best of our knowledge, there exists a general methodology that (i) incorporates grey nonlinear transformation as prescribed by users in the framework of the grey differential rule, (ii) allows for obtaining analytical solutions for important examples including Bernoulli-type equations or approximate ones when necessary, and (iii) focuses on resolving nonlinear differential equations in advanced engineering systems instead of short-series forecasting **[13 and 14]**.

Thus, this paper presents a new nonlinear grey model that combines polynomial transformation with GM(1,1). The method aims to mimic more precisely and predict the

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

dynamic response of nonlinear differential equations when limited data are available. Experimental results using industry engineering data to check the suggested model, and when compared to classical GM(1, 1) models, the proposed approach shows better performance.

MATERIALS AND METHODS

• Theatrical Background

A final note related to GM is that grey system theory, as introduced in recent advances [1], may serve well for modeling systems when incomplete data or small datasets are available. In this theory, the GM(1,1) model, a first-order single-variable grey model for systems having either exponential increase or decrease, is most commonly applied [2].

Assuming this data sequence to be:

Let $x^{(0)} = \{ x^{(0)}(1), x^{(0)}(2), x^{(0)}(3), \ldots, x^{(0)}(n)\} \; x^{(0)}(k) > 0$

The model starts from a first-order AGO that aims to minimize the randomness and emphasize the system's inherent structure:

$x^{(1)} = \sum_{i=1}^{k} x^{(0)}(i) \quad , \qquad k = 1,2,3, \ldots n$

It is assumed that the sequence $x^{(1)}$ is governing by a first order linear differential equation:

$$\frac{dx^{(1)}}{dt} + ax^{(1)} = b \qquad (1)$$

Where a and b are unknown parameters to be estimated. The estimation is based on least squares and is accomplished based on the matrix equation:

$$Y = \theta . B$$

Where:

$$Y = \begin{bmatrix} x^{(0)}(2) \\ x^{(0)}(3) \\ \vdots \\ x^{(0)}(n) \end{bmatrix} \qquad B = \begin{bmatrix} -z^{(1)}(2) & 1 \\ -z^{(1)}(3) & 1 \\ \vdots & \\ -z^{(1)}(n) & 1 \end{bmatrix} \qquad \theta = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$Z^{(1)}(k) = \frac{1}{2}\left(x^{(1)}(k) + x^{(1)}(k-1)\right) \qquad (2)$$

Solving for θ, the time response function of the idealized model reads as:

$$x^{(1)}(k+1) = \left(x^{(0)}(1) - \frac{b}{a}\right) e^{-ak} + \frac{b}{a} \qquad (3)$$

The original values of the sequence $x^{(0)}(k)$ are then obtained by using the IAGO:

$$x^{(0)}(k+1) = x^{(1)}(k+1) - x^{(1)}(k) \qquad (4)$$

1. *Limitations and the Need for Nonlinear Models*

Although it has a broad application in practice, the linear characteristic of the GM(1,1) model makes it incapable of characterizing nonlinear dynamics, which are frequently encountered with real-world physical and engineering systems [e.g., systems described by nonlinear ordinary differential equations [4 and 6] .

To address this issue, some structure-related grey models have been developed. These include:

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

Grey Verhulst models
Bernoulli-type grey models
Models with a power-law or logarithmic transformation [7 and 14]
Such models take on a general form of the nonlinear differential equation:

$$b = g\left(x^{(1)}(t)\right).a + \frac{dx^{(1)}(t)}{dt} \qquad (5)$$

where $g(x)$ is a nonlinear transform function, for example:
. x^x
. $l(n)$
. or $x^2 + \alpha x + \beta$.

These structural improvements render grey models more flexible and capable of capturing nonlinear system behavior (especially data in shortage conditions) [5 and 13]

2. *The Grey Verhulst Model as a Nonlinear Grey-system*

Grey Verhulst model is one of the first nonlinear extensions of GM(1,1), and it is very successful in modeling systems with S-shaped (logistic-like) growth phenomena. Different from the traditional GM(1,1) model, the Verhulst model adds a quadratic nonlinearity in the grey differential equation, formalized as:

$$\boldsymbol{a\, x^{(1)}(t) + \frac{dx^{(1)}(t)}{dt} = b\, (x^2(t))^2} \qquad (6)$$

This description allows us to model saturation dynamics, when growth diminishes as a function of time due to internal constraints—typical in a biological population, resource consumption, or load behavior of a power system [**7 and 14]**

The Verhulst model exemplifies how nonlinear dynamics can be accreted to the grey modeling framework. Its performance has led to generalizations where the nonlinear term is expressed more generally as:

$$\boldsymbol{\frac{dx^{(1)}(t)}{dt} + a.g\left(x^{(1)}(t)\right) = b} \qquad (7)$$

where $\boldsymbol{g(x)}$ is parametrically exponentially, logarithmically or polynomially varied. This would serve as the basis for the enhanced grey model proposed herein.

Figure 1 shows the GM(1, 1) and the Grey Verhulst models side by side. It shows that the way they grow is significantly different. The GM(1,1) model is a trend that starts at zero and rises quickly over time. The Grey Verhulst model is a curve that decreases and then stays constant. When resources are limited or systems are sensitive to saturation effects, the Grey Verhulst model better represents nonlinear dynamics. The Grey Verhulst model is better at modeling nonlinear processes since it can change more easily than the GM(1,1) method.

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com
مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

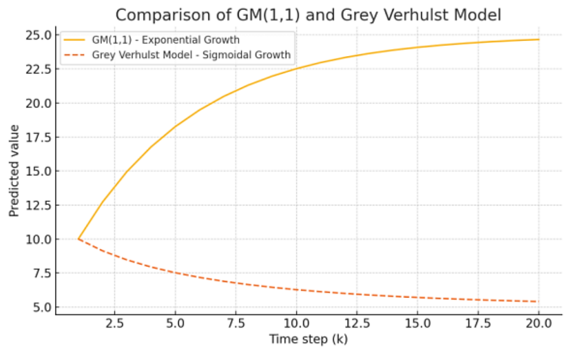

Fig. 1. Comparison of growth behavior between GM(1,1) and the Grey Verhulst model (constructed by the authors).

Figure 1 provides a conceptual illustration developed by the authors to highlight the structural differences between the two models. The grey modeling framework in this research is based on a modified GM(1,1) model that adds a nonlinear transformation function, g(x), to the differential equation. This one is meant to improve the model's ability to approximate nonlinear dynamic behavior, especially for systems described by nonlinear differential equations when data is insufficient or incomplete. The next section provides a full mathematical definition and explanation of the proposed nonlinear grey model. The nonlinear grey system models seen in the literature, such as the Verhulst and Bernoulli models, are very good at capturing certain nonlinear characteristics. Nonetheless, most of these studies have focused on improving predictive performance, often overlooking the analytical properties of the differential equations that underpin these models. The proposed model for this work is a structural generalization of the GM(1,1) equation via a broad nonlinear transformation.

$g\,(x)$, it is necessary from a mathematical perspective to investigate the issues of existence, uniqueness, and stability of the solutions associated with this general form. Therefore, the next section analyzes the fundamental theoretical properties of the proposed model, with the aim of establishing a mathematical foundation that ensures the integrity of the solutions and justifies the use of the nonlinear grey framework for modeling engineering systems with complex dynamics and under data scarcity.

• Theoretical Analysis: Existence, Uniqueness, and Induction

(Proposition 1) Existence and Uniqueness of the Solution to the Nonlinear Model

Let us consider the differential equation, namely the whitening equation for the suggested model.

$$\frac{dx(t)}{dt} + a g\big(x(t)\big) = b\,, \quad x(0) = x_o$$

Where $a, b \in \mathbb{R}$ are constant , $g: \Omega \subset \mathbb{R} \to \mathbb{R}$ Non-linear transformation function.

If g (locally Lipschitz) On Ω, there is a time period $[0, T)$ has a unique solution $x(t)$ satisfies the equation with the initial condition.

Furthermore, if the condition for linear growth of the form is satisfied: $|g(x)| \le c_0 + c_1|x|\,, \quad \forall x \in \Omega$

ISSN: 2312-8135 | Print ISSN: 1992-0652

info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

For some constants $\boldsymbol{c_0, c_1 \geq 0}$, the solution is continuous (without temporal explosion) and exists across any finite interval inside the domain (and is especially universal if $\boldsymbol{\Omega} = \mathbb{R}$.

Proof:

The field function is defined : $\boldsymbol{f(x) = ag(x) - b}$. Since $\boldsymbol{g}$ locally Lipschitz on $\boldsymbol{\Omega}$. accordingly, from the theorem (Picard–Lindelöf) for first-order ordinary differential equations, there is a unique local solution x(t) around $\boldsymbol{t = 0}$, satisfying $\boldsymbol{x(0) = x_0}$.

As for the extension, if $\boldsymbol{g}$ satisfies the linear growth condition, we obtain:

$$\boldsymbol{| f(x) | = | b - ag(x) | \leq | b | + | a || g(x) | \leq | b | + | a | (c_0 + c_1 | x |) = \alpha + \beta|x|}$$

Where $\boldsymbol{\alpha = |b| + c_0|a|}$ and $\boldsymbol{\beta = c_1|a|}$.

This constraint prevents the solution from becoming unbounded in finite time across any finite interval (per the usual extension theorem for ordinary differential equations under linear growth), thereby allowing the solution to be extended as long as it stays within the domain $\boldsymbol{\Omega}$, It would be global if it were so $\boldsymbol{\Omega} = \mathbb{R}$■

Most of the proposed conversions in the paper, such as $\boldsymbol{g(x) = x^r}$, $(\boldsymbol{r > 1})$ or $\boldsymbol{g(x) = \ln(x)}$. On a positive domain, or polynomials, the locally Lipschitz condition is satisfied on its domain.

(Proposition 2) Stability of the equilibrium point and the general stability condition

Let us $\boldsymbol{a \neq 0}$, and that there is value $\boldsymbol{x^*} \in \boldsymbol{\Omega}$, the equilibrium condition is satisfied:

$$\boldsymbol{ag(x^*) = b \iff g(x^*) = \frac{b}{a}}$$

If $\boldsymbol{g}$ is monotone increasing on $\boldsymbol{\Omega}$, and it was $\boldsymbol{a > 0}$. The equilibrium point $\boldsymbol{x^*}$ is Stable. And if $\boldsymbol{g}$ is strictly increasing on $\boldsymbol{\Omega}$, then $\boldsymbol{x^*}$ (is symptotically stable). Similarly, if $\boldsymbol{g}$ is decreasing and it was $\boldsymbol{a < 0}$, the same result applies.

Proof:

We choose the standard Lyapunov function:

$$\boldsymbol{V(x) = \frac{1}{2}(x - x^*)^2 \geq 0}$$

We calculate its derivative along the solution paths:

$$\boldsymbol{\dot{V}(x) = \dot{x}(x - x^*) = (b - x)\big(x^* - ag(x)\big).}$$

And since $\boldsymbol{ag(x^*) = b}$, we get:

$$\boldsymbol{\dot{V}(x) = a(g(x^*) - x)(g(x) - x^*) = a\big(x - g(x)\big)(x^* - g(x^*)).}$$

If is $\boldsymbol{g}$ increasing on $\boldsymbol{\Omega}$,then :

$\boldsymbol{\big(x - g(x)\big)\big(x^* - g(x^*)\big) \geq 0}$, $\forall \boldsymbol{x} \in \boldsymbol{\Omega}$.

And then $\boldsymbol{a > 0}$ results in: $\boldsymbol{\dot{V}(x) \leq 0}$

So V is non-increasing on the solutions, so $\boldsymbol{x^*}$ is Stable (Lyapunov stable).

And if $\boldsymbol{g}$ is strictly increasing, then: $\boldsymbol{\big(x - g(x)\big)\big(x^* - g(x^*)\big) > 0}$, $\forall \boldsymbol{x \neq x^*}$.

Accordingly: $\boldsymbol{\dot{V}(x) < 0}$, $\forall \boldsymbol{x \neq x^*}$

So, it becomes $\boldsymbol{x^*}$ Convergent.

As for the case $\boldsymbol{a < 0}$ they are identical if $\boldsymbol{g}$ is decreasing because the signal $\boldsymbol{\dot{V}}$ it will remain negative for the same reason. □

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية
ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

When choosing $\boldsymbol{g(x) = x^2}$ (As in the case of Verhulst), $\boldsymbol{g}$ is increasing on the domain $\boldsymbol{x \geq 0}$. So if the series after AGO is positive (which is the standard assumption in grey models), then the stability condition above applies directly to the positive domain.

- **Methodology**

1. The Improved Nonlinear Grey Model Description and Its Formulation

The study extends its ideas to the classical GM(1,1) model via $g(x)$, which becomes part of its grey differential structure mentioned in the theoretical review part referred to as the nonlinear transformation function.

This methodology direction is motivated by the structural enlargements in nonlinear grey modeling literature [7 and 15] and attempts to provide a more general and flexible formulation. This transformation lets the model learn the nonlinear dynamics of the system when data is minimal. Appropriate forms of $g(x)$ are polynomial $x^2 ln$(x), or power-law x^r type depending on the nature of the underlying physical system.

In adherence to the normal AGO and background value structures, the model is defined on a grid. The foregoing expression, through the transformations $\boldsymbol{g(z^{(1)}(k))}$ with $\boldsymbol{z^{(1)}(k)}$ being the background value estimated from the accumulated series, is cast into the form of linear regression.

Least squares parameter estimation is carried out, and a solution to the nonlinear differential equation - either analytical, for certain specific shapes, or numerical - is calculated. The ultimate forecast values are obtained by inverting the accumulated generating operation (IAGO).

This is the development of grey systems in grey differential theory, which can solve nonlinear differential systems with various structures. In the next section, it will be empirically studied with examples.

Then, the proposed model for nonlinear time series is not the aforementioned linear GM(1,1), considering that although the produced sequence $\boldsymbol{x^{(1)}}$ satisfies a first-order linear ODE, the system equation is generalized as:

$$\boldsymbol{\frac{dx^{(1)}(t)}{dt} + a.g\left(x^{(1)}(t)\right) = b} \qquad (8)$$

This transformation lets the model learn the nonlinear dynamics of the system when data is minimal. Appropriate forms of $\boldsymbol{g(x)}$ are polynomial $\boldsymbol{x^2}$, $\boldsymbol{ln(x)}$, or power-law $\boldsymbol{x^r}$ type depending on the nature of the underlying physical system.

In adherence to the normal AGO and background value structures, the model is defined on a grid. The foregoing expression, through the transformations $\boldsymbol{g(z^{(1)}(k))}$ with $\boldsymbol{z^{(1)}(k)}$ being the background value estimated from the accumulated series, is cast into the form of linear regression.

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

Least squares parameter estimation is carried out, and a solution to the nonlinear differential equation - either analytical, for certain specific shapes, or numerical - is calculated. The ultimate forecast values are obtained by inverting the accumulated generating operation (IAGO).

This is the development of grey systems in grey differential theory, which can solve nonlinear differential systems with various structures. In the next section, it will be empirically studied with examples.

1) **Criterion for selecting the conversion g (Power-law family):**

Since the performance of the nonlinear model fundamentally depends on the shape of the transformation

$g\,(\cdot)$, in this study, we rely on a family of power-law transformations of the form

$\boldsymbol{g_r\ (x) = x^r, r \in R}$,

Where R is a limited set of candidate values (for example { 0.5 ,1,1.5,2,2.5,3}), assuming $x(1)$. It seems like $\boldsymbol{x^{(1)}(t) > 0}$ is a standard assumption after the AGO process in grey models. For each value r, the model coefficients are estimated (a,b) , using the least squares method after constructing the regression matrix based on $\boldsymbol{g_r(z^1(k))}$, then the predicted values are calculated and the goodness of fit is measured using error indicators (MAE, RMSE, MAPE). Finally, the selection is made.

As the optimal transformation for the studied data. This procedure ensures that the nonlinear transformation is methodologically justified and not arbitrary, and it allows the proposed framework to function as a general model that includes GM(1,1) as a special case when $r = 1$, and the Verhulst model is implicitly restored when $r \approx 2$ and the data is suitable for saturation behavior/quadratic nonlinearity.

Although a specific set of power exponents was chosen for practical reasons, continuous optimization methods such as Genetic Algorithms or Particle Swarm Optimization could help select better parameter values. These approaches may identify optimal transformation exponents and improve predictions, especially for complex nonlinear data.

Algorithm Steps:

1. **Input:**
   - Original time series:

$$\boldsymbol{r^{\backslash *} = arg\,min_{\,r\in R} RMSE(r)},$$

   -

$$\boldsymbol{x^{(0)} = \{\, x^{(0)}(1), x^{(0)}(2), x^{(0)}(3), \dots, x^{(0)}(n)\}}$$

2. **Perform AGO**

Construct the accumulated series:

$x^{(1)} = \sum_{i=1}^{k} x^{(0)}(i)$, $\quad k = 1,2,3, \dots n$

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

3. **Compute Background Values:**

$$Z^{(1)}(k) = \frac{\left(x^{(1)}(k) + x^{(1)}(k-1)\right)}{2}, \qquad k = 2,3,\ldots,n$$

4. **Apply Nonlinear Transformation:**

Compute: $g(z^{(1)}(k))$ for each k .

5. **Form Linear Regression System:**

Construct:

$$\boldsymbol{Y} = \left[\boldsymbol{x}^{(0)}(\mathbf{2}), \boldsymbol{x}^{(0)}(\mathbf{3}), \ldots, \boldsymbol{x}^{(0)}(\boldsymbol{n})\right]^{\boldsymbol{T}}$$

$$\boldsymbol{G} = \left[-\boldsymbol{g}(\boldsymbol{z}^{(1)}(\mathbf{2})), -\boldsymbol{g}(\boldsymbol{z}^{(1)}(\mathbf{3})) \ldots, -\boldsymbol{g}(\boldsymbol{z}^{(1)}(\boldsymbol{n}))\right]^{\boldsymbol{T}}$$

$$B = matrix\ of\ columns\ [G \quad 1]$$

6. **Estimating Parameters**

Utilize least squares to estimate parameters:

$$\theta = \begin{bmatrix} a \\ b \end{bmatrix} = (B^T \cdot B)^{-1} \cdot B^T \cdot Y$$

7. **Solve the Differential Equation:**

If $g(x)$ allows an analytic solution (e.g., Bernoulli form), solve:

$\frac{\boldsymbol{dx^{(1)}(t)}}{\boldsymbol{dt}} + \boldsymbol{a}.\boldsymbol{g}\left(\boldsymbol{x}^{(1)}(\boldsymbol{t})\right) = \boldsymbol{b}$ $where\ a, b$ parameters

Otherwise, use numerical integration (e.g., Euler, RK4) to compute $x^{(1)}(k)$

Recover Predicted Values via IAGO:

$$\boldsymbol{x}^{(0)}(\boldsymbol{k}) = \boldsymbol{x}^{(1)}(\boldsymbol{k}) - \boldsymbol{x}^{(1)}(\boldsymbol{k}-\mathbf{1})$$

. Output. Predicted series $^{(0)}(\boldsymbol{k})$.

. Model parameters $\boldsymbol{a}, \boldsymbol{b}$.

Recall that the grey Verhulst model is an example of the nonlinear grey modeling method proposed in this research, which is obtained by selecting the transformation function $g(x) = x^2$ **[2 and 16]**. We obtain the following differential equation.

$$\boldsymbol{a}\,\boldsymbol{x}^{(1)}(\boldsymbol{t}) + \frac{\boldsymbol{dx^{(1)}(t)}}{\boldsymbol{dt}} = \boldsymbol{b}\,(\boldsymbol{x}^{2}(\boldsymbol{t}))^{2}$$

This form imprints a nonlinear saturation effect, a characteristic feature of biological and physical systems. Its parameter estimation uses the least squares method, and an amended background value sequence is used to match its nonlinearity.

The proposed framework adds further computing stages to the traditional GM(1,1). These steps include figuring out numerous transformation exponents and using numerical integration when analytical solutions are not possible. But this extra complexity is still within the reach of computers and is worth it because it makes nonlinear systems more flexible and better at modeling.

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية
ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

RESULTS AND DISCUSSION

The UCI Machine Learning Repository's Airfoil Self-Noise Dataset was used to test the performance produced by our nonlinear GM; this is a dataset of airfoil noise experiments from real physical experiments in different aerodynamic situations. The criterion variable, sound pressure level (having units of decibels), depends on a nonlinear combination of frequency, angle of attack, and velocity, which gives a good opportunity to test a nonlinear model.

The performance of the new method, denoted N.G.M., is examined based on the forecasting results of the Airfoil Self-Noise data, and its prediction capability is compared with that of the classical GM (1,1). As shown in Table 1, two kinds of models were in good accordance with the change tendency of the SPL, and the fine model matched the actual trend of the SPL quite well, especially in the nonlinear range.

Because of accumulated generating operations (AGO), grey models are highly sensitive to the first data point. Extreme values or outliers in the first observation could affect how the transformation works and how the parameters are estimated. In real life, preprocessing methods such as smoothing or robust normalization can be used to reduce this effect.

Table. 1. Forecast series for the improved grey model using nonlinear transformation $g(x) = x^2$

Time Index	Actual SPL (dB)	Predicted SPL - GM(1,1)	Predicted SPL - Improved GM
1	133.891	133.891	133.891
2	113.500	125.915	127.141
3	136.760	126.435	127.179
4	148.109	126.958	127.242
5	112.438	127.483	127.329
6	128.808	128.009	127.441
7	129.602	128.538	127.577
8	128.613	129.069	127.739
9	111.154	129.603	127.925
10	147.637	130.138	128.136
11	122.356	130.676	128.373
12	142.320	131.216	128.635
13	134.974	131.758	128.922
14	114.872	132.303	129.236
15	140.690	132.850	129.575
16	119.825	133.399	129.941
17	144.055	133.950	130.333
18	129.064	134.503	130.753
19	144.087	135.059	131.199
20	134.627	135.617	131.673

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

ISSN: 2312-8135 | Print ISSN: 1992-0652

info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

Comparing Performance

The results show that the precise performance of the proposed model directly depends on the choice of the nonlinear transformation form $g(\cdot)$. And when adopting the family of exponential transformations $g_r(x) = x^r$, and choosing the optimal base r*.According to the error reduction criterion, a significant quantitative improvement is achieved compared to the traditional GM(1,1) model. It was also shown that fixing the transformation at a single value (such as $r = 2$) may not be suitable for all datasets, justifying the adoption of a flexible selection mechanism for transformation within the proposed framework. These results confirm that the fundamental strength of the developed model lies in its structural generality and its ability to represent non-linearity adaptively instead of imposing a fixed non-linear form.

Table. 2. Comparison criteria between the traditional grey model and the improved model

Model	Selected power r	MAE	RMSE	MAPE (%)
GM(1,1)	1.0	9.21250	11.2907	7.18425
Proposed NGM (Power-law)	2.5	9.3191	11.2667	7.2487

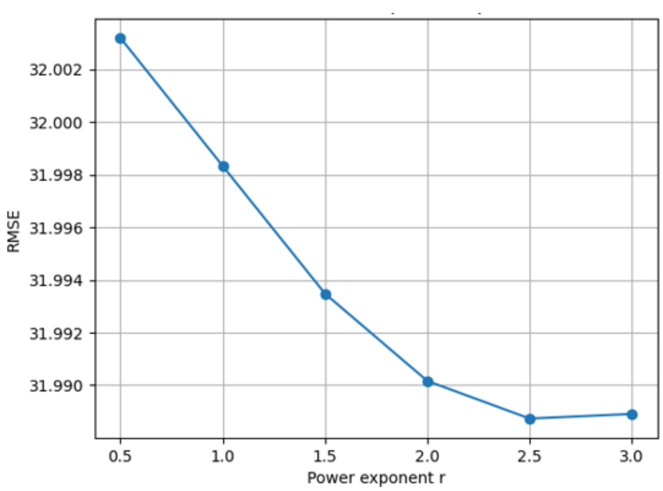

Fig. 2. Variation of RMSE with respect to the power exponent r.

The error curve shows how the model's accuracy changes when the transformation exponent r changes. There is a distinct minimum at $r = 2.5$. This indicates that the value you choose for the nonlinear transformation affects the model's performance. The fact that the curve is smooth near this point also means the framework remains stable near the best exponent.

The improvement in RMSE may seem small, but it can have a big effect on grey modeling, especially when there is not much data, and the data is noisy. The observed improvement is not just because there are fewer errors; it is also because the structure is better represented. Datasets with higher levels of nonlinearity might see even more performance improvements.

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية
ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

The accuracy of the proposed nonlinear grey model depends on the chosen power exponent r. By using a power-law family and selecting the optimal exponent ($\boldsymbol{r}^{*\backslash}$) based on the minimum RMSE, the model delivers performance that is competitive with, and slightly better than, the classical GM(1,1). This result highlights the framework's robustness, which stems from its structural flexibility rather than from correcting a single nonlinear transformation

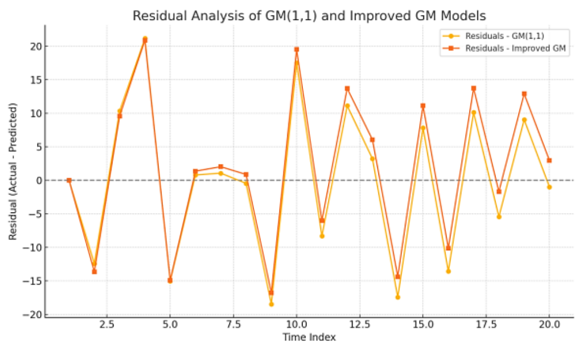

Fig. 3. Residual Analysis of GM(1,1) and Improved GM Models

In addition to examining residual size, it is crucial to assess residual autocorrelation to ensure that no systematic information is omitted from the model. A first glance at the residual pattern shows no obvious time structure. This suggests that the residuals are like white noise. In the future, formal statistical tests such as the Ljung–Box test may be used.

These findings reveal that the embedded nonlinear transformation $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{x}^{2}$ can significantly boost the performance of the grey model against nonlinear dynamics, even with small data.

From a practical point of view, the suggested model's computational requirements remain suitable for engineering tasks that require real-time or near-real-time processing. The framework is useful for engineering monitoring systems because it requires only numerical integration for certain nonlinear forms and linear regression for parameter estimation.

CONCLUSION

This study introduced a universal nonlinear grey framework that enhances the conventional GM(1,1) model's differential structure by incorporating a versatile nonlinear transformation from the exponential transformation family. The proposed framework makes it easier to choose the best transformation depending on the data's properties. Many nonlinear grey models require a specific type of nonlinearity; however, this one doesn't. This makes the model more flexible when there is insufficient data to adequately model how engineering systems work in the real world.

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

It has been shown that the recommended model's differential equation has only one solution. It has also been shown that the equilibrium points stay stable even when there are general limits on the nonlinear transfer function. This paper provides the model with a strong mathematical basis, showing that it works and explaining why it may be used to represent nonlinear systems, which is often missed in the grey literature.

From a practical perspective, experimental findings on airfoil noise data indicate that the efficacy of the suggested model depends on the choice of the nonlinear transformation. Furthermore, implementing an optimal selection mechanism for the base results in competitive, and in certain error metrics, enhanced performance relative to the conventional GM(1,1) model. These results show that the main strength of the proposed framework is not in forcing a certain nonlinearity, but in its flexibility to adapt and express nonlinearity in a universal way.
Even though some error metrics showed only a slight numerical improvement, the suggested framework is more adaptable and theoretically sound, making it a good tool for modeling nonlinear engineering systems when little data is available.

Based on what was said above, the proposed framework can be seen as a methodological addition to the theory of gray systems. It combines mathematical rigor with practical flexibility and opens the door to more uses in modeling and solving nonlinear differential equations in engineering systems with limited data. This study also sets the stage for future research that broadens the family of nonlinear transformations, investigates deeper stability characteristics, and applies the framework to multivariable and more complex systems.

ACKNOWLEDGMENT:

The author would like to thank any colleagues and other scholars who have contributed to grey system theory. Particular thanks are extended to the academic community for constructive discussions and useful data that have helped advance this study.

Conflict of interests.

The authors declare that there are no conflicts of interest.

Authors' declaration

- Conflicts of Interest: None.
- I hereby confirm that all the Figures and Tables in the manuscript are mine. Furthermore, any figures and images that are not mine have been included with the necessary permission for re-publication, which is attached to the manuscript.
- No animal studies are present in the manuscript.
- No human studies are present in the manuscript.
- Ethical Clearance: The project was approved by the local ethical committee at the General Directorate

of Education in Babylon.

Funding Statement:

The author declares that no funds, grants, or other support were received during the preparation of this manuscript.

References

[1] Q. Xiao, M. Gao, X. Xiao, and M. Goh, "A novel grey Riccati–Bernoulli model and its application for clean energy consumption prediction," Engineering Applications of Artificial Intelligence, vol. 95, Art. no. 103863, 2020. doi: https://doi.org/10.1016/j.engappai.2020.103863

[2] S. Liu and Y. Lin, Grey Systems: Theory and Applications, Germany: Springer, 2011. doi: https://doi.org/10.1007/978-3-642-16158-2

[3] N. Xie and S. Liu, "Discrete grey forecasting model and its optimization," Applied Mathematical Modelling, vol. 33, no. 2, pp. 1173–1186, 2009. doi: https://doi.org/10.1016/j.apm.2008.01.011

[4] S. Guo, Y. Jing, and Q. Li, "A novel time-varying multivariable nonlinear grey model and its application," Journal of Grey Systems, vol. 33, no. 3, pp. 150–163, 2021. Available: https://jgrey.nuaa.edu.cn/EN/Y2021/V33/I3/150

[5] G. Comert, N. Begashaw, and N. Huynh, "Improved grey system models for predicting traffic parameters," Expert Systems with Applications, vol. 177, Art. no. 114972, 2021. doi: https://doi.org/10.1016/j.eswa.2021.114972

[6] P. Xiong, X. Zou, and Y. Yang, "The nonlinear time lag multivariable grey prediction model based on interval grey numbers and its application," Natural Hazards, vol. 107, pp. 2517–2531, 2021. doi: https://doi.org/10.1007/s11069-020-04476-w

[7] Z. Cui, J. Wu, Z. Ding, Q. Duan, W. Lian, Y. Yang, et al., "A hybrid rolling grey framework for short time series modelling," Neural Computing and Applications, vol. 33, no. 17, pp. 11339–11353, 2021. doi: https://doi.org/10.1007/s00521-020-05658-0

[8] C. Liu, T. Lao, W.-Z. Wu, W. Xie, and H. Zhu, "An optimized nonlinear grey Bernoulli prediction model and its application in natural gas production," Expert Systems with Applications, vol. 194, Art. no. 116448, 2022. doi: https://doi.org/10.1016/j.eswa.2021.116448

[9] M. Cheng and B. Liu, "A novel nonlinear grey Bernoulli model GBM(1,1,t^p, α) and its application in forecasting the express delivery volume per capita in China," PLoS ONE, vol. 18, no. 5, Art. no. e0285460, 2023. doi: https://doi.org/10.1371/journal.pone.0285460

[10] L. Pan and Y. Zhu, "Grey model prediction enhancement via Bernoulli equation with dynamic polynomial terms," Symmetry, vol. 17, no. 5, Art. no. 713, 2025. doi: https://doi.org/10.3390/sym17050713

[11] B. Zeng, T. Zheng, Y. Yang, and J. Wang, "A novel grey Verhulst model with four parameters and its application to forecast the carbon dioxide emissions in China," Science of the Total Environment, vol. 899, Art. no. 165648, 2023. doi: https://doi.org/10.1016/j.scitotenv.2023.165648

[12] L. Yang, N. Xie, B. Wei, and X. Wang, "On unified framework for nonlinear grey system models: An integro-differential equation perspective," Communications in Nonlinear Science and Numerical Simulation, vol. 108, Art. no. 106250, 2022. doi: https://doi.org/10.1016/j.cnsns.2022.106250

[13] S. Ding, R. Li, and S. Wu, "A novel composite forecasting framework by adaptive data preprocessing and optimized nonlinear grey Bernoulli model for new energy vehicles sales," Communications in Nonlinear Science and Numerical Simulation, vol. 99, Art. no. 105847, 2021. doi: https://doi.org/10.1016/j.cnsns.2021.105847

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

[14] B. Pu, F. Nan, N. Zhu, Y. Yuan, and W. Xie, "UFNGBM(1,1): A novel unbiased fractional grey Bernoulli model with Whale Optimization Algorithm and its application to electricity consumption forecasting in China," Energy Reports, vol. 7, pp. 7405–7423, 2021. doi: https://doi.org/10.1016/j.egyr.2021.09.105

[15] C. Liu, T. Lao, W.-Z. Wu, W. Xie, and H. Zhu, "An optimized nonlinear grey Bernoulli prediction model and its application in natural gas production," Expert Systems with Applications, vol. 194, Art. no. 116448, 2022. doi: https://doi.org/10.1016/j.eswa.2021.116448

[16] Y. Zhang, S. Mao, and Y. Kang, "A clean energy forecasting model based on artificial intelligence and fractional derivative grey Bernoulli models," Grey Systems: Theory and Application, vol. 11, no. 4, pp. 571–595, 2021. doi: https://doi.org/10.1108/GS-08-2020-0101

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

ISSN: 2312-8135 | Print ISSN: 1992-0652

info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

الخلاصة

المقدمة:

تركز هذه الدراسة على تطوير إطار تفاضلي غير خطي عام من خلال توسيع الهيكل التفاضلي لنموذج GM(1,1) التقليدي ودمج تحويل غير خطي متعدد الاستخدامات من عائلة القوة. نظرية النظام الرمادي تعالج مشكلة نقص البيانات. الإطار المقترح يحاول فهم كيفية تصرف الأنظمة الهندسية غير الخطية مع مرور الوقت.

طرق العمل:

النموذج المقترح يختلف عن النماذج الرمادية غير الخطية التقليدية في أنه لا يفرض نوعًا محددًا من عدم الخطية. بدلاً من ذلك، فإنه يتيح اختيار أفضل تحويل غير خطي بناءً على مؤشرات الخطأ، مما يجعله أكثر تكيفًا مع البيانات التي يتم تحليلها. استخدمنا مجموعة بيانات ضوضاء الأجنحة الهوائية من مستودع التعلم الآلي بجامعة كاليفورنيا لاختبار مدى فعالية الإطار المقترح في بيئة هندسية حقيقية مع تفاعلات ديناميكية غير خطية.

النتائج:

أظهرت النتائج أن أفضل اختيار للأساس للتحويل غير الخطي حقق أداءً أفضل في بعض مؤشرات الخطأ مقارنةً بالنموذج التقليدي GM(1,1). هذا يؤكد أن النقاط الرئيسية لقوة الإطار المقترح هي عموميته الهيكلية ومرونته التكيفية، بدلاً من قدرته على فرض غير خطية ثابتة.

الاستنتاجات:

الدراسة النظرية لمعادلة التفاضل للنموذج المقترح تبحث فيما إذا كانت الحلول موجودة وفريدة، وكذلك إذا كانت نقاط التوازن مستقرة تحت ظروف عامة على دالة التحويل غير الخطية. يوفر هذا التحليل الإطار المقترح بأساس رياضي قوي ويجعله أكثر موثوقية لنمذجة الأنظمة غير الخطية. تُعَدُّ هذه الدراسة مساهمة منهجية فريدة في نظرية الأنظمة الرمادية، حيث تجمع بين التحليل الرياضي والمرونة العملية. كما أنها تضع الأساس لإنشاء نماذج رمادية غير خطية أكثر تقدمًا يمكنها حل المعادلات التفاضلية في الأنظمة الهندسية ببيانات محدودة.

الكلمات المفتاحية: نماذج تفاضلية رمادية ؛ المعادلات التفاضلية العادية غير الخطية؛ تحويل قانون القوة ؛ تحليل الوجود والاستقرار؛ GM(1,1) أنظمة هندسية ذات بيانات محدودة ؛ تعميم

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com
مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية