Article
JOURNAL OF UNIVERSITY OF BABYLON
For Pure and Applied Sciences (JUBPAS)
Vol.34 ; No.1 | 2026

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

qIntroducing $\widehat{P}$-Open Sets and Related Topological Operators

Nehmat K.Ahmed* , Rekar A. Jalal **

College of Education University of Salahaddin, Erbil / Iraq

nehmat.ahmed@su.edu.krd , rekar.jalal@su.edu.krd

تقديم المجموعات-$\widehat{P}$ المفتوحة والمؤثرات الطوبولوجية المرتبطة بها

نعمت خضر أحمد * ريكار عبدالخالق جلال**

قسم الرياضيات, كلية التربية , جامعة صلاح الدين, اربيل/ العراق

Accepted:	15/2/2026		Published:	31/3/2026

ABSTRACT

This paper introduces a new class of open sets, referred to as $\hat{P}$-*open sets*, within the framework of general topology. It is shown that the family of all $\hat{P}$-*open sets* generates a topology that lies precisely between the classes of θ-*open sets* and ordinary open sets. Based on this concept, several associated topological operators are defined and investigated, namely the $\hat{P}$-*interior*, the $\hat{P}$-*closure*, and the $\hat{P}$-*derived set*. The study highlights the fundamental properties of these operators, the interrelationships among them, and their significance in extending the theory of generalized open sets and the corresponding topological structures.

Keywords: $\hat{P}$-Open set, pre-closed set, open set, θ-open set, $\hat{P}$-limit point.

1. INTRODUCTION

In point-set topology, an open set is one of the most important basic concepts. It provides the framework for defining ideas like continuity, convergence, compactness, connectedness, and other key topological concepts. By using open sets, topology studies spaces in an abstract way without depending on distance, while still capturing the ideas of closeness and continuity. Therefore, open sets form the foundation of topology and are the main tool for understanding the structure of topological spaces. A systematic study of open and closed sets

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

ISSN: 2312-8135 | Print ISSN: 1992-0652

info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

was first formalized by Kuratowski in 1922 **[1]**. Departing from this classical framework, Norman Levine proposed the broader notion of semi-open sets in 1963 **[2]**. Levine's relaxation was soon complemented by the introduction of pre-open sets, put forward by Mashhour et al. in 1982 **[3]**. Addressing certain inadequacies of ordinary open sets, particularly in questions of continuity and separation, Velicko defined θ-open sets in 1968 **[4]**. Later refinements were provided by Njåstad, who in 1966 introduced α-open sets **[5]**, further enriching the hierarchy of generalized open concepts. Furthermore, several types of open sets are defined (see **(6-11)**). Motivated by these developments, this study introduces a new class of open sets, called $\hat{P}$-open sets, which constitute a stronger form of ordinary open sets. Their fundamental properties are investigated, and their role within the framework of generalized topology is examined.

2. PRELIMINARIES

Within this section, we delve into pertinent definitions and present some novel findings. Let $(\breve{W}, \rho)$ or simply $\breve{W}$ represent a topological space. We use $Cl(\breve{T})$ to denote the closure of a set $\breve{T}$, and $Int(\breve{T})$ to denote its interior. A subset $\breve{T}$ of $\breve{W}$is defined as pre-open**[3]** if and only if $\breve{T} \subset int(cl(\breve{T}))$. The complement of a pre-open set is referred to as pre-closed.

Definition2.1.[12] In the context of a topological space $(\breve{W}, \rho)$. A subset $\breve{T}$ is defined as a θ-open set within $\breve{W}$ if for every element $r \in \breve{T}$, there exists an open set $\breve{H}$ in $\breve{W}$ satisfying the condition that $r \in \breve{H} \subseteq cl(\breve{H}) \subseteq \breve{T}$.

Definition2.2.[13] The intersection of pre-closed sets containing a set $\breve{T}$ is called the pre-closure of $\breve{T}$ and is denoted by $pCl(\breve{T})$, which is the smallest pre-closed set in $\breve{W}$ containing $\breve{T}$.

Definition2.3.[14] A semi-open set $\breve{T}$ of a space $\breve{W}$ is S_p-open set if for each $r \in \breve{T}$, there exists a pre-closed set $\breve{S}$ such that $r \in \breve{S} \subset \breve{T}$. The family of all S_p-open subsets of $\breve{W}$ is denoted by $S_pO(\breve{W})$.

Definition2.4.[15] In the context of a space $\breve{W}$, a subset $\breve{T}$ is designated as P_p-open if, for every element r within $\breve{T}$, there exists a pre-closed set $\breve{S}$ such that r is contained in $\breve{S}$ and $\breve{S}$ is a subset of $\breve{T}$.

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

Definition2.5.[16] In a space $\breve{W}$, A subset $\breve{T}$ is referred to as a pre-θ-open set if, for every element r in $\breve{T}$, there exists a pre-open set $\breve{L}$ such that $r \in \breve{L} \subset pCl(\breve{L}) \subset \breve{T}$. The collection of all pre-θ-open sets in $\breve{W}$ is denoted by $P\theta O(\breve{W})$.

Theorem2.6.[17] A topological space $(\breve{W}, \rho)$ is defined as a pre-regular space (or P-regular space) if and only if for every closed set $\breve{S}$ and every point $r \notin \breve{S}$, there exists a pre-open sets $\breve{H}$ and $\breve{G}$ such that $r \in \breve{H}$, $\breve{S} \subseteq \breve{G}$, and $\breve{H} \cap \breve{G} = \emptyset$.

Lemma2.7.[13] Let $(\breve{W}, \rho)$ be a topological space, and let $\breve{T}$ and $\breve{Y}$ be subsets of $\breve{W}$. The following characteristics are subsequently fulfilled:

1) $r \in pCl(\breve{T})$ if and only if $\breve{T} \cap \breve{G} \neq \emptyset$ for every $\breve{G} \in PO(\breve{W}, \rho), r \in \breve{G}$.
2) $\breve{T}$ is pre-closed in $(\breve{W}, \rho)$ if and only if $\breve{T} = pCl(\breve{T})$.
3) $pCl(\breve{T}) \subset pCl(\breve{Y})$ if $\breve{T} \subset \breve{Y}$.
4) $pCl\left(pCl(\breve{T})\right) = pCl(\breve{T})$.

Lemma2.8.[18] In a space $\breve{W}$, a subset $\breve{T}$ is classified as pre-open if and only if it serves as a pre-neighborhood for every point within $\breve{T}$.

Theorem2.9.[14] A space $\breve{W}$ is submaximal if and only if every pre-open subset of $\breve{W}$ is open.

Theorem2.10. For a topological space $(\breve{W}, \rho)$ the following are equivalent:

1. $\breve{W}$ is extremely disconnected**[19]**.
2. Every semi-open subset of $\breve{W}$ is pre-open **[20]**.
3. $cl(\breve{T}) \cap cl(\breve{Y}) = cl(\breve{T} \cap \breve{Y})$ for all subsets $\breve{T}$ and $\breve{Y}$ of $\breve{W}$**[21]**.
4. Every regular closed subset of $\breve{W}$ is regular open and every regular open subset of $\breve{W}$ is regular closed **[22]**.

Lemma2.11.[23] Let $\breve{W}$ be a space, the following holds:

(a) $SO(\breve{W}, \rho) = SO(\breve{W}, \rho_\alpha)$; $SC(\breve{W}, \rho) = SC(\breve{W}, \rho_\alpha)$.

(b) $PO(\breve{W}, \rho) = PO(\breve{W}, \rho_\alpha)$; $PC(\breve{W}, \rho) = PC(\breve{W}, \rho_\alpha)$.

(c) $RO(\breve{W}, \rho) = RO(\breve{W}, \rho_\alpha)$; $RC(\breve{W}, \rho) = RC(\breve{W}, \rho_\alpha)$.

Definition2.12.[13] In a topological space $(\breve{W}, \rho)$, regularity is defined as follows: For any closed set $\breve{S}$ and any point r not in $\breve{S}$, there exists disjoint open sets $\breve{H}$ and $\breve{G}$ such that $r \in \breve{H}$, $\breve{S} \subseteq \breve{G}$, and $\breve{H} \cap \breve{G} = \emptyset$.

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

Definition2.13.[24] A space $\breve{W}$ is considered locally indiscrete if each of its open subsets is also closed. Conversely; this can be understood as a scenario where every closed set in $\breve{W}$ is simultaneously open.

Lemma2.14.[24] In a topological space the collection of pre-open sets is closed under arbitrary unions but not under finite intersections.

Theorem2.15.[25] For any spaces $\breve{W}$ and $\breve{Q}$. If $\breve{T} \subset \breve{W}$ and $\breve{Y} \subset \breve{Q}$ then:

1. $pInt_{\breve{W}\times\breve{Q}}(\breve{T} \times \breve{Y}) = pInt_{\breve{W}}(\breve{T}) \times pInt_{\breve{Q}}(\breve{Y})$.
2. $Int_{\breve{W}\times\breve{Q}}(\breve{T} \times \breve{Y}) = Int_{\breve{W}}(\breve{T}) \times Int_{\breve{Q}}(\breve{Y})$.

Lemma2.16.[24] The intersection of a pre-open set and an α-open set in a space $\breve{W}$ is pre-open set in $\breve{W}$.

Lemma2.17.[24] If $\breve{Q}$ is an α-open subset of a space $\breve{W}$, then a subset $\breve{H}$ of $\breve{Q}$ is pre-open in $(\breve{Q}, \rho_{\breve{Q}})$ if and only if $\breve{H}$ is pre-open in $\breve{W}$.

Theorem2.18[26] Let $\breve{Q}$ be an open subspace of a topological space $(\breve{W}, \rho)$. A subset $\breve{T}$ of $\breve{Q}$ is an open set in $\breve{W}$ if and only if $\breve{T}$ is open in $\breve{Q}$.

Proposition2.19. Let $(\breve{Q}, \rho_{\breve{Q}})$ be an open subspace of $(\breve{W}, \rho)$. If $\breve{S} \in PC(\breve{W}, \rho)$ and $\breve{S} \subset \breve{Q}$, then $\breve{S} \cap \breve{Q} \in PC(\breve{Q}, \rho_{\breve{Q}})$.

Proof: Given that $\breve{Q}$ is an open subset of $\breve{W}$ it follows that $int_{\breve{Q}}(\breve{Y}) = int(\breve{Y})$ for every subset $\breve{Y}$ within$\breve{Q}$. Consequently, we can derive that $cl_{\breve{Q}}(int_{\breve{Q}}(\breve{S} \cap \breve{Q})) = cl_{\breve{Q}}(int(\breve{S} \cap \breve{Q})) = cl(int\,(\breve{S} \cap \breve{Q}) \cap \breve{Q}) \subset cl(int(\breve{S}) \cap int(\breve{Q})) \cap \breve{Q} = cl(int(\breve{S})) \cap \breve{Q} \subset \breve{S} \cap \breve{Q} \in PC(\breve{Q}, \rho_{\breve{Q}})$.

Lemma2.20. [26] Let $\breve{Q}$ be a subspace of a topological space $\breve{W}$. Then for any subset $\breve{V} \subset \breve{Q}$, the closure of $\breve{V}$ in $\breve{Q}$, denoted $cl_{\breve{Q}}(\breve{V})$, is equal to the intersection of $\breve{Q}$ and the closure of $\breve{V}$ in $\breve{W}$, denoted $\breve{Q} \cap cl_{\breve{W}}(\breve{V})$.

Definition2.21. [26] A subset $\breve{T}$ of a space $\breve{W}$ is considered dense if its closure encompasses the entire space $\breve{W}$.

Corollary2.22.[5] A topology ρ is an α-topology if and only if $\rho = \rho^{\alpha}$. Thus, an α-topology belongs to the α-class of all its determining topologies, and is the finest topology of this class.

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

Theorem2.23.[25] Let $(\breve{Y}, \rho_{\breve{Y}})$ be a subspace of a space $(\breve{W}, \rho)$. If $\breve{A} \in PO(\breve{W}, \rho)$ and $\breve{A} \subseteq \breve{Y}$, then $\breve{A} \in PO(\breve{Y}, \rho_{\breve{Y}})$. Moreover, if $\breve{Y}$ is an α-open subspace of $\breve{W}$, $\breve{F} \in PC(\breve{W}, \rho\ )$and $\breve{F} \subseteq \breve{Y}$, then $\breve{F} \in PC(\breve{Y}, \rho_{\breve{Y}})$.

3. $\hat{P}$-open sets

Definition3.1. A subset $\breve{T}$ of a topological space $(\breve{W}, \rho)$ is said to be $\hat{P}$-open if for each $r \in \breve{T}$, there exists a pre-closed set $\breve{S}$ such that $r \in \breve{S} \subset \breve{T}$. The family of all $\hat{P}$-open subsets of a topological space $(\breve{W}, \rho)$ is denoted by $\hat{P}O(\breve{W}, \rho)$ or $\hat{P}O(\breve{W})$.

Proposition3.2. A subset $\breve{T}$ in a topological space $\breve{W}$ is $\hat{P}$-open if and only if $\breve{T}$ is open and it is a union of pre-closed sets.

Proof: Clear from the definition 3.1.

It is clear from the definition that every $\hat{P}$-open subset of a space $\breve{W}$ is open but the converse is not true in general and every $\hat{P}$-open subset of a space $\breve{W}$ is not pre-closed, as shown in the following examples:

Example3.3. Let $\breve{W} = \{a, b, c, d\}$ with topology $\rho = \left\{\emptyset, \breve{W}, \{a\}, \{a, b\}, \{a, b, c\}\right\}$. Then $PO(\breve{W}) = \left\{\emptyset, \breve{W}, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\right\}$ and $PC(\breve{W}) = \left\{\emptyset, \breve{W}, \{b, c, d\}, \{c, d\}, \{b, d\}, \{b, c\}\{d\}, \{c\}, \{b\}\right\}$. Therefore, $\hat{P}O(\breve{W}) = \{\emptyset, \breve{W}\}$. Hence $\{a\}$ and $\{a, b, c\}$ are open sets but they are not $\hat{P}$-open.

Example3.4. In the topological space $(\mathbb{R}, U)$, where U denotes the usual topology on $\mathbb{R}$, consider the set $\breve{A} = (0,1)$. Since $\breve{A}$ is an open interval in the usual topology, it follows that $\breve{A} \in \hat{P}O(\mathbb{R})$; that is, $\breve{A}$ is $\hat{P}$-open. However, $cl(int(\breve{A})) = cl((0,1)) = [0,1]$, and clearly $[0,1] \not\subset (0,1)$. Therefore, $\breve{A}$ is not pre-closed in $(\mathbb{R}, U)$.

Definition3.5. A subset $\breve{Y}$ of a topological space $(\breve{W}, \rho)$ is $\hat{P}$-closed if $\breve{W} \backslash \breve{Y}$ is a $\hat{P}$-open set. The family of all $\hat{P}$-closed sets of a topological space $(\breve{W}, \rho)$ is denoted by $\hat{P}C(\breve{W}, \rho)$ or $\hat{P}C(\breve{W})$.

Corollary3.6. A subset $\breve{Y}$ of a topological space $(\breve{W}, \rho)$ is $\hat{P}$-closed if and only if $\breve{Y}$ is closed and it is an intersection of pre-open sets.

Proof: Follows from Proposition 3.2.

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

ISSN: 2312-8135 | Print ISSN: 1992-0652

info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

The following result shows that any union of $\hat{P}$-open sets in a topological space $(\breve{W},\rho)$ is $\hat{P}$-open.

Proposition3.7. Let $\{\breve{T}_\gamma;\gamma\in\nabla\}$ be a family of $\hat{P}$-open sets in a topological space$(\breve{W},\rho)$. Then $\cup\{\breve{T}_\gamma;\gamma\in\nabla\}$ is a $\hat{P}$-open set.

Proof. Let $\breve{T}_\gamma$ be $\hat{P}$-open set for each γ, then $\breve{T}_\gamma$ is open and hence $\cup\{\breve{T}_\gamma;\gamma\in\nabla\}$ is open. Let $r\in\cup\{\breve{T}_\gamma;\gamma\in\nabla\}$, there exists $\gamma_0\in\nabla$ such that $r\in\breve{T}_{\gamma_0}$. Since $\breve{T}_{\gamma_0}$ is $\hat{P}$-open set for each γ_0, there exists a pre-closed set $\breve{S}$ such that $r\in\breve{S}\subset\breve{T}_{\gamma_0}\subset\cup\{\breve{T}_\gamma;\gamma\in\nabla\}$, so $r\in\breve{S}\subset\cup\{\breve{T}_\gamma;\gamma\in\nabla\}$. Therefore, $\cup\{\breve{T}_\gamma;\gamma\in\nabla\}$ is $\hat{P}$-open set.

From Proposition 3.7, it is clear that any intersection of $\hat{P}$-closed sets of a topological space $(\breve{W},\rho)$ is an $\hat{P}$-closed set.

Proposition3.8. Any finite intersection of $\hat{P}$-open sets is an $\hat{P}$-open set.

Proof: Suppose $\breve{T}_k$ be $\hat{P}$-open for $k=1,2,3,\dots,m$, in a space $\breve{W}$. Then $\cap\breve{T}_k$ is an open in $\breve{W}$. Let $r\in\cap\breve{T}_k$, then $r\in\breve{T}_k$ for each $k=1,2,3,\dots,m$. But $\breve{T}_k$ is an $\hat{P}$-open, hence there exists a pre-closed $\breve{S}_k$ for each $k=1,2,3,\dots$, m, such that $r\in\breve{S}_k\subset\breve{T}_k$. Thus $r\in\cap\breve{S}_k\subset\cap\breve{T}_k$. Therefore $\cap\breve{T}_k$ is an $\hat{P}$-open set for each $k=1,2,3,\dots,m$. Hence the intersection of finite $\hat{P}$-open sets is $\hat{P}$-open.

From Proposition 3.8, it is clear that any finite union of $\hat{P}$-closed sets of a topological space $(\breve{W},\rho)$ is an $\hat{P}$-closed set.

From Proposition 3.7 and Proposition 3.8 we get that the family of all $\hat{P}$-open sets is a topology.

Proposition3.9. The set $\breve{L}$ is $\hat{P}$-open in the space $\breve{W}$ if and only if for each $r\in\breve{L}$, there exists a $\hat{P}$-open set $\breve{C}$ such that $r\in\breve{C}\subset\breve{L}$.

Proof: Let $\breve{L}$ be an $\hat{P}$-open set in $\breve{W}$, then for each $r\in\breve{L}$, put $\breve{L}=\breve{C}$ is an $\hat{P}$-open set containing r such that $r\in\breve{C}\subset\breve{L}$. Conversely; suppose that for each $r\in\breve{L}$, there exist an $\hat{P}$-open set $\breve{C}$ such that $r\in\breve{C}\subset\breve{L}$, then $\cup_{r\in\breve{L}}\{r\}\subset\cup_{r\in\breve{L}}\breve{C}_r\subset\cup_{r\in\breve{L}}\breve{L}$, thus $\breve{L}=\cup_{r\in\breve{L}}\breve{C}_r$ where, $\breve{C}_r\in\hat{P}O(\breve{W})$ for each r. Hence by Proposition 3.7 $\breve{L}\in\hat{P}O(\breve{W})$.

Proposition3.10. For any subset $\breve{T}$ of a space $(\breve{W},\rho)$. If $\breve{T}\in\theta O(\breve{W})$, then $\breve{T}\in\hat{P}O(\breve{W})$.

مجلة جامعة بابل للعلوم الصرفة والتطبيقية
مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية
ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

Proof: Suppose that $\breve{T} \in \theta O(\breve{W})$. If $\breve{T} = \emptyset$, then $\breve{T} \in \hat{P}O(\breve{W})$. If $\breve{T} \neq \emptyset$, then for each $r \in \breve{T}$, there exist an open set $\breve{L}$ such that $r \in \breve{L} \subset cl(\breve{L}) \subset \breve{T}$, so $\breve{T} = \cup \{r; r \in \breve{T}\} \subset \cup \breve{L}_r \subset \cup\, cl(\breve{L}_r) \subset \breve{T}$ for each $r \in \breve{T}$. This implies that $\breve{T} = \cup\, cl(\breve{L})$. Since $cl(\breve{L})$ is closed, hence it is pre-closed, so $\breve{T}$ can be written a union of pre-closed sets by Proposition 3.2, we get $\breve{T} \in \hat{P}O(\breve{W})$.

Proposition3.11. In the usual topology $(\mathbb{R}, U)$, $U = \hat{P}O(\mathbb{R})$.

Proof: It is clear that by Definition 3.1, every $\hat{P}$-open sets is open.

Conversely; Let $\breve{L}$ be any open set in usual topology on $\mathbb{R}$ and let $q \in \breve{L}$. Since $\{q\}$ is closed in the usual topology on $\mathbb{R}$. Hence $\{q\}$ is pre-closed. Moreover $\{q\} \subset \breve{L}$. Hence for each $q \in \breve{L}$, there exists a pre-closed set $q \in \{q\} = \breve{S} \subset \breve{L}$. It follows that $\breve{L} \in \hat{P}O(\mathbb{R})$. Therefore, $U \subset \hat{P}O(\mathbb{R})$. We conclude that $U = \hat{P}O(\mathbb{R})$.

Theorem3.12.

1) Every $\hat{P}$-open subset of a space $\breve{W}$ is S_p-open.
2) Every $\hat{P}$-open subset of a space $\breve{W}$ is P_p-open.

Proof: By definition 3.1, every $\hat{P}$-open is open set. Also, all open sets are S_p-open, and P_p-open.

$$
\begin{array}{ccccccccc}
 & & S_P O(\breve{W}) & & & & & & \\
 & & \uparrow & & & & & & \\
\theta O(\breve{W}) & \rightarrow & \hat{P}O(\breve{W}) & \longrightarrow & \rho & \longrightarrow & \alpha O(\breve{W}) & \longrightarrow & PO(\breve{W}) \\
 & & \downarrow & & & & & & \\
 & & P_P O(\breve{W}) & & & & & &
\end{array}
$$

Diagram

The reverse implication in the above diagram may not hold in general as it is shown in the following example:

Example3.13. Consider $\breve{W} = \{a, b, c, d\}$ with the topology $\rho = \{\emptyset, \breve{W}, \{a, b\}\}$. Then $PO(\breve{W}) = \{\emptyset, \breve{W}, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\} = P_P O(\breve{W})$, $\hat{P}O(\breve{W}) = \{\emptyset, \breve{W}, \{a, b\}\}$, $\theta O(\breve{W}) = \{\emptyset, \breve{W}\}$ and $S_p O(\breve{W}) = \{\emptyset, \breve{W}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Hence $\{a, b\} \in \hat{P}O(\breve{W})$ but $\{a, b\} \notin \theta O(\breve{W})$. Also $\{a\}, \{b\} \in P_P O(\breve{W})$ but they did not belong to $\hat{P}$-open set and $\{a, b, c\}$ is S_p-open set but it is not $\hat{P}$-open set.

Corollary3.14. Let $(\breve{W}, \rho)$ be a submaximal space, then every P_p-open set is $\hat{P}$-open set.

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

ISSN: 2312-8135 | Print ISSN: 1992-0652 | www.journalofbabylon.com | info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq

Proof: Let $\breve{T} \in P_pO(\breve{W})$, thus $\breve{T}$ is pre-open set and for each $r \in \breve{T}$, there exists a pre-closed set $\breve{S}$ such that $r \in \breve{S} \subset \breve{T}$. Since $\breve{W}$ is submaximal then every pre-open set is open, hence $\breve{T}$ is open. Thus $\breve{T} \in \hat{P}O(\breve{W})$.

Corollary3.15. Let $(\breve{W}, \rho)$ be an extremely disconnected and submaximal space, then every S_p-open set is $\hat{P}$-open set.

Proof: Obvious.

Corollary3.16. Let $(\breve{W}, \rho)$ be extremely disconnected. If $\breve{T} \in \hat{P}O(\breve{W})$ and $\breve{Y} \in \hat{P}O(\breve{W})$, then $int(cl(\breve{T} \cap \breve{Y})) = int(cl(\breve{T})) \cap int(cl(\breve{Y}))$.

Proof: Let $\breve{T} \in \hat{P}O(\breve{W})$ and $\breve{Y} \in \hat{P}O(\breve{W})$, then $\breve{T} \in \rho$ and $\breve{Y} \in \rho$. Since $\breve{W}$ is extremely disconnected, then $cl(\breve{T} \cap \breve{Y}) = cl(\breve{T}) \cap cl(\breve{Y})$. Therefore, $int(cl(\breve{T} \cap \breve{Y})) = int(cl(\breve{T})) \cap int(cl(\breve{Y}))$.

Corollary3.17. Let $(\breve{W}, \rho)$ be an extremely disconnected space, if $\breve{T}, \breve{Y} \subset \breve{W}$ such that $\breve{T} \in \hat{P}O(\breve{W})$ and $\breve{Y} \in RO(\breve{W})$, then $\breve{T} \cap \breve{Y} \in \hat{P}O(\breve{W})$.

Proof: Let $\breve{T} \in \hat{P}O(\breve{W})$ and $\breve{Y} \in RO(\breve{W})$, then $\breve{T}$ is open. Thus $\breve{T} \cap \breve{Y}$ is open. Now let $r \in \breve{T} \cap \breve{Y}$, then $r \in \breve{T}$ and $r \in \breve{Y}$, therefore, there exists a pre-closed set $\breve{S}$ such that $r \in \breve{S} \subset \breve{T}$. Since $\breve{W}$ is extremely disconnected, then $\breve{Y}$ is regular closed, and then $\breve{Y}$ is pre-closed. This implies that by Lemma 2.14. $\breve{S} \cap \breve{Y}$ is pre-closed. Therefore, $r \in \breve{S} \cap \breve{Y} \subset \breve{T} \cap \breve{Y}$. Thus $\breve{T} \cap \breve{Y}$ is $\hat{P}$-open set in $\breve{W}$.

Proposition3.18. If a topological space $(\breve{W}, \rho)$ is regular, then $\rho = \hat{P}O(\breve{W})$.

Proof: Let $\breve{T}$ be any subset of a space $\breve{W}$ and $\breve{T} \in \rho$. If $\breve{T} = \emptyset$, then $\breve{T} \in \hat{P}O(\breve{W})$. Let $\breve{T} \neq \emptyset$, since the space $\breve{W}$ is regular, then for each $r \in \breve{T} \subset \breve{W}$ there exists an open set $\breve{H}$ such that $r \in \breve{H} \subset cl(\breve{H}) \subset \breve{T}$ by Definition 2.12 we have $r \in cl(\breve{H}) \subset \breve{T}$. Since $\breve{T} \in \rho$ therefore, $\breve{T} \in \hat{P}O(\breve{W})$. Hence $\rho \subset \hat{P}O(\breve{W})$.

Corollary3.19. If a topological space $(\breve{W}, \rho)$ is pre-regular, then $\rho = \hat{P}O(\breve{W})$.

Proof: Follows directly from Theorem 2.6 and Proposition 3.18.

Proposition3.20. In any space $\breve{W}$. If $\rho = \{\emptyset, \breve{W}\}$, then $\hat{P}O(\breve{W}) = \{\emptyset, \breve{W}\}$.

Proof: Obvious.

The converse of Proposition 3.20 is not true in general as it is shown in the following example:

Example3.21. Let $\breve{W} = \{a, b, c\}$ and $\hat{P}O(\breve{W}) = \{\emptyset, \breve{W}\}$, where $\rho = \{\emptyset, \breve{W}, \{a\}\}$.

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com | ISSN: 2312-8135 | Print ISSN: 1992-0652

Proposition3.22. Let $(\breve{W}, \rho)$ be a topological space, the family of all $\hat{P}$-open sets is $\mathcal{P}(\breve{W})$ if and only if ρ is discrete.

Proof: If ρ is discrete, then it is obvious that the family of all $\hat{P}$-open set is $\mathcal{P}(\breve{W})$. Suppose that the family of all $\hat{P}$-open sets is $\mathcal{P}(\breve{W})$, this means that every subset of $\breve{W}$ is $\hat{P}$-open set and hence every singleton $\{r\}$ where for every $r \in \breve{W}$ is $\hat{P}$-open and hence it is open and pre-closed. Which implies that ρ is discrete.

In general if $\breve{Q}$ is a subspace of a space $\breve{W}$ and $\breve{T}$ is an $\hat{P}$-open set in $\breve{W}$, then $\breve{T} \cap \breve{Q}$ may not be $\hat{P}$-open in $\breve{Q}$, as it is shown in the following example:

Example3.23. Let $\breve{W} = \{a, b, c, d\}$ and $\rho = \{\emptyset, \breve{W}, \{c\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Then $\hat{P}O(\breve{W}) = \{\emptyset, \breve{W}, \{c\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$, and let $\breve{Q} = \{a, c, d\}$, then $\rho_{\breve{Q}} = \{\emptyset, \breve{Q}, \{a\}, \{c\}, \{a, c\}, \{a, d\}\}$ is relative topology on $\breve{Q}$ so, $\hat{P}O(\breve{Q}) = \{\emptyset, \breve{Q}, \{c\}, \{a, d\},\}$. Then $\{a, b\} \in \hat{P}O(\breve{W})$ but $\{a, b\} \cap \breve{Q} = \{a\} \notin \hat{P}O(\breve{Q})$.

Proposition3.24. If $\breve{Q}$ is a clopen subset of a space $\breve{W}$ and $\breve{T}$ is an $\hat{P}$-open set in $\breve{W}$, then $\breve{T} \cap \breve{Q} = \hat{P}O(\breve{Q})$.

Proof: Let $\breve{T}$ be $\hat{P}$-open set in $\breve{W}$, so $\breve{T}$ is open and $\breve{Q}$ is open and closed in $\breve{W}$, then $\breve{T} \cap \breve{Q}$ is open in $\breve{W}$, let $r \in \breve{T} \cap \breve{Q}$, this implies that $r \in \breve{T}$ and $r \in \breve{Q}$, since $\breve{T}$ is $\hat{P}$-open there exists a pre-closed set $\breve{S}$ in $\breve{W}$ such that $r \in \breve{S} \subset \breve{T}$, also $\breve{Q}$ is closed then $\breve{Q}$ is pre-closed, hence $\breve{Q} \cap \breve{S}$ is pre-closed set. Therefore, $r \in \breve{S} \cap \breve{Q} \subset \breve{T} \cap \breve{Q}$, this implies that $\breve{T} \cap \breve{Q}$ is an $\hat{P}$-open set in $\breve{Q}$.

Corollary3.25. Let $(\breve{W}, \rho)$ be a locally indiscrete topological space and $\breve{T}, \breve{Y} \subset \breve{W}$. If $\breve{T} \in \hat{P}O(\breve{W})$ and $\breve{Y}$ is open, then $\breve{T} \cap \breve{Y} \in \hat{P}O(\breve{W})$.

Proof: Follows from Proposition 3.24.

If $(\breve{Q}, \rho_{\breve{Q}})$ is a subspace of the space $(\breve{W}, \rho)$ and if a subset $\breve{T}$ is an $\hat{P}$-open set relative to $\breve{Q}$, then $\breve{T}$ may not be $\hat{P}$-open set in $\breve{W}$ as it is shows in the following example:

Example3.26. Consider the set $\breve{W} = \{a, b, c, d\}$ and the topology $\rho = \{\emptyset, \breve{W}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$ so $\hat{P}O(\breve{W}) = \{\emptyset, \breve{W}, \{b\}, \{c\}, \{a, d\}, \{b, c\}, \{a, b, d\}, \{a, c, d\}\}$, let $\breve{Q} = \{b, c, d\}$, then $\rho_{\breve{Q}} =$

$\{\emptyset, \breve{Q}, \{b\}, \{c\}, \{d\}, \{b,c\}, \{b,d\}, \{c,d\}\}$ is relative topology on $\breve{Q}$, $\hat{P}O(\breve{Q}) = \mathcal{P}(\breve{Q}) = \rho_{\breve{Q}}$ then $\{d\}$ is an $\hat{P}$-open set on $\breve{Q}$, but $\{d\}$ is not an $\hat{P}$-open set in $\breve{W}$.

Proposition3.27. Let $\breve{Q}$ is an open sub space of $\breve{W}$, if $\breve{T} \in \hat{P}O(\breve{W})$, and $\breve{T} \subset \breve{Q}$, and then $\breve{T} \in \hat{P}O(\breve{Q})$.

Proof: Let $\breve{T} \in \hat{P}O(\breve{W})$, then $\breve{T}$ is an open subset of $\breve{W}$ and for each r $\in \breve{T}$, there exists a pre-closed set $\breve{S}$ in $\breve{W}$ such that $r \in \breve{S} \subset \breve{T}$. Since $\breve{T}$ is an open subset of $\breve{W}$ and $\breve{T} \subset \breve{Q}$. Then by Theorem 2.18, $\breve{T}$ is an open set in $\breve{Q}$. Since $\breve{S}$ pre-closed set in $\breve{W}$ and $\breve{T} \subset \breve{Q}$. Then by Theorem 2.23, $\breve{S}$ is pre-closed set in $\breve{Q}$. Hence $\breve{T} \in \hat{P}O(\breve{Q})$.

Proposition3.28. Let $\breve{T} \subset \breve{Q} \subset \breve{W}$, and $\breve{T} \in \hat{P}O(\breve{Q}, \rho_{\breve{Q}})$, If $\breve{Q}$ is a regular open subset of $\breve{W}$, then $\breve{T} \in \hat{P}O(\breve{W}, \rho)$.

Proof: Let $\breve{T} \in \hat{P}O(\breve{Q}, \rho_{\breve{Q}})$ then $\breve{T} \in (\breve{Q}, \rho_{\breve{Q}})$ and for each $r \in \breve{T}$ there exists a pre-closed set $\breve{S}$ in $\breve{Q}$ such that $r \in \breve{S} \subset \breve{T}$. Since $\breve{Q}$ is regular open then $\breve{Q}$ is open in $\breve{W}$ and since $\breve{T} \in (\breve{Q}, \rho_{\breve{Q}})$, then by Theorem 2.18, $\breve{T} \in (\breve{W}, \rho)$. Again since $\breve{S}$ is pre-closed in $\breve{Q}$, then by Proposition 2.19, $\breve{S}$ is pre-closed in $\breve{W}$. Hence $\breve{T} \in \hat{P}O(\breve{W}, \rho)$.

Corollary3.29. Let $\breve{T} \subset \breve{Q} \subset \breve{W}$. If $\breve{T}$ is $\hat{P}$-open set in $\breve{Q}$, and $\breve{Q}$ clopen set in $\breve{W}$, then $\breve{T}$ is an $\hat{P}$-open set in $\breve{W}$.

Proof: Follows from Proposition 3.28.

Theorem3.30. If $(\breve{Q}, \rho_{\breve{Q}})$ is an open subspace of a space $(\breve{W}, \rho)$ and $\breve{T} \in \hat{P}O(\breve{W})$, then $\breve{T} \cap \breve{Q} \in \hat{P}O(\breve{Q})$.

Proof: Let $\breve{T} \in \hat{P}O(\breve{W})$, then $\breve{T}$ is an open set in $\breve{W}$ and for all $r \in \breve{T}$ there exists a pre-closed set $\breve{S}$ such that $r \in \breve{S} \subset \breve{T}$. Therefore, $r \in \breve{S} \cap \breve{Q} \subset \breve{T} \cap \breve{Q}$ hence $\breve{T} \cap \breve{Q}$ is an open set in $\breve{Q}$. Since $\breve{Q}$ is an open subspace of $\breve{W}$, then by Proposition 2.19, $\breve{S} \cap \breve{Q} \in PC(\breve{Q})$. This implies that $\breve{T} \cap \breve{Q} \in \hat{P}O(\breve{Q})$.

Proposition3.31. Let $(\breve{W}, \rho)$ be a topological space and $\breve{T}, \breve{Y} \subseteq \breve{W}$. If $\breve{T} \in \hat{P}O(\breve{W})$ and $\breve{Y}$ is clopen then $\breve{T} \cap \breve{Y} \in \hat{P}O(\breve{W})$.

Proof: Let $\breve{T} \in \hat{P}O(\breve{W})$ and $\breve{Y}$ is clopen, then $\breve{T}$ is open set. This implies that $\breve{T} \cap \breve{Y} \in \rho$. Now let $r \in \breve{T} \cap \breve{Y}$, then $r \in \breve{T}$ and $r \in \breve{Y}$. Therefore, there exists a pre-closed $\breve{S}$ such that $r \in \breve{S} \subset \breve{T}$.

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

ISSN: 2312-8135 | Print ISSN: 1992-0652

info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

Since $\breve{Y}$ is clopen, so $\breve{Y}$ is closed, hence $\breve{Y}$ is pre-closed implies that $\check{S} \cap \breve{Y}$ is pre-closed. Therefore, $r \in \check{S} \cap \breve{Y} \subset \breve{T} \cap \breve{Y}$. Thus $\breve{T} \cap \breve{Y} \in \hat{P}O(\breve{W})$.

Remark3.32. If a topological space $(\breve{W}, \rho)$ is T_1-space then $\rho = \hat{P}O(\breve{W})$.

Proposition3.33. If a space $\breve{W}$ is T_1-space then $PO(\breve{W}) = \hat{P}O(\breve{W})$.

Proof: To show that $PO(\breve{W}) \subset \hat{P}O(\breve{W})$. Let $\breve{T}$ be any subset of a space $\breve{W}$ and $\breve{T} \in PO(\breve{W})$. If $\breve{T} = \emptyset$, then $\breve{T} \in \hat{P}O(W)$. If $\breve{T} \neq \emptyset$, now let $r \in \breve{T}$. Since a space $\breve{W}$ is T_1, then every singleton is pre-closed and hence $r \in \{r\} \subset \breve{T}$. Therefore, $\breve{T} \subset \hat{P}O(\breve{W})$. But $\hat{P}O(\breve{W}) \subset PO(\breve{W})$. Therefore, $PO(\breve{W}) = \hat{P}O(\breve{W})$.

Corollary3.34. For any space $\breve{W}$, $\hat{P}O(\breve{W}, \rho) = \hat{P}O(\breve{W}, \rho_\alpha)$.

Proof: Let $\breve{T}$ be any subset of a space $\breve{W}$ and $\breve{T} \in \hat{P}O(\breve{W}, \rho)$. If $\breve{T} = \emptyset$, then $\breve{T} \in \hat{P}O(\breve{W}, \rho_\alpha)$. If $\breve{T} \neq \emptyset$, and since $\breve{T} \in \hat{P}O(\breve{W}, \rho)$, then $\breve{T}$ is ρ-open set and $\breve{T} = \bigcup \check{S}_\kappa$ where $\check{S}_\kappa$ is pre-closed for each κ. Since $\breve{T}$ is ρ-open, then by Corollary 2.22, $\breve{T}$ is ρ_α-open set. Since $\check{S}_\kappa$ is pre-closed in $(\breve{W}, \rho)$ for each κ, then by Lemma 2.11, $\check{S}_\kappa$ is pre-closed in $(\breve{W}, \rho_\alpha)$ for each κ. Therefore, by Proposition 3.2, $\breve{T} \in \hat{P}O(\breve{W}, \rho_\alpha)$. So $\hat{P}O(\breve{W}, \rho) \subseteq \hat{P}O(\breve{W}, \rho_\alpha)$. By the same way we can prove $\hat{P}O(\breve{W}, \rho_\alpha) \subset \hat{P}O(\breve{W}, \rho)$. Hence we get $\hat{P}O(\breve{W}, \rho) = \hat{P}O(\breve{W}, \rho_\alpha)$.

Theorem3.35. Let $(\breve{W}_1, \rho_1)$ and $(\breve{W}_2, \rho_2)$ be two topological spaces and $\breve{W}_1 \times \breve{W}_2$ be the topological product. Let $\breve{T}_1 \in \hat{P}O(\breve{W}_1)$ and $\breve{T}_2 \in \hat{P}O(\breve{W}_2)$, then $\breve{T}_1 \times \breve{T}_2 \in \hat{P}O(\breve{W}_1 \times \breve{W}_2)$.

Proof: Let $(r_1, r_2) \in \breve{T}_1 \times \breve{T}_2$, then $r_1 \in \breve{T}_1$ and $r_2 \in \breve{T}_2$. Since $\breve{T}_1 \in \hat{P}O(\breve{W}_1)$ and $\breve{T}_2 \in \hat{P}O(\breve{W}_2)$, then $\breve{T}_1$ is open in $\breve{W}_1$ and $\breve{T}_2$ is open in $\breve{W}_2$. Also there exists $\check{S}_1 \in PC(\breve{W}_1)$ and $\check{S}_2 \in PC(\breve{W}_2)$ such that $r_1 \in \check{S}_1 \subset \breve{T}_1$ and $r_2 \in \check{S}_2 \subset \breve{T}_2$. Therefore, $(r_1, r_2) \in \check{S}_1 \times \check{S}_2 \subset \breve{T}_1 \times \breve{T}_2$. Since $\breve{T}_1$ is open in $\breve{W}_1$ and $\breve{T}_2$ is open in $\breve{W}_2$, hence $\breve{T}_1 \times \breve{T}_2$ is open in $\breve{W}_1 \times \breve{W}_2$.Then we get $\check{S}_1 \times \check{S}_2 = pcl_{\breve{W}_1}(\check{S}_1) \times pcl_{\breve{W}_2}(\check{S}_2) = pcl_{\breve{W}_1 \times \breve{W}_2}(\check{S}_1 \times \check{S}_2)$. Therefore, $\breve{T}_1 \times \breve{T}_2 \in \hat{P}O(\breve{W}_1 \times \breve{W}_2)$.

Lemma3.36. Let $\breve{W}$ be a space and $\breve{T}$ be closed in $\breve{W}$, then$\breve{T}$ is $\hat{P}$-closed in $\breve{W}$ if and only if for each $r \notin \breve{T}$, there exists a pre-closed set $\check{S}$ and a pre-open set $\check{C}$ in $\breve{W}$ such that $r \in \check{S}$ and $\breve{T} \subset \check{C}$ and $\check{S} \cap \check{C} = \emptyset$.

Proof: Suppose that $\breve{T}$ is $\hat{P}$-closed and let $r \notin \breve{T}$, then $r \in \breve{W} \backslash \breve{T}$ and $\breve{W} \backslash \breve{T}$ is $\hat{P}$-open set, then there exists a pre-closed set $\check{S}$ such that $r \in \check{S} \subset \breve{W} \backslash \breve{T}$, this implies that $\breve{T} \subset \breve{W} \backslash \check{S}$ and $\breve{W} \backslash \check{S}$ is pre-open, so let $\breve{W} \backslash \check{S} = \check{C}$, thus $r \in \check{S}$ and $\breve{T} \subset \check{C}$ and $\check{S} \cap \check{C} = \emptyset$.

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

ISSN: 2312-8135 | Print ISSN: 1992-0652

info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

ISSN: 2312-8135 | Print ISSN: 1992-0652

info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

Conversely; let $r \in \breve{W}\backslash\breve{T}$, then $r \notin \breve{T}$, therefore, by hypothesis there exists a pre-closed set $\breve{S}$ and a pre-open set $\breve{C}$ such that $r \in \breve{S}$ and $\breve{T} \subset \breve{C}$ and $\breve{S} \cap \breve{C} = \emptyset$, and this implies that $\breve{S} \subset \breve{W}\backslash\breve{C}$ and $\breve{W}\backslash\breve{C} \subset \breve{W}\backslash\breve{T}$. Therefore, $\breve{S} \subset \breve{W}\backslash\breve{C} \subset \breve{W}\backslash\breve{T}$.Thus for each $r \in \breve{W}\backslash\breve{T}$ there exists a pre-closed set $\breve{S}$ such that $r \in \breve{S} \subset \breve{W}\backslash\breve{T}$.This implies that $\breve{W}\backslash\breve{T}$ is $\hat{P}$-open. Therefore, $\breve{T}$ is $\hat{P}$-closed in $\breve{W}$.

4. $\hat{P}$-operators on a set

Definition4.1. In a topological space $(\breve{W}, \rho)$, consider a subset $\breve{T} \subset \breve{W}$. A point $r \in \breve{W}$ is defined as a $\hat{P}$-limit point of $\breve{T}$ if, for every $\hat{P}$-open set $\breve{L}$ that contains r, the intersection $\breve{T} \cap (\breve{L}\backslash\{r\})$ is non-empty. The collection of all such $\hat{P}$-limit points of $\breve{T}$ is referred to as the $\hat{P}$-derived set of $\breve{T}$, denoted by $\hat{P}D(\breve{T})$.

Remark4.2. A point r is considered a $\hat{P}$-limit point of $\breve{T}$ if every $\hat{P}$-open set that includes r also contains at least one point from $\breve{T}$ that is distinct from r.

The theorem outlined below elucidates several key characteristics of the $\hat{P}$-derived set.

Theorem4.3. If $\breve{T}, \breve{Y}$ and $\breve{V}$ are subsets of a topological space $(\breve{W}, \rho)$, then $\hat{P}$-derived set has the following properties:

1. $\hat{P}D(\emptyset) = \emptyset$.
2. $\hat{P}D(\breve{T}) \subset \hat{P}D(\breve{Y})$, if $\breve{T} \subset \breve{Y}$.
3. $\hat{P}D(\breve{T} \cap \breve{Y}) \subset \hat{P}D(\breve{T}) \cap \hat{P}D(\breve{Y})$.
4. $\hat{P}D(\breve{T}) \cup \hat{P}D(\breve{Y}) = \hat{P}D(\breve{T} \cup \breve{Y})$.
5. $r \in \hat{P}D(\breve{V}\backslash\{r\})$, if $r \in \hat{P}D(\breve{V})$.

Proof: Obvious.

As illustrated by the case below, the statement in (3) lacks a valid converse.

Example4.4. Consider the set $\breve{W} = \{a, b, c\}$ endowed with the topology $\rho = \{\breve{W}, \emptyset, \{a\}, \{a, b\}\}$. From this topology, the collection of $\hat{P}$-open subsets is $\hat{P}O(\breve{W}) = \{\emptyset, \breve{W}\}$. Take the subsets $\breve{T} = \{a, c\}$ and $\breve{Y} = \{b, c\}$. Their intersection is $\breve{T} \cap \breve{Y} = \{c\}$. Now, the corresponding $\hat{P}$-derived sets are: $\hat{P}D(\breve{T}) = \{a, b, c\}$, $\hat{P}D(\breve{Y}) = \{a, b, c\}$, $\hat{P}D(\breve{T} \cap \breve{Y}) = \{a, b\}$. Consequently, $\hat{P}D(\breve{T}) \cap \hat{P}D(\breve{Y}) = \{a, b, c\}$ is not a subset of $\hat{P}D(\breve{T} \cap \breve{Y}) = \{a, b\}$.

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com ISSN: 2312-8135 | Print ISSN: 1992-0652

Theorem4.5. Let ρ_1 and ρ_2 be two topologies on the same set $\breve{W}$ such that $\hat{P}(\breve{W},\rho_1) \subset \hat{P}(\breve{W},\rho_2)$. For any subset $\breve{T} \subset \breve{W}$, every $\hat{P}$-limit point of $\breve{T}$ determined by ρ_2 is also a $\hat{P}$-limit point of $\breve{T}$ determined by ρ_1.

Proof: Assume $\boldsymbol{r}$ is a $\hat{\boldsymbol{P}}$-limit point of $\breve{\boldsymbol{T}}$ with respect to $\boldsymbol{\rho_2}$. By definition, for every set $\breve{\boldsymbol{L}} \in \hat{\boldsymbol{P}}(\breve{\boldsymbol{W}},\boldsymbol{\rho_2})$ that contains $\boldsymbol{r}$, we have $(\breve{\boldsymbol{L}} \cap \breve{\boldsymbol{T}})\backslash\{\boldsymbol{r}\} \neq \emptyset$. Since $\hat{\boldsymbol{P}}(\breve{\boldsymbol{W}},\boldsymbol{\rho_1}) \subset \hat{\boldsymbol{P}}(\breve{\boldsymbol{W}},\boldsymbol{\rho_2})$, this property must also hold for every $\breve{\boldsymbol{L}} \in \hat{\boldsymbol{P}}(\breve{\boldsymbol{W}},\boldsymbol{\rho_1})$ with $\boldsymbol{r} \in \breve{\boldsymbol{L}}$. Therefore, $\boldsymbol{r}$ qualifies as a $\hat{\boldsymbol{P}}$-limit point of $\breve{\boldsymbol{T}}$ with respect to $\boldsymbol{\rho_1}$.

The converse of Theorem 4.5, does not hold true in general, as demonstrated by the following example:

Example4.6. Consider topology $\rho_1 = \left\{\breve{W},\emptyset,\{a,b\},\{a,b,c\}\right\} = \hat{P}O(\breve{W},\rho_1)$ and $\rho_2 = \left\{\breve{W},\emptyset,\{c\},\{a,b\},\{a,b,c\},\{a,b,d\}\right\} = \hat{P}O(\breve{W},\rho_2)$ on a set $\breve{W} = \{a,b,c,d\}$. Note that $\hat{P}O(\breve{W},\rho_1) \subset \hat{P}O(\breve{W},\rho_2)$ and c is a $\hat{P}$-limit point of $\breve{T} = \{a,b\}$ with respect to ρ_1, but it is not a $\hat{P}$-limit point of $\breve{T}$ with respect to ρ_2.

Remark4.7. By leveraging De Morgan's Laws and the three axioms that define a topological space, we can derive the following properties pertaining to $\hat{P}$-closed sets:

1. The empty set and the entire space are both classified as $\hat{P}$-closed.
2. The union of any two $\hat{P}$-closed sets results in a set that is also $\hat{P}$-closed.
3. The intersection of an arbitrary collection of $\hat{P}$-closed sets yields a set that remains $\hat{P}$-closed.

Definition4.8. Let $(\breve{W},\rho)$ be a topological space. A set $\breve{S} \subset \breve{W}$ is said to be $\hat{P}$-closed if it contains all of its $\hat{P}$-limit points. Thus $\breve{S}$ is $\hat{P}$-closed if and only if $\hat{P}D(\breve{S}) \subset \breve{S}$.

Theorem4.9. If $r \notin \breve{S}$, where $\breve{S}$ is a $\hat{P}$-closed subset of a topological space $(\breve{W},\rho)$, then there exists an $\hat{P}$-open set $\breve{L}$ such that $r \in \breve{L} \subset \breve{S}^c$.

Proof: Suppose no such $\hat{P}$-open sets exists. Then $r \in \breve{L} \in \hat{P}O(\breve{W})$ would imply that $\breve{L} \cap \breve{S} \neq \emptyset$. Since $r \notin \breve{S}, \breve{L} \cap \breve{S}\backslash\{r\} \neq \emptyset$. This implies that r is a $\hat{P}$-limit point of $\breve{S}$, that is $r \in \hat{P}D(\breve{S})$. $\breve{S}$ however is a $\hat{P}$-closed set and so $\hat{P}D(\breve{S}) \subset \breve{S}$, so that r must belong to $\breve{S}$. But this is contradiction to $r \notin \breve{S}$. This contradiction shows that such an $\hat{P}$-open set must exists.

Corollary4.10. If $\breve{S}$ is a $\hat{P}$-closed set, then $\breve{S}^c$ is $\hat{P}$-open.

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

ISSN: 2312-8135 | Print ISSN: 1992-0652

info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

Proof: If $r \in \breve{S}^c$, then $r \notin \breve{S}$ where $\breve{S}$ is a $\hat{P}$-closed set. By Theorem 4.9, there exists an $\hat{P}$-open set $\breve{L}_r$ such that $r \in \breve{L}_r \subset \breve{S}^c$. But then $\breve{S}^c = \cup \{r; r \in \breve{S}^c\} \subset \cup \{L_r; r \in \breve{S}^c\} \subset \breve{S}^c$. Thus $\breve{S}^c = \cup \{\breve{L}_r; r \in \breve{S}^c\}$ which is the union of $\hat{P}$-open sets and hence an $\hat{P}$-open set.

Corollary4.11. If $\breve{S}^c$ is an $\hat{P}$-open set, then $\breve{S}$ is $\hat{P}$-closed.

Proof: Suppose r is a $\hat{P}$-limit point of $\breve{S}$ and let $r \notin \breve{S}$. Then $r \in \breve{S}^c$, and $\breve{S} \cap \breve{S}^c \backslash \{r\} = \emptyset$ which implies that r is not a $\hat{P}$-limit point of $\breve{S}$. Hence the assumption that $r \notin \breve{S}$ is wrong. Therefore, every $\hat{P}$-limit point of $\breve{S}$ is in $\breve{S}$and so $\breve{S}$ is $\hat{P}$-closed.

Theorem4.12. If $\hat{P}D(\breve{S}) \subset \breve{T} \subset \breve{S}$ and $\breve{S}$ is a $\hat{P}$-closed set, then $\breve{T}$ is $\hat{P}$-closed.

Proof: Since $\breve{T} \subset \breve{S}$ then $\hat{P}D(\breve{T}) \subset \hat{P}D(\breve{S})$. But $\breve{S}$ is $\hat{P}$-closed, hence $\hat{P}D(\breve{T}) \subset \hat{P}D(\breve{S}) \subset \breve{T} \subset \breve{S}$, that is, $\hat{P}D(\breve{T}) \subset \breve{T}$ which implies that $\breve{T}$ is $\hat{P}$-closed.

Corollary4.13. The derived set of a $\hat{P}$-closed set is $\hat{P}$-closed.

Proof: Let $\breve{S}$ be a $\hat{P}$-closed set we have to prove that $\hat{P}D(\breve{S})$ is $\hat{P}$-closed. Now as $\breve{S}$ is $\hat{P}$-closed, $\hat{P}D(\breve{S}) \subset \breve{S}$ this implies that $\hat{P}D(\hat{P}D(\breve{S})) \subset \hat{P}D(\breve{S}) \subset \breve{S}$. Thus by the above theorem, $\hat{P}D(\breve{S})$ is $\hat{P}$-closed.

Definition4.14. Let $(\breve{W}, \rho)$ be a topological space and $\breve{V} \subset \breve{W}$. Then $\hat{P}$-closure of $\breve{V}$ denoted by $\hat{P}cl(\breve{V})$ is defined by $\hat{P}cl(\breve{V}) = \cap \{all\ \hat{P} - closed\ sets\ containing\ \breve{V}\}$. Since by Remark 4.7, the family of $\hat{P}$-closed sets containing $\breve{V}$ is non-void and by Remark 4.7, the intersection of all elements of this family is $\hat{P}$-closed. Hence $\hat{P}$-closure of a set is $\hat{P}$-closed and it is the smallest $\hat{P}$-closed set containing $\breve{V}$.

Theorem4.15. For any set $\breve{V}$in a topological space, $\hat{P}cl(\breve{V}) = \breve{V} \cup \hat{P}D(\breve{V})$.

Proof: Suppose $r \notin \breve{V} \cup \hat{P}D(\breve{V})$, so that $r \notin \breve{V}$ and $r \notin \hat{P}D(\breve{V})$, there exists an $\hat{P}$-open set $\breve{L}_r$ containing r such that $\breve{V} \cap \breve{L}_r \backslash \{r\} = \emptyset$. Since $r \notin \breve{V}$, this actually means that $\breve{V} \cap \breve{L}_r = \emptyset$ so $\breve{L}_r \subset \breve{V}^c$. Since $\breve{L}_r$ is $\hat{P}$-open set disjoint from $\breve{V}$, no point of $\breve{L}_r$ can be a $\hat{P}$-limit point of $\breve{V}$, that is $\breve{L}_r \subset (\hat{P}D(\breve{V}))^c$, Thus $(\breve{V} \cup \hat{P}D(\breve{V}))^c = \cup \left\{\breve{L}_r; r \notin \left(\breve{V} \cup \hat{P}D(\breve{V})\right)\right\}$ which is an $\hat{P}$-open set. Since an arbitrary union of $\hat{P}$-open sets is $\hat{P}$-open. Therefore, $\breve{V} \cup \hat{P}D(\breve{V})$ is $\hat{P}$-closed, which obviously contains $\breve{V}$. Hence $\hat{P}cl(\breve{V})$ being the smallest $\hat{P}$-closed set containing $\breve{V}$, we have $\hat{P}cl(\breve{V}) \subset \breve{V} \cup \hat{P}D(\breve{V})$.

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

ISSN: 2312-8135 | Print ISSN: 1992-0652

info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

Conversely; suppose that $r \in \breve{V} \cup \hat{P}D(\breve{V})$ and $\breve{S}$ is any $\hat{P}$-closed set containing $\breve{V}$. If $r \in \hat{P}D(\breve{V})$, then $r \in \hat{P}D(\breve{S})$ and so $r \in \breve{S}$ [since $\hat{P}D(\breve{V}) \subset \hat{P}D(\breve{S})$, if $\breve{V} \subset \breve{S}$]. But if $r \in \breve{V}$, then again we have $r \in \breve{S}$ since $\breve{V} \subset \breve{S}$. Thus r belongs to any $\hat{P}$-closed set containing $\breve{V}$ and hence to the intersection of all such sets, which is the $\hat{P}$-closure of $\breve{V}$. Thus $\breve{V} \cup \hat{P}D(\breve{V}) \subset \hat{P}cl(\breve{V})$. Hence $\hat{P}cl(\breve{V}) = \breve{V} \cup \hat{P}D(\breve{V})$.

Theorem4.16. $\breve{V}$ is $\hat{P}$-closed if and only of $\breve{V} = \hat{P}cl(\breve{V})$.

Proof: We suppose first that $\breve{V}$ is $\hat{P}$-closed. Then $\hat{P}D(\breve{V}) \subset \breve{V}$. Since $\hat{P}cl(\breve{V}) = \breve{V} \cup \hat{P}D(\breve{V})$, therefore, it follows that if $\breve{V}$ is $\hat{P}$-closed, then $\hat{P}cl(\breve{V}) = \breve{V} \cup \hat{P}D(\breve{V}) = \breve{V}$.

Conversely; if $\breve{V} = \hat{P}cl(\breve{V})$, then $\hat{P}cl(\breve{V})$ being the intersection of all $\hat{P}$-closed sets containing $\breve{V}$ is $\hat{P}$-closed. Hence $\breve{V}$ is $\hat{P}$-closed.

Here are some properties of the $\hat{P}$-closure:

Proposition4.17. Let $(\breve{W}, \rho)$ be a topological space. And $\breve{T}, \breve{Y}$ be two subsets of $\breve{W}$, then:

1) $\hat{P}cl(\breve{T})$ is the smallest $\hat{P}$-closed set which contains $\breve{T}$.
2) $\hat{P}cl(\emptyset) = \emptyset$.
3) $\hat{P}cl(\hat{P}cl(\breve{T}) = \hat{P}cl(\breve{T})$.
4) If $\breve{T} \subset \breve{Y}$, then $\hat{P}cl(\breve{T}) \subset \hat{P}cl(\breve{Y})$.
5) $\hat{P}cl(\breve{T}) \cup \hat{P}cl(\breve{Y}) = \hat{P}cl(\breve{T} \cup \breve{Y})$.
6) $\hat{P}cl(\breve{T} \cap \breve{Y}) \subset \hat{P}cl(\breve{T}) \cap \hat{P}cl(\breve{Y})$.

Proof: Obvious.

The converse of statement (6) does not generally hold true, as demonstrated by the following example:

Example4.18. Let $\breve{W} = \{a, b, c, d\}$ with topology $\rho = \{\emptyset, \breve{W}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$, then $\hat{P}O(\breve{W}) = \{\emptyset, \breve{W}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$, $\hat{P}C(\breve{W}) = \{\emptyset, \breve{W}, \{c, d\}, \{d\}, \{c\}\}$. Let $\breve{T} = \{c, d\}$, $\breve{Y} = \{a, b\}$ and $\breve{T} \cap \breve{Y} = \emptyset$, then $\hat{P}cl(\breve{T}) = \{c, d\}$, $\hat{P}cl(\breve{Y}) = \breve{W}$ and $\hat{P}cl(\breve{T} \cap \breve{Y}) = \emptyset$. Therefore, $\hat{P}cl(\breve{T}) \cap \hat{P}cl(\breve{Y}) \not\subset \hat{P}cl(\breve{T} \cap \breve{Y})$.

Theorem4.19. Let $\breve{T}$ and $\breve{Y}$ be subsets of $\breve{W}$. If $\breve{T} \in \hat{P}O(\breve{W})$, then $\breve{T} \cap \hat{P}cl(\breve{Y}) \subset \hat{P}cl(\breve{T} \cap \breve{Y})$.

Proof: Let $r \in \breve{T} \cap \hat{P}cl(\breve{Y})$. Then $r \in \breve{T}$ and $r \in \hat{P}cl(\breve{Y}) = \breve{Y} \cup \hat{P}D(\breve{Y})$. If $r \in \breve{Y}$, then $r \in \breve{T} \cap$

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

ISSN: 2312-8135 | Print ISSN: 1992-0652

info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

$\breve{Y} \subseteq \hat{P}cl(\breve{T} \cap \breve{Y})$. If $r \notin \breve{Y}$, then $r \in \hat{P}D(\breve{Y})$ and so $\breve{L} \cap \breve{Y} = \emptyset$ for all $\hat{P}$-open set $\breve{L}$ containing r. Since $\breve{T} \in \hat{P}O(W)$, then $\breve{L} \cap \breve{T}$ is also $\hat{P}$-open set containing r. Hence $\breve{L} \cap (\breve{T} \cap \breve{Y}) = (\breve{L} \cap \breve{T}) \cap \breve{Y} = \emptyset$ and consequently, $r \in \hat{P}D(\breve{T} \cap \breve{Y}) \subset \hat{P}cl\ (\breve{T} \cap \breve{Y})$. Therefore, $\breve{T} \cap \hat{P}cl\ (\breve{Y}) \subset \hat{P}cl\ (\breve{T} \cap \breve{Y})$.

Theorem4.20. If $\breve{Q}$ is an open subset of a topological space $\breve{W}$ and $\breve{T} \subset \breve{Q}$, then $\hat{P}cl_{\breve{Q}}(\breve{T}) \subset \hat{P}cl_{\breve{W}}(\breve{T})$.

Proof: Let $r \notin \hat{P}cl_{\breve{W}}(\breve{T})$, then there exists an $\hat{P}$-open set $\breve{H}$ in $\breve{W}$ containing r such that $\breve{H} \cap \breve{T} = \emptyset$. Since $\breve{T} \subset \breve{Q}$, then $\breve{H} \cap \breve{Q} \cap \breve{T} = \emptyset$. Put $\breve{L} = \breve{H} \cap \breve{Q}$, since $\breve{H} \in \hat{P}O(\breve{W})$ and $\breve{Q}$ is open then by Corollary 3.29, we have $\breve{L} = \breve{H} \cap \breve{Q} \in \hat{P}O(\breve{Q})$ and $\breve{L} \cap \breve{T} = \emptyset$. Thus $r \notin \hat{P}cl_{\breve{Q}}(\breve{T})$. Therefore, we obtain $\hat{P}cl_{\breve{Q}}(\breve{T}) \subset \hat{P}cl_{\breve{W}}(\breve{T})$.

Theorem4.21. Let $\breve{W}$ be a topological space. If $\breve{Q} \in RO(\breve{W})$ and $\breve{T} \subset \breve{Q} \subset \breve{W}$, then $\hat{P}cl_{\breve{W}}(\breve{T}) \cap \breve{Q} = \hat{P}cl_{\breve{Q}}(\breve{T})$.

Proof: From previous result $\hat{P}cl_{\breve{Q}}(\breve{T}) \subset \hat{P}cl_{\breve{W}}(\breve{T})$. Now, let $r \in \hat{P}cl_{\breve{W}}(\breve{T}) \cap \breve{Q}$ and $\breve{G} \in \hat{P}O(\breve{Q})$ containing r. Since $\breve{Q} \in RO(\breve{W})$ then by Proposition 3.28, $\breve{G} \in \hat{P}O(\breve{W})$ contains r and hence $\breve{G} \cap \breve{T} \neq \emptyset$ therefore, $r \in \hat{P}cl_{\breve{Q}}(\breve{T})$. Hence $\hat{P}cl_{\breve{W}}(\breve{T}) \cap \breve{Q} \subset \hat{P}cl_{\breve{Q}}(\breve{T})$. This implies that $r \in \hat{P}cl_{\breve{W}}(\breve{T}) \cap \breve{Q}$. Thus $\hat{P}cl_{\breve{Q}}(\breve{T}) \subset \hat{P}cl_{\breve{W}}(\breve{T}) \cap \breve{Q}$. Therefore, $\hat{P}cl_{\breve{W}}(\breve{T}) \cap \breve{Q} = \hat{P}cl_{\breve{Q}}(\breve{T})$.

Theorem4.22. In a discrete topological space $\breve{W}$, if $\breve{T}$ is a subset of $\breve{W}$, then the $\hat{P}D(\breve{T})$ is the empty set $\emptyset$.

Proof: Let r be an arbitrary element of $\breve{W}$. Given that every subset of $\breve{W}$ is open and consequently $\hat{P}$-open, the singleton set $\breve{L} = \{r\}$ is $\hat{P}$-open. Since $r \in \breve{L}$ and $\breve{L} \cap \breve{T} = \{r\} \cap \breve{T} \subset \{r\}$, it follows that r is not a $\hat{P}$-limit point of $\breve{T}$. Therefore, $\hat{P}D(\breve{T}) = \emptyset$.

Theorem4.23. Suppose $\breve{T}$ is a subset of $\breve{W}$. If a point r in $\breve{W}$ is a $\hat{P}$-limit point of $\breve{T}$, then r is still a $\hat{P}$-limit point of the set $\breve{T}$ when r is removed from $\breve{T}$.

Proof: Straightforward.

Corollary4.24. Let $\breve{T}$ be a set in a space $\breve{W}$. A point $r \in \hat{P}cl(\breve{T})$ if and only if $\breve{T} \cap \breve{H} \neq \emptyset$ for every $\hat{P}$-open set $\breve{H}$ containing r.

Proof: Let $r \notin \hat{P}cl(\breve{T})$. Then $r \notin \cap \breve{S}$, where $\breve{S}$ is $\hat{P}$-closed with $\breve{T} \subset \breve{S}$. So $r \in \breve{W} \backslash \cap \breve{S}$ and $\breve{W} \backslash \cap \breve{S}$ is a $\hat{P}$-open set containing r and hence $(\breve{W} \backslash \cap \breve{S}) \cap \breve{T} \subset (\breve{W} \backslash \cap \breve{S}) \cap (\cap \breve{S}) = \emptyset$.

Conversely; suppose that there exists a $\hat{P}$-open set containing r with $\breve{T} \cap \breve{H} = \emptyset$. Then $\breve{T} \subset \breve{W} \backslash \breve{H}$ and $\breve{W} \backslash \breve{H}$ is a $\hat{P}$-closed with $r \notin \breve{W} \backslash \breve{H}$. Hence $r \notin \hat{P}cl(\breve{T})$.

Proposition4.25. Let $\breve{T}$ be any subset of a space $\breve{W}$. If $\breve{T} \cap \breve{S} \neq \emptyset$ for every closed set $\breve{S}$ of $\breve{W}$ containing r, then the point r is in the $\hat{P}$-closure of $\breve{T}$.

Proof: Suppose that $\breve{H}$ be any $\hat{P}$-open set containing r, then by Definition 3.1, there exists a closed set $\breve{S}$ such that $r \in \breve{S} \subset \breve{H}$. So by hypothesis $\breve{T} \cap \breve{S} \neq \emptyset$ implies $\breve{T} \cap \breve{H} \neq \emptyset$ for every $\hat{P}$-open set $\breve{H}$ containing r. Therefore, $r \in \hat{P}cl(\breve{T})$ by Corollary 4.24.

Definition4.26. In the context of a topological space $(\breve{W}, \rho)$. A subset $\breve{T}$ is defined to be $\hat{P}$-dense in $\breve{W}$ if the $\hat{P}$-closure of $\breve{T}$ equals $\breve{W}$, denoted as $\hat{P}cl(\breve{T}) = \breve{W}$. This implies that $\breve{T}$ is sufficiently spread out within $\breve{W}$ such that its $\hat{P}$-closure encompasses the entire space $\breve{W}$. As a straightforward example, the set $\breve{W}$ itself is inherently $\hat{P}$-dense in $\breve{W}$ because $\hat{P}cl(\breve{W}) = \breve{W}$. Additionally, the set of rational numbers $\mathbb{Q}$ is $\hat{P}$-dense in the set of real numbers $\mathbb{R}$, as $\hat{P}cl(\mathbb{Q}) = \mathbb{R}$.

When considering the finite complement topology ρ on $\mathbb{R}$, it is observed that every infinite subset of $\mathbb{R}$ is $\hat{P}$-dense in $\mathbb{R}$.

Theorem4.27. A subset $\breve{T}$ of a topological space $(\breve{W}, \rho)$ is $\hat{P}$-dense in $\breve{W}$if and only if for every nonempty $\hat{P}$-open subset $\breve{Y}$ of $\breve{W}$, $\breve{T} \cap \breve{Y} \neq \emptyset$.

Proof: Suppose $\breve{T}$ is $\hat{P}$-dense in $\breve{W}$and $\breve{Y}$is a non empty $\hat{P}$-open set in $\breve{W}$. If $\breve{T} \cap \breve{Y} = \emptyset$, then $\breve{T} \subset \breve{W} \backslash \breve{Y}$ implies that $\hat{P}cl(\breve{T}) \subset \breve{W} \backslash \breve{Y}$ since $\breve{W} \backslash \breve{Y}$ is $\hat{P}$-closed. But then $\breve{W} \backslash \breve{Y} \subset \breve{W}$ contradicts that $\hat{P}cl(\breve{T}) = \breve{W}$ [since $\hat{P}cl(\breve{T}) \subset \breve{W} \backslash \breve{Y} \subset \breve{W}$]. Conversely; assume that $\breve{T}$ meets every non-empty $\hat{P}$-open subset of $\breve{W}$. Thus the only $\hat{P}$-closed set containing $\breve{T}$ is $\breve{W}$ and consequently, $\hat{P}cl(\breve{T}) = \breve{W}$. Hence $\breve{T}$ is $\hat{P}$-dense in $\breve{W}$.

Theorem4.28. In a topological space $(\breve{W}, \rho)$.

i. Any set $\breve{C}$ containing a $\hat{P}$-dense set $\breve{D}$ is a $\hat{P}$-dense set.
ii. If $\breve{T}$ is a $\hat{P}$-dense set and $\breve{Y}$ is $\hat{P}$-dense on $\breve{T}$, then $\breve{Y}$ is also a $\hat{P}$-dense set.

Proof: (i) Since $\breve{D} \subset \breve{C}$, then $\hat{P}cl(\breve{D}) \subset \hat{P}cl(\breve{C})$. But $\hat{P}cl(\breve{D}) = \breve{W}$ hence $\breve{W} \subset \hat{P}cl(\breve{C})$ also $\hat{P}cl(\breve{C}) \subset \breve{W}$ so that $\hat{P}cl(\breve{C}) = \breve{W}$. Thus $\breve{C}$ is $\hat{P}$-dense in $(\breve{W}, \rho)$.

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

ISSN: 2312-8135 | Print ISSN: 1992-0652

info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

(ii) Since $\breve{T}$ is $\hat{P}$-dense, $\hat{P}cl(\breve{T}) = \breve{W}$. Also $\breve{Y}$ is $\hat{P}$-dense on $\breve{T}$, then $\breve{T} \subset \hat{P}cl(\breve{Y})$. Thus $\hat{P}cl(\breve{T}) \subset \hat{P}cl(\hat{P}cl(\breve{Y})) = \hat{P}cl(\breve{Y})$ (By closure property). Then $\hat{P}cl(\breve{T}) \subset \hat{P}cl(\breve{Y})$ this implies that $cl(\breve{W}) = \hat{P}cl(\breve{T}) \subset \hat{P}cl(\breve{Y})$. Thus $\breve{Y}$ is $\hat{P}$-dense in $(\breve{W}, \rho)$.

Definition4.29. Let $(\breve{W}, \rho)$ be a topological space. A subset $\breve{M} \subset \breve{W}$ is referred to as a $\hat{P}$-neighborhood of a subset $\breve{T} \subset \breve{W}$ if there exists an $\hat{P}$-open set $\breve{H}$ such that $\breve{T} \subset \breve{H} \subset \breve{M}$. Specifically, when $\breve{M} = \{r\}$ for some $r \in \breve{W}$, the set $\breve{M}$ is designated as a $\hat{P}$-neighborhood of the point r.

Proposition4.30. Let $\breve{W}$ be a space and let $\breve{T}, \breve{Y} \subset \breve{W}$, then:

i. If $\breve{T} \subset \breve{Y}$, where $\breve{T}$ is $\hat{P}$-neighborhood of $r \in \breve{W}$, then $\breve{Y}$ is also $\hat{P}$-neighborhood of r.
ii. Intersection of two $\hat{P}$-neighborhoods is also $\hat{P}$-neighborhood.
iii. An arbitrary union of $\hat{P}$-neighborhoods of a point $r \in \breve{W}$ is also $\hat{P}$-neighborhood of r.
iv. $\breve{T}$ is $\hat{P}$-open set if and only if is $\hat{P}$-neighborhood of each of it′s points.

Proof: The proofs are clear by definition 4.29.

Definition4.31. A point $r \in \breve{W}$ is said to be an $\hat{P}$-interior point of a subset $\breve{T}$ of $\breve{W}$, if there exists an $\hat{P}$-open set $\breve{H}$ containing r such that $r \in \breve{H} \subset \breve{T}$. The set of all $\hat{P}$-interior points of $\breve{T}$ is said to be $\hat{P}$-interior of $\breve{T}$.

As a direct consequence of the definition we get:

Proposition4.32. Let $\breve{T}$ be any subset of a space $\breve{W}$. If a point r is in the $\hat{P}$-interior of $\breve{T}$, then there exists a pre-closed set $\breve{S}$ of $\breve{W}$ containing r such that $\breve{S} \subset \breve{T}$.

Proof: Let $r \in \hat{P}int(\breve{T})$. The point r is $\hat{P}$-interior of $\breve{T}$ if and only if $\breve{T}$ is $\hat{P}$-neighborhood of r, then there exists an $\hat{P}$-open set $\breve{H}$ such that $r \in \breve{H} \subset \breve{T}$, that is there is a pre-closed set $\breve{S}$, such that $r \in \breve{S} \subset \breve{H}$, this implies that $r \in \breve{S} \subset \breve{T}$.

The subsequent theorem elucidates several characteristics of $\hat{P}$-interior operators on a given set.

Theorem4.33. For any subset $\breve{T}$ of a topological space $(\breve{W}, \rho)$, the $\hat{P}$-interior of $\breve{T}$ is the union of all $\hat{P}$-open sets contained in $\breve{T}$.

Proof: The proof is clear by definition 4.31.

Proposition4.34. Let $(\breve{W}, \rho)$ be a topological space and let $\breve{T}, \breve{Y} \subset \breve{W}$, then:

1. $\hat{P}int(\emptyset) = \emptyset$ and $\hat{P}int(\breve{W}) = \breve{W}$.
2. $\hat{P}int(\breve{T}) = \breve{T} \backslash \hat{P}D(\breve{W} \backslash \breve{T})$.

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

3. $\hat{P}int(\breve{T})$ is the largest $\hat{P}$-open set contained in $\breve{T}$.
4. If $\breve{T} \subset \breve{Y}$, then $\hat{P}int(\breve{T}) \subset \hat{P}int(\breve{Y})$.
5. $\breve{T}$ is $\hat{P}$-open if and only if $\breve{T} = \hat{P}int(\breve{T})$.
6. $\breve{T}$ is $\hat{P}$-neighborhood of $r \in \breve{W}$ if and only if $r \in \hat{P}int(\breve{T})$.
7. $\hat{P}int\left(\hat{P}int(\breve{T})\right) = \hat{P}int(\breve{T})$.
8. $\hat{P}int(\breve{T} \cap \breve{Y}) = \hat{P}int(\breve{T}) \cap \hat{P}int(\breve{Y})$.
9. $\hat{P}int(\breve{T} \cup \breve{Y}) \supset \hat{P}int(\breve{T}) \cup \hat{P}int(\breve{Y})$.
10. Let $\{\breve{T}_k, k \in \nabla\}$ be a family of subsets of a topological space $(\breve{W}, \rho)$, then $\cup_{k\in\nabla} \hat{P}int\,(\breve{T}_k) \subset \hat{P}int\,(\cup_{k\in\nabla} \breve{T}_k)$.

Proof: Straightforward.

In general $\hat{P}int(\breve{T} \cup \breve{Y}) \neq \hat{P}int(\breve{T}) \cup \hat{P}int(\breve{Y})$ as shown in the following example:

Example4.35. Let $\breve{W} = \{a, b, c, d\}$ and $\rho = \left\{\breve{W}, \emptyset, \{c\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\right\} = \hat{P}O(\breve{W}, \rho)$. Let $\breve{T} = \{a, b, c\}$, $\breve{Y} = \{c, d\}$ and $\breve{T} \cup \breve{Y} = \breve{W}$, then $\hat{P}int(\breve{T}) = \{a, b, c\}$, $\hat{P}int(\breve{Y}) = \{c\}$. So $\hat{P}int(\breve{T}) \cup \hat{P}int(\breve{Y}) = \{a, b, c\}$ and $\hat{P}int(\breve{T} \cup \breve{Y}) = \breve{W}$. Therefore, $\hat{P}int(\breve{T}) \cup \hat{P}int(\breve{Y})$ is a proper sub set of $\hat{P}int(\breve{T} \cup \breve{Y})$. Hence $\hat{P}int(\breve{T}) \cup \hat{P}int(\breve{Y}) \neq \hat{P}int(\breve{T} \cup \breve{Y})$.

The statement that the converse in (10) is not generally true can be illustrated by the following example:

Example4.36. Let $\breve{W} = \{a, b, c, d\}$ with topology $\rho = \left\{\emptyset, \breve{W}, \{c\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\right\}$, then $\hat{P}O(\breve{W}) = \left\{\emptyset, \breve{W}, \{c\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\right\}$. Let $\breve{T}_1 = \{a\}$, $\breve{T}_2 = \{b\}$, $\breve{T}_3 = \{a, b\}$, $\breve{T}_4 = \{a, c\}$ and $\breve{T}_5 = \{b, c, d\}$ thus $\hat{P}int(\breve{T}_1) = \emptyset$, $\hat{P}int(\breve{T}_2) = \emptyset$, $\hat{P}int(\breve{T}_3) = \{a, b\}$, $\hat{P}int(\breve{T}_4) = \{c\}$ and $\hat{P}int(\breve{T}_5) = \{c\}$.Therefore, $\cup_{k=1}^{5} \hat{P}int(\breve{T}_k) = \{a, b, c\}\ but\ \cup_{k=1}^{5} \breve{T}_k = \{a, b, c, d\}$ and $\hat{P}int(\cup_{k=1}^{5} \breve{T}_k) = \{a, b, c, d\}$. Therefore, $\hat{P}int\,(\cup_{k\in\nabla}\ \breve{T}_k) \not\subset \cup_{k\in\nabla} \hat{P}int\,(\breve{T}_k)$.

Lemma4.37. Let $\breve{W}$ be a space and $\breve{T} \subset \breve{W}$, then:

1. $\breve{W} \setminus \hat{P}int(\breve{T}) = \hat{P}cl(\breve{W} \setminus \breve{T})$.
2. $\breve{W} \setminus \hat{P}cl(\breve{T}) = \hat{P}int(\breve{W} \setminus \breve{T})$.
3. $\hat{P}int(\breve{T}) = \breve{W} \setminus \hat{P}cl(\breve{W} \setminus \breve{T})$.

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

ISSN: 2312-8135 | Print ISSN: 1992-0652

info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

Proof: 3. Let $r \in \hat{P}int(\breve{T})$, then $\hat{P}int(\breve{T})$ is itself an $\hat{P}$-open set containing r which is disjoint from $\breve{T}$ and so $r \notin \hat{P}D(\breve{T}^c)$. But $r \notin \breve{T}^c, r \notin \hat{P}D(\breve{T}^c)$. This implies that $r \notin \breve{T}^c \cup \hat{P}D(\breve{T}^c)$ thus $r \notin \hat{P}cl(\breve{W} \setminus \breve{T})$ then $r \in \breve{W}\setminus\hat{P}cl(\breve{W} \setminus \breve{T})$. Hence $\hat{P}int(\breve{T}) \subset \breve{W}\setminus\hat{P}cl(\breve{W} \setminus \breve{T})$.

Conversely; suppose that $r \in \breve{W}\setminus\hat{P}cl(\breve{W} \setminus \breve{T})$ then $r \notin \hat{P}cl(\breve{W} \setminus \breve{T})$ thus $r \notin \breve{T}^c \cup \hat{P}D(\breve{T}^c)$, so $r \notin \breve{T}^c$ and $r \notin \hat{P}D(\breve{T}^c)$. Thus $r \in \breve{T}$ and r is not a limit point of $\breve{T}^c$. Thus there exists an open set $\breve{L}$ containing r such that $\breve{T}^c \cap \breve{L}\setminus\{r\} = \emptyset$. Since $r \notin \breve{T}^c$, we have $\breve{T}^c \cap \breve{L} = \emptyset$ and so $\breve{L} \subset \breve{T}$. Thus $r \in \breve{L} \subset \breve{T}$ for some open set $\breve{L}$and so r belongs to the union of all open sets contained in $\breve{T}$, which is $\hat{P}int(\breve{T})$. Thus $\breve{W}\setminus\hat{P}cl(\breve{W} \setminus \breve{T}) \subset \hat{P}int(\breve{T})$. Hence it follows that $\hat{P}int(\breve{T}) = \breve{W}\setminus\hat{P}cl(\breve{W} \setminus \breve{T})$.

Definition4.38. For any subset $\breve{T}$ of a space $\breve{W}$, the set $\hat{P}b(\breve{T}) = \breve{T}\setminus\hat{P}int\ (\breve{T})$ is called the $\hat{P}$-boundary of $\breve{T}$, and the set $\hat{P}F_r(\breve{T}) = \hat{P}cl\ (\breve{T})\setminus\hat{P}int\ (\breve{T})$ is called the $\hat{P}$-frontier of $\breve{T}$.

Note that if $\breve{T}$ is a $\hat{P}$-closed subset of $\breve{W}$, then $\hat{P}b(\breve{T}) = \hat{P}F_r(\breve{T})$, in general $\hat{P}b(\breve{T}) \neq \hat{P}F_r(\breve{T})$.

Example4.39. Given the set $\breve{W} = \{a, b, c, d\}$ with topology $\rho = \{\emptyset, \breve{W}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$, we can derive the following: The set $\hat{P}O(\breve{W})$ is defined as $\{\emptyset, \breve{W}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$, then $\hat{P}C(\breve{W}) = \{\emptyset, \breve{W}, \{c, d\}, \{d\}, \{c\}\}$. For the subset $\breve{T} = \{a, b\}$:

1. $\hat{P}int(\breve{T})$ is $\{a, b\}$.
2. $\hat{P}cl(\breve{T})$ is $\breve{W}$.
3. $\hat{P}F_r(\breve{T})$ is $\{c, d\}$.
4. $\hat{P}b(\breve{T}) = \emptyset$.

Proposition4.40. The following assertions are true for a subset $\breve{T}$ of $\breve{W}$:

1. $\breve{T} = \hat{P}int(\breve{T}) \cup \hat{P}b(\breve{T})$.
2. $\hat{P}int(\breve{T}) \cap \hat{P}b(\breve{T}) = \emptyset$.
3. $\breve{T}$ is $\hat{P}$-open set if and only if $\hat{P}b(\breve{T}) = \emptyset$.
4. $\hat{P}b\left(\hat{P}int(\breve{T})\right) = \emptyset$.
5. $\hat{P}int\left(\hat{P}b(\breve{T})\right) = \emptyset$.
6. $\hat{P}b\left(\hat{P}b(\breve{T})\right) = \hat{P}b(\breve{T})$.

7. $\hat{P}b(\breve{T}) = \breve{T} \cap \hat{P}cl(\breve{W} \backslash \breve{T})$.
8. $\hat{P}b(\breve{T}) = \breve{T} \cap \hat{P}D(\breve{W} \backslash \breve{T})$.

Proof: Obvious.

Lemma4.41. For a subset $\breve{T}$ of $\breve{W}$, $\breve{T}$ is $\hat{P}$-closed if and only if $\hat{P}F_r(\breve{T}) \subset \breve{T}$.

Proof: Assume that $\breve{T}$ is $\hat{P}$-closed. The $\hat{P}F_r(\breve{T}) = \hat{P}cl(\breve{T}) \backslash \hat{P}int(\breve{T}) = \breve{T} \backslash \hat{P}int(\breve{T}) \subset \breve{T}$. Conversely; suppose that $\hat{P}F_r(\breve{T}) \subset \breve{T}$. Then $\hat{P}cl(\breve{T}) \backslash \hat{P}int(\breve{T}) \subset \breve{T}$ and so $\hat{P}cl(\breve{T}) \subset \breve{T}$ since $\hat{P}int(\breve{T}) \subset \breve{T}$. Noting that $\breve{T} \subset \hat{P}cl(\breve{T})$, we have $\breve{T} = \hat{P}cl(\breve{T})$. Therefore, $\breve{T}$ is $\hat{P}$-closed.

Theorem4.42. The following claims hold for a subset $\breve{T}$ of $\breve{W}$:

1. $\hat{P}cl(\breve{T}) = \hat{P}int(\breve{T}) \cup \hat{P}F_r(\breve{T})$.
2. $\hat{P}int(\breve{T}) \cap \hat{P}F_r(\breve{T}) = \emptyset$.
3. $\hat{P}b(\breve{T}) \subset \hat{P}F_r(\breve{T})$.
4. $\hat{P}F_r(\breve{T}) = \hat{P}b(\breve{T}) \cup \left(\hat{P}D(\breve{T}) \backslash \hat{P}int(\breve{T})\right)$.
5. $\breve{T}$ is $\hat{P}$-open set if and only if $\hat{P}F_r(\breve{T}) = \hat{P}b(\breve{W} \backslash \breve{T})$.
6. $\hat{P}F_r(\breve{T}) = \hat{P}cl(\breve{T}) \cap \hat{P}cl(\breve{W} \backslash \breve{T})$.
7. $\hat{P}F_r(\breve{T}) = \hat{P}F_r(\breve{W} \backslash \breve{T})$.
8. $\hat{P}F_r(\breve{T})$ is pre-closed.
9. $\hat{P}F_r(\hat{P}F_r\breve{T}) \subset \hat{P}F_r(\breve{T})$.
10. $\hat{P}F_r\left(\hat{P}int(\breve{T})\right) \subset \hat{P}F_r(\breve{T})$.
11. $\hat{P}F_r\left(\hat{P}cl(\breve{T})\right) \subset \hat{P}F_r(\breve{T})$.
12. $\hat{P}int(\breve{T}) = \breve{T} \backslash \hat{P}F_r(\breve{T})$.

Proof: Straightforward.

5. CONCLUSION

In this paper, we introduced a new class of generalized open sets, called $\hat{P}$-open sets, in topological spaces. We showed that they form a topology, representing a stronger version of ordinary open sets. Their relationships with other generalized open sets were examined, and fundamental properties were established. We also defined related operators $\hat{P}$-interior, $\hat{P}$-closure,

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

and $\hat{P}$-derived set, and studied their properties. These results show that $\hat{P}$-open sets extend the theory of generalized open sets and enhance our understanding of topological structures.

This work paves the way for further research on separation axioms, continuity, and other topological properties using $\hat{P}$-open sets, with potential applications across topology.

Acknowledgment

The author expresses gratitude to Salahaddin University-Erbil for its institutional assistance, which was crucial to making this work a reality.

Conflict of interests.

The authors decelerate that there is no conflict of interest.

References

[1] C. Kuratowski, "Sur l'opération A de l'analysis situs," *Fundamenta Mathematicae,* vol. 3, no. 1, pp. 182-199, 1922.

[2] N. Levine, "Semi-open sets and semi-continuity in topological spaces," *The American mathematical monthly,* vol. 70, no. 1, pp. 36-41, 1963.

[3] A.S. Mashhour, I.A. Hasanein, and S.N. El-Deeb, "On preopen sets and precontinuous mappings," Bull. Math. Soc. Sci. Math. Roumanie, vol. 26, pp. 45–56, 1982.

[4] N. Velicko, "On θ-open sets in topological spaces," Mat. Sb., vol. 75, pp. 134–146, 1968.

[5] O. Njåstad, "On some classes of nearly open sets," *Pacific journal of mathematics,* vol. 15, no. 3, pp. 961-970, 1965.

[6] N. K. Ahmed and O. T. Pirbal, "A NEW TYPE OF MINIMAL AND MAXIMAL SETS," *Palestine Journal of Mathematics,* vol. 13, no. 4, 2024.

[7] R. P. Stanley, "On the number of open sets of finite topologies," *Journal of Combinatorial Theory, Series A,* vol. 10, no. 1, pp. 74-79, 1971.

[8] S. Al Ghour, "Between the classes of soft open sets and soft omega open sets," *Mathematics,* vol. 10, no. 5, p. 719, 2022.

[9] P. Şaşmaz and M. Özkoç, "On the Topology of $\delta_ {\omega} $-Open Sets and Related Topics," *Konuralp Journal of Mathematics,* vol. 10, no. 1, pp. 203-209, 2022.

[10] L. A. Añora, C. Paran, L. L. Butanas, and M. Labendia, "Theta^ omega-Open Set and its Corresponding Topological Concepts," *The Mindanawan Journal of Mathematics,* vol. 5, no. 2, pp. 75-90, 2023.

[11] L. A. Steen and J. A. Seebach, "Counterexamples in topology, Holt, Rinehard and Winston," *Inc., New York,* 1970.

[12] T. Noiri and S. On, "closed and S-perfect functions," *Atti della Acad. Delle Sci. Torino,* pp. 71-79, 1986.

[13] N. El-Deeb, I. Hasanein, A. Mashhour, and T. Noiri, "On p-regular spaces," *Bull. Math. Soc. Sci. Math. R. S. Roumanie*, pp. 311–315, 1983.

[14] H. Shareef, *On Sp-open sets, Sp-continuity and Sp-compactness in Topological spaces*. LAP LAMBERT Academic Publishing, 2011.

[15] A. B. Khalaf and S. M. Mershkhan, "Pp-Open Sets and Pp-Continuous Functions," *Gen,* vol. 20, no. 1, pp. 34-51, 2014.

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية
ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

[16] J. Dontchev, M. Ganster, and T. Noiri, "On p-closed spaces," *International Journal of Mathematics and Mathematical Sciences,* vol. 24, no. 3, pp. 203-212, 2000.
[17] A. R. Sadek, "On pre Open Regular Spaces," *Baghdad Science Journal,* vol. 21, no. 12, pp. 3831-3835, 2024.
[18] A. Kar and P. Bhattacharyya, "Some weak separation axioms," *Bull. Calcutta Math. Soc,* vol. 82, no. 5, pp. 415-422, 1990.
[19] T. Noiri, "A note on extremally disconnected spaces," *Proceedings of the American Mathematical Society,* vol. 79, no. 2, pp. 327-330, 1980.
[20] M. Stone, "Algebraic characterizations of special Boolean rings," *Fundamenta Mathematicae,* vol. 29, no. 1, pp. 223-303, 1937.
[21] V. V. Gautam, "Some equivalent of extremally disconnected spaces," *arXiv preprint math/0206122,* 2002.
[22] T. Y. Guo, "A characterization of extremally disconnected spaces," *J. Central China Normal Univ. Natur. Sci,* vol. 21, no. 2, pp. 169-170, 1981.
[23] G. Navalagi, "Further properties of pre-To, pre-T1 and pre-T2 spaces," 2011.
[24] J. Dontchev, "Survey on preopen sets," *arXiv preprint math/9810177,* 1998.
[25] N. Ahmed, "On some types of separation axioms," *M. Bc. Thesis, Salahaddin University, Arbil,* 1990.
[26] J. N. Sharma, *Topology*. Meerut (U.P.), India: Krishna Prakashan Mandir, 1977.

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

الخلاصة

يقدّم هذا البحث فئةً جديدة من المجموعات المفتوحة تُعرف باسم المجموعات-$\hat{P}$ المفتوحة ضمن إطار الطوبولوجيا العامة . وتُظهر الدراسة أن هذه الفئة تولّد طوبولوجيا تقع بين فئتي المجموعات -$\boldsymbol{\theta}$ المفتوحة **و**المجموعات المفتوحة الاعتيادية .كما تم تعريف ودراسة المؤثرات الطوبولوجية المرتبطة، وهي الداخل$\hat{P}$ ، والإغلاق $\hat{P}$، **و**المجموعة المشتقة $\hat{P}$ ، مع بيان خصائصها الأساسية والعلاقات المتبادلة بينها، بما يساهم في تطوير نظرية المجموعات المفتوحة المعممة وتوسيع بنيتها الطوبولوجية.