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A Theoretical Study On Adjoint Drazin Inverse Hyponormal Operators Based On Fuzzy Soft Set Theory

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دراسة نظرية حول المرافق للمؤثرات فوق السوية بالاعتماد على نظرية الاعداد الضبابية الناعمة

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ABSTRACT

Background: The aim of this article is to introduce novel traits of a new version of the fuzzy soft hyponormal operator which namely the fuzzy soft adjoint Drazin inverse hyponormal operator symbolized by (FS-(D, ∗)- hyponormal operator), in a fuzzy soft Hilbert space

Materials and Methods: this study in the realm of Operator Theory related to Fuzzy Soft Theory (FST). Some analytical properties have been discovered via theorems related in this concept, moreover submitted the conditions required to prove the direct sum and tensor produced.

Results: In this paper, a new class of fuzzy soft hyponormal operators, which is closely related to the Drazin inverse is introduced, namely the fuzzy soft adjoint Drazin inverse hyponormal operator. The necessary conditions for a new operator are investigated. The direct sum and tensor product are also examined

Conclusion: In this work, we obtain significant results based on the newly introduced definition fuzzy soft (D, ∗)-hyponormal operator, such as a theorem equivalent to the definition under certain conditions as well as the sum and product of two operators.

Key words: Hilbert space, Soft set ,Fuzzy soft set, Fuzzy Soft Hilbert space, Fuzzy Soft Drazin invertible, FS-(D, ∗)- hyponormal operator.

INTRODUCTION

The theory of operators is one of the functional analysis and the theory of linear spaces. It is closely related to the algebraic and analytical operations of vector space, especially complex Hilbert spaces. The study of operators has received wide attention due to its important fundamental applications in spectral analysis, dynamical systems. Analysis, and the treatment of problems. Researchers also use hyponormal to examine the geometric properties of operators in Hilbert spaces. The study of normed operators, hyponormal operators, and Drazin operators has

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opened new horizons for understanding the structural behavior of operators. In 1980, Halmos [1] first put his idea of hyponormality under the different term, subnormality. Furthermore,many properties of hyponormal operators are explained. In general, A hyponormal operator is based on the concept of adjoint, for instance, the unilateral shift operator. In 1984, M.Putinar [2] proposed a universal functional model for hyponormal operators, where it was shown that every hyponormal operator is a subscalar operator .In 2019,Okelo [3] provided a detailed description of hyponormal operators that achieved new results for norm. In 2019, Sitate [4] introduced a generalized class $\boldsymbol{G_A}$of A-unitary, A-normal, and A-hyponormal operators, and the properties of this class of operators were explained and proven A-unitary equivalence is an equivalence relation it also proves many results in terms of polar analysis of an operator T. Afterwards, In 2020, Mohsen [5] introduced a new generalization for hyponormal operators, which gave the solvability of the $\boldsymbol{\lambda}$-commuting operators equation, and studied some properties of these operators. In 2020, Chellali and ali [6] studied a new class of operators defined on complex Hilbert space, named (n,m)-power-D-hyponormal operators, which is related to the Drazin inverse through which the proposed operator is defined. They also studied some of its advantages and presented some practical examples. Whereas, in 2020, Dana and Yousefi [7] studied the generalized class of D-hyponormal operators and extended the concept of D-normality and achieved fundamental results in proving many theorems to include a wider range of cases within Hilbert spaces. In the same year, Dharmarha and Ram [8] studied two generalized classes of linear operators on Hilbert spaces, namely (m,n)-paranormal and $(m,n)^*$-paranormal operators. In 2021,Mortad [9] presented three counterexamples to illustrate that supernatural operators are densely defined , but some of these may not be closeable. In 2016, Bachir [10] studied a class of (p,k)-quasiposinormal operators and concluded that the generalized derivative resulting from proving Putnam-Fugede theorem is orthogonal to its kernel. In 2022, Mesbah and Messaoudene [11] studied a special class of D-hyponrmal and D-quasi-hyponrmal operators and investigated some properties of these operators, they also presented a generalized study of the Fuglede-Putnam theorem. In the same year. Benali [12] studied some basic properties of a new class of (n,m)-power-D-quasi-hyponormal operators generalizations closely related to the Drazin inverse. Whereas, in 2023, Mohsen [13] considered the notion of an M-hyponormal operator, a novel extension of the hyponormal operator, and discussed several crucial outcomes related to it. Moreover, Shen et al. [14] introduced a class of p-quasi-n-hyponormal operators and demonstrated their structural properties via the Hansen inequality and Lowner-Heinz inequality. Afterwards, in 2024, Mohsen [15] established the notion of k Quasi (ʎ −M) -hyponormal Operator. In addition, Al-Shammari [16] formulated another generalization related to the Drazin inverse called (n,m)-Drazin normal operators. In 2024, Mohsen [17] introduced a completely new class of quasi operators, named (n,D)-quasi operators on the Hilbert spaces using Drazin inverse, also studied the scalar and power of this concept and proved of the Tensor product and direct product for this class. Whereas, In 2025, Dana et al. [18] likewise provided some findings on the inequalities and generalized outcomes of the D-hyponormal operator. However, in 2004, Caradus [19] provided and discussed the Drazin inverse operator, as a generalization of a hyponormal operator, it is used in solving many in Cauchy problems, differential equations, and analyzing Markov chains, [20,21].Whereas, in 2006,Liu and Wang [22] proposed a concept of linear operators between normed linear spaces and defined a type of inverse called a bounded quasi-linear generalized inverse. In past years, numerous researchers, such as Duggal and Kim [23], Chen and Sheibani [24], and Qin and Lu [25] have proposed other generalization and important developments in Drazin inverse theory and its extensions in different fields, such as

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Weyl's theorem and the Spectral Mapping Theorem they have also discussed several other properties that coud be generalizations defined that open up horizons for applications in the analysis of normal and hyponormal operators and its implementations within Hilbert spaces and Banach algebras. Within this framework, in 1965, Zadeh [26] introduced a significant mathematical concept called fuzzy sets. Thereafter, in 1999, Molodsov [27] established Soft Set Theory (SS-T) as an elegant mathematical tool for treating uncertainty and assorted complex problems. Whereas, in 2001, Maji [28] combined the principles of fuzzy and soft into an interesting idea, which is named Fuzzy Soft Set (FSS). Fuzzy Soft Set Theory (FSs-T) is a substantial theme. Many investigators have studied numerous elegant implementations of FSs-T in varied disciplines and contributed to the expansion and highlighting of considerable new characteristics on this topic. Among their pivotal contributions are, for instance [29-36]. In 2023, Mohsen [37] developed a new idea named fuzzy soft b-metric, and developed some basic theories of this concept and defined a fuzzy soft open ball and fuzzy soft Hausdorff b-metric space. Subsequently, In 2020, Faried et al. [38] studied several generalized concepts on fuzzy soft (FS) systems, where they presented a generalization of the fuzzy soft orthogonal family and the fuzzy soft orthonormal family also studied fuzzy soft spectral and fuzzy soft spectral radius with other types of fuzzy soft Linear operators ,also presented a definition of fuzzy soft shift from right and left and they concluded that fuzzy soft Hilbert space is fuzzy soft self-dual.). In 2020, Faried et al. [39] investigated the fuzzy soft Hilbert space principle alongside its attributes and distinct new outcomes. In 2022, Mohsen [40] formulated an advanced class within fuzzy soft Hilbert space called fuzzy soft quasi normal operator and discussed some properties of this concept. Afterwards, in 2023, Mohsen [41] proposed a new class of fuzzy soft operator in fuzzy soft Hilbert space, named fuzzy soft $(\widehat{\boldsymbol{k}}^{*}-\widehat{\boldsymbol{A}})$-quasinormal operator. Whereas, in 2024, Radharamani and Nagajothi [42] proposed a first of its kind class of fuzzy soft paranormal operators and they also studied some of the concepts and theories related to this regard. In this paper, a new class of fuzzy soft hyponormal operators, which is closely related to the Drazin inverse is introduced, namely the fuzzy soft adjoint Drazin inverse hyponormal operator. The necessary conditions for a new operator are investigated. The direct sum and tensor product are also examined.

MATERIALS AND METHODS

This section introduces the key concepts in the study of fuzzy soft Hilbert space, which will be employed to get the main findings.

Definition 2.1. [26] The set of order pair $\widetilde{\boldsymbol{\mathcal{G}}}$= {(Y, $\mu_{\tilde{\mathcal{G}}}$(Y))| Y ∈ X , $\mathcal{A}$(Y) ∈ $\mathcal{I}$} named fuzzy set on X with a membership function $\mu_{\tilde{\mathcal{G}}}$:X ⟶ Γ, where Γ = [0,1]. A set $\widetilde{\boldsymbol{\mathcal{G}}}$ is sometimes be written as $\widetilde{\boldsymbol{\mathcal{G}}}=\left\{\frac{\mu_{\tilde{\mathcal{G}}}(\mathrm{Y})}{\mathrm{Y}}\ \mathrm{Y} \in \mathrm{X}\right\}$. The real number, $\mu_{\tilde{\mathcal{G}}}$(Y) , namely the membership of Y in $\widetilde{\boldsymbol{\mathcal{G}}}$.

Definition 2.2. [27] The set $\mathfrak{f}_{\mathcal{G}}$= {𝔣 ($\omega$) ∈ (X): ω ∈ $\mathcal{G}$} named soft set over X ,with a set of parameters Σ and (X) the set of all subsets $\mathcal{X}$ such that $\mathcal{G}$⊆ Σ, 𝔣: $\mathcal{G}$⟶ X , represented by $\mathfrak{f}_{\mathcal{G}}$.

Definition 2.3. [28] The soft set $\mathfrak{f}_{\mathcal{G}}$ is named fuzzy soft set (FSs) over $\mathcal{X}$, where 𝔣: $\mathcal{G}$ ⟶Γ^{X}, has the range {(ω) ∈Γ^{X}: ω ∈ $\mathcal{G}$}, and the class of all (FSs), indicated by FSs($\tilde{X}$).

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Definition 2.4. [39] The set fuzzy soft vector is FS-vector space , according with the following two operations below:

i. $\tilde{v}^1_{f_1 G(e_1)} + \tilde{v}^2_{f_2 G(e_2)} = (\widetilde{v^1 + v^2})_{(f_{1_{G(e_1)}} + f_{2_{G(e_2)}})}$ for all $\tilde{v}^1_{f_1 G(e_1)}, \tilde{v}^2_{f_2 G(e_2)} \in$ FS-V.

ii. $\tilde{r}\tilde{v}_{f_{G(e)}} = \widetilde{(rv)}_{f_{G(e)}}$ for all $\tilde{v}_{f_{G(e)}} \in$ FS-V and $\forall \tilde{r} \in \tilde{R}_B$.

Definition 2.5. [39] Assume $\tilde{H}$ be FS-V and $\mathcal{R}(\mathcal{A})$ be fuzzy soft real set. A mapping $\widetilde{\|.\|}: \tilde{H} \to \mathcal{R}(\mathcal{A})$ namely fuzzy soft norm (FS-N) on $\tilde{H}$ if $\widetilde{\|.\|}$ achieves the following:

i. for all $\tilde{v}_{f_{G(e)}} \in \tilde{H}$ $\widetilde{\left\|\tilde{v}_{f_{G(e)}}\right\|} \gtrsim \tilde{0}$;

ii. $\left\|\tilde{v}_{f_{G(e)}}\right\| \cong \tilde{0}$ if and only if $\tilde{v}_{f_{G(e)}} \cong \tilde{0}$;

iii. for all $\tilde{v}_{f_{G(e)}} \in \tilde{H}$, and $\tilde{r} \in \mathcal{C}(\mathcal{A})$ $\widetilde{\left\|\tilde{r}\tilde{v}_{f_{G(e)}}\right\|} \cong |\tilde{r}| \widetilde{\left\|\tilde{v}_{f_{G(e)}}\right\|}$;

iv. for all $\tilde{v}_{f_{G(e_1)}}$, $\tilde{u}_{f_{G(e_2)}} \in \tilde{H}$ obtain $\widetilde{\left\|\tilde{v}_{f_{G(e_1)}} + \tilde{u}_{f_{G(e_2)}}\right\|} \lesssim \widetilde{\left\|\tilde{v}_{f_{G(e_1)}}\right\|} + \widetilde{\left\|\tilde{u}_{f_{G(e_2)}}\right\|}$.

The FS-V $\tilde{H}$ with $\widetilde{\|.\|}$ indicated by ($\tilde{H}, \widetilde{\|.\|}$),namely fuzzy soft normed space (FS-N space).

Definition 2.6. [38] Assume $\tilde{H}$ be FS-V and $\mathcal{C}(\mathcal{A})$ be fuzzy soft complex set. Then, the mapping $\widetilde{\langle .,. \rangle}: \tilde{H} \times \tilde{H} \to \mathcal{C}(\mathcal{A})$, called fuzzy soft inner product (FS-IP) on $\tilde{H}$ if $\widetilde{\langle .,. \rangle}$ achieves the following:

i. for all $\tilde{v}_{f_{G(e)}} \in \tilde{H}$ yields $\widetilde{\langle \tilde{v}_{f_{G(e)}}, \tilde{v}_{f_{G(e)}} \rangle} \gtrsim \tilde{0}$;

ii. $\widetilde{\langle \tilde{v}_{f_{G(e)}}, \tilde{v}_{f_{G(e)}} \rangle} \cong \tilde{0}$ if and only if $\tilde{v}_{f_{G(e)}} \cong \tilde{0}$;

iii. for all $\tilde{v}^1_{f_1 G(e_1)}, \tilde{v}^2_{f_2 G(e_2)} \in \tilde{H}$ gains $\widetilde{\langle \tilde{v}^1_{f_1 G(e_1)}, \tilde{v}^2_{f_2 G(e_2)} \rangle} \cong \overline{\widetilde{\langle \tilde{v}^2_{f_2 G(e_2)}, \tilde{v}^1_{f_1 G(e_1)} \rangle}}$;

iv. for all $\tilde{v}^1_{f_1 G(e_1)}, \tilde{v}^2_{f_2 (e_2)} \in \tilde{H}$ *and* $\tilde{r} \in \mathcal{C}(\mathcal{A})$, attains $\widetilde{\langle \tilde{r}\, \tilde{v}^1_{f_1 G(e_1)}, \tilde{v}^2_{f_2 G(e_2)} \rangle} \cong \tilde{r} \widetilde{\langle \tilde{v}^1_{f_1 G(e_1)}, \tilde{v}^2_{f_2 G(e_2)} \rangle}$;

v. for all $\tilde{v}^1_{f_1 G(e_1)}, \tilde{v}^2_{f_2 G(e_2)}, \tilde{v}^3_{f_3 G(e_3)}, \in \tilde{H}$ we yield $\widetilde{\langle \tilde{v}^1_{f_1 G(e_1)} + \tilde{v}^2_{f_2 G(e_2)}, \tilde{v}^3_{f_3 G(e_3)} \rangle} \cong \widetilde{\langle \tilde{v}^1_{f_1 G(e_1)}, \tilde{v}^3_{f_3 G(e_3)} \rangle} + \widetilde{\langle \tilde{v}^2_{f_2 G(e_2)}, \tilde{v}^3_{f_3 G(e_3)} \rangle}$.

The FS-V $\tilde{H}$ with $\widetilde{\langle .,. \rangle}$ indicated by $(\tilde{H}, \widetilde{\langle .,. \rangle})$ namely fuzzy soft inner product space (FS-IP space).

Definition 2.7.[39] Assume $(\tilde{H}, \widetilde{\langle .,. \rangle})$ be a fuzzy soft inner product space, then is fuzzy soft Hilbert space when its fuzzy soft complete, denoted by FSH-space.

Definition 2.8. [38] Let $\tilde{H}$ be FSH-space, the operator $\tilde{\Lambda}: \tilde{H} \to \tilde{H}$, named fuzzy soft linear operator (FS-operator), if for all $\tilde{v}^1_{f_1 G(e_1)}, \tilde{v}^2_{f_2 G(e_2)} \in \tilde{H}$ and $\tilde{\sigma}, \tilde{\rho} \in \mathcal{C}(\mathcal{A})$, then

$$\tilde{\Lambda}(\tilde{\sigma}\, \tilde{v}^1_{f_1 G(e_1)} +, \tilde{\rho}\, \tilde{v}^2_{f_2 G(e_2)}) \cong \tilde{\sigma}\tilde{\Lambda}\left(\tilde{v}^1_{f_1 G(e_1)}\right) + \tilde{\rho}\tilde{\Lambda}\left(\tilde{v}^2_{f_2 G(e_2)}\right).$$

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Definition 2.9.[39] Assume $\widetilde{H}$ be a fuzzy soft Hilbert spase and Let $\widetilde{\Lambda}: \widetilde{H} \to \widetilde{H}$ be a fuzzy soft linear operator. the operator $\widetilde{\Lambda}$ named fuzzy soft bounded operator (FSB-operator), if there exists a positive real constant $\widetilde{\Sigma} \in \tilde{R}_A$ such that $\left\|\widetilde{\Lambda}\left(\widetilde{\tilde{v}_{f_{G(e)}}}\right)\right\| \lesseqqgtr \widetilde{\Sigma}\left\|\widetilde{\tilde{v}_{f_{G(e)}}}\right\|$ for all $\tilde{v}_{f_{G(e)}} \in \widetilde{H}$.

Definition 2.10. [38] Assume $\widetilde{H}$ is fuzzy soft Hilbert space and $\widetilde{\Lambda}: \widetilde{H} \to \widetilde{H}$ is FSB-operator, then the fuzzy soft adjoint operator of $\widetilde{\Lambda}$ denoted by $\widetilde{\Lambda}^*$is defined by

$$\langle \widetilde{\Lambda}\, \widetilde{\tilde{v}^1_{f_1 G(e_1)}, v^2_{f_2 G(e_2)}} \rangle \cong \langle \tilde{v}^1_{f_1 G(e_1)}, \widetilde{\widetilde{\Lambda}^*}\, \tilde{v}^2_{f_2 G(e_2)} \rangle, \ \forall\ \tilde{v}^1_{f_1 G(e_1)}, \tilde{v}^2_{f_2 G(e_2)} \in \widetilde{H}.$$

RESULTS AND DISCUSSION

In this section we present a new class of fuzzy soft hyponormal operators by utilizing the Drazin inverse is. at first, the definition show in [11], it generalized the down.

Definition 3.1.

Assume Fuzzy soft operator $\widetilde{\Lambda} \in \mathrm{FSB}(\widetilde{H})$ is Fuzzy soft Drazin invertible if there exists a unique Fuzzy soft operator $\widetilde{\Lambda}^D \in \mathrm{FSB}(\widetilde{H})$ ($\widetilde{\Lambda}^D$ is the fuzzy soft Drazin invertible of $\widetilde{\Lambda}$)
$\widetilde{\Lambda}\widetilde{\Lambda}^D \cong \widetilde{\Lambda}^D\widetilde{\Lambda}$, $\widetilde{\Lambda}^D\widetilde{\Lambda}\widetilde{\Lambda}^D \cong \widetilde{\Lambda}^D$, $\widetilde{\Lambda}^{n+1} \cong \widetilde{\Lambda}^n$ for some n∈N and $\mathrm{FSB}(\widetilde{H})^D$.

Definition 3.2. Assume operator $\widetilde{\Lambda} \mathrel{\widetilde{\in}} \widetilde{B}(\widetilde{H})$, named fuzzy soft (D,∗)-normal operator if $\widetilde{\Lambda}\widetilde{\Lambda}^{*D} - \widetilde{\Lambda}^{*D}\widetilde{\Lambda} \cong \widetilde{0}$.

Definition 3.3. Assume $\widetilde{\Lambda} \in \widetilde{\mathrm{B}}(\widetilde{H})^D$ named fuzzy soft (D ,∗)-hyponormal operator if $\widetilde{\Lambda}\widetilde{\Lambda}^{*D} \gtreqqless \widetilde{\Lambda}^{*D}\widetilde{\Lambda}$, and shortly FS-(D,*)-hyponormal operator .

Example 3.4. $\widetilde{\Lambda} = \begin{bmatrix} (0.4,0) & (0.1,0) \\ (0.2,0) & (0.6,0) \end{bmatrix}$ is FS-(D,*)-hyponormal operator, since

$$\widetilde{\Lambda}^* = \begin{bmatrix} (0.6,0) & -(0.2,0) \\ -(0.1,0) & (0.4,0) \end{bmatrix} \text{ and } \widetilde{\Lambda}^{*D} = \begin{bmatrix} (0.75,0) & (0,0) \\ (0.12,0) & (1,0) \end{bmatrix} \text{ then } \widetilde{\Lambda}\widetilde{\Lambda}^{*D} - \widetilde{\Lambda}^{*D}\widetilde{\Lambda} \gtreqqless \widetilde{0}.$$

Proposition 3.5. Assume $\widetilde{\Lambda}: \widetilde{H} \to \widetilde{H}$ be FS-(D ,∗)-hyponormal operator on FSH-space Then $(\widetilde{\Lambda}\,\widetilde{\Lambda}^{*D})^m \gtreqqless (\widetilde{\Lambda}^{*D}\widetilde{\Lambda})^m$ is FS-(D ,∗)-hyponormal operator.

Proof: By utilizing mathematical induction, since $\widetilde{\Lambda}$ be FS-(D, ∗)- hyponormal operator, then $(\widetilde{\Lambda}\,\widetilde{\Lambda}^{*D})^m \gtreqqless (\widetilde{\Lambda}^{*D}\widetilde{\Lambda})^m$ is FS-(D ,∗)-hyponormal operator.

For m=1 then $(\widetilde{\Lambda}\,\widetilde{\Lambda}^{*D})^1 \gtreqqless (\widetilde{\Lambda}^{*D}\widetilde{\Lambda})^1$ ……………. (i)

Given this outcome is attained for m=k, consequently

$(\widetilde{\Lambda}\,\widetilde{\Lambda}^{*D})^k \gtreqqless (\widetilde{\Lambda}^{*D}\widetilde{\Lambda})^k$…………… (ii)

Then, to demonstrate the validity of the outcome for m=k+1,

$(\widetilde{\Lambda}\,\widetilde{\Lambda}^{*D})^{k+1} \gtreqqless (\widetilde{\Lambda}^{*D}\widetilde{\Lambda})^k(\widetilde{\Lambda}^{*D}\widetilde{\Lambda})^1$,

from Eq. (i) and Eq. (ii), we yield

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$(\widetilde{\Lambda}\,\widetilde{\Lambda}^{*D})^{k+1} \gneqq (\widetilde{\Lambda}^{*D}\widetilde{\Lambda})^{k+1}$

Therefore $(\widetilde{\Lambda}\,\widetilde{\Lambda}^{*D})^{m} \gneqq (\,\widetilde{\Lambda}^{*D}\widetilde{\Lambda})^{m}$ is FS-(D ,∗)-hyponormal operator.

Theorem 3.6. Assume $\widetilde{\Lambda}$ and $\tilde{k}$ be FS-(D ,∗)-hyponormal operator on FSH-space $\widetilde{H}$. If $\tilde{k}\widetilde{\Lambda}^{*D} \cong \widetilde{\Lambda}^{*D}\tilde{k}$ *and* $\widetilde{\Lambda}^{*D}\tilde{k} \cong \tilde{k}\widetilde{\Lambda}^{*D}$ Then $(\widetilde{\Lambda} + \tilde{k})$ is FS-(D ,∗)-hyponormal operator.

Proof: Since $\widetilde{\Lambda}$ and $\tilde{k}$ are FS-(D ,∗)-hyponormal operator then
$\tilde{k}\widetilde{\Lambda}^{*D} \cong \widetilde{\Lambda}^{*D}\tilde{k}$ *and* $\widetilde{\Lambda}^{*D}\tilde{k} \cong \tilde{k}\widetilde{\Lambda}^{*D}$

$$\begin{aligned}(\widetilde{\Lambda} + \tilde{k})\,(\widetilde{\Lambda} + \tilde{k})^{*D} &\cong (\widetilde{\Lambda} + \tilde{k})(\widetilde{\Lambda}^{*D} + \tilde{k}^{*D}) \\ &\cong \widetilde{\Lambda}\widetilde{\Lambda}^{*D} + \widetilde{\Lambda}\tilde{k}^{*D} + \tilde{k}\widetilde{\Lambda}^{*D} + \tilde{k}\tilde{k}^{*D} \\ &\gneqq \widetilde{\Lambda}^{*D}\widetilde{\Lambda} + \tilde{k}^{*D}\widetilde{\Lambda} + \widetilde{\Lambda}^{*D}\tilde{k} + \tilde{k}^{*D}\tilde{k} \\ &\cong (\widetilde{\Lambda}^{*D} + \tilde{k}^{*D})(\widetilde{\Lambda} + \tilde{k}) \\ &\cong (\widetilde{\Lambda} + \tilde{k})^{*D}(\widetilde{\Lambda} + \tilde{k})\end{aligned}$$

Hence, $\widetilde{\Lambda} + \tilde{k}$ *is* FS-(D, ∗)-hyponormal operator.

Theorem 3. 7. Assume $\widetilde{\Lambda}$ and $\tilde{k}$ be FS-(D ,∗)-hyponormal operator on FSH-space $\widetilde{H}$ If $\tilde{k}\widetilde{\Lambda}^{*D} \cong \widetilde{\Lambda}^{*D}\tilde{k}$ *and* $\widetilde{\Lambda}^{*D}\tilde{k} \cong \tilde{k}\widetilde{\Lambda}^{*D}$. Then $\widetilde{\Lambda}\tilde{k}$ is FS-(D ,∗)-hyponormal operator.

Proof: Since $\widetilde{\Lambda}$ and $\tilde{k}$ are FS-(D ,∗)-hyponormal operator then $\tilde{k}\widetilde{\Lambda}^{*D} \cong \widetilde{\Lambda}^{*D}\tilde{k}$ *and* $\widetilde{\Lambda}^{*D}\tilde{k} \cong \tilde{k}\widetilde{\Lambda}^{*D}$.

$$\begin{aligned}(\widetilde{\Lambda}\tilde{k})\,(\widetilde{\Lambda}\tilde{k})^{*D} &\cong (\widetilde{\Lambda}\,\tilde{k})(\tilde{k}^{*D}\widetilde{\Lambda}^{*D}) \\ &\cong \widetilde{\Lambda}\,(\tilde{k}\,\tilde{k}^{*D})\widetilde{\Lambda}^{*D} \\ &\gneqq \widetilde{\Lambda}\,(\,\tilde{k}^{*D}\tilde{k})\widetilde{\Lambda}^{*D} \\ &\cong (\,\tilde{k}^{*D}\tilde{k})(\widetilde{\Lambda}\,\widetilde{\Lambda}^{*D}) \\ &\cong \tilde{k}^{*D}\tilde{k}\widetilde{\Lambda}(\,\widetilde{\Lambda}^{*D}) \\ &\cong \tilde{k}^{*D}(\,\widetilde{\Lambda}^{*D})\tilde{k}\widetilde{\Lambda} \\ &\cong (\,\tilde{k}^{*D}\widetilde{\Lambda}^{*D})(\widetilde{\Lambda}\tilde{k}\,) \\ &\cong (\widetilde{\Lambda}\tilde{k})^{*D}(\widetilde{\Lambda}\tilde{k})\end{aligned}$$

Hence, $\widetilde{\Lambda}\tilde{k}$ *is* FS-(D ,∗)-hyponormal operator.

Theorem 3.8. Assume $\widetilde{\Lambda}_1, \widetilde{\Lambda}_2, \widetilde{\Lambda}_3, \ldots\ldots, \widetilde{\Lambda}_m$ are FS-(D ,∗)-hyponormal operator in $\tilde{B}(\widetilde{H})^{D}\ \forall\ k = 1,2,3,\ldots\ldots, m$. Then $(\widetilde{\Lambda}_1 \otimes \widetilde{\Lambda}_2 \otimes \ldots\ldots \otimes \widetilde{\Lambda}_m)$ is FS-(D ,∗)-hyponormal operator.

Proof:

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$(\widetilde{\Lambda}_1 \otimes \widetilde{\Lambda}_2 \otimes \dots \dots \otimes \widetilde{\Lambda}_m)(\widetilde{\Lambda}_1 \otimes \widetilde{\Lambda}_2 \otimes \dots . \otimes \widetilde{\Lambda}_m)^{*D} \cong (\widetilde{\Lambda}_1 \otimes \widetilde{\Lambda}_2 \otimes \dots \dots . \otimes \widetilde{\Lambda}_m)(\widetilde{\Lambda}_1^{*D} \otimes \widetilde{\Lambda}_2^{*D} \otimes \dots \dots \otimes \widetilde{\Lambda}_m^{*D})$

$$\cong (\widetilde{\Lambda}_1\widetilde{\Lambda}_1^{*D} \otimes \widetilde{\Lambda}_2\widetilde{\Lambda}_2^{*D} \otimes \dots \dots \otimes \widetilde{\Lambda}_m\widetilde{\Lambda}_m^{*D})$$

$$\gneqq (\widetilde{\Lambda}_1^{*D}\widetilde{\Lambda}_1 \otimes \widetilde{\Lambda}_2^{*D}\widetilde{\Lambda}_2 \otimes \dots . \otimes \widetilde{\Lambda}_m^{*D}\widetilde{\Lambda}_m)$$

$$\cong (\widetilde{\Lambda}_1^{*D} \otimes \widetilde{\Lambda}_2^{*D} \otimes \dots . \otimes \widetilde{\Lambda}_m^{*D})(\widetilde{\Lambda}_1 \otimes \widetilde{\Lambda}_2 \otimes \dots \dots \otimes \widetilde{\Lambda}_m)$$

$$\cong (\widetilde{\Lambda}_1 \otimes \widetilde{\Lambda}_2 \otimes \dots . \otimes \widetilde{\Lambda}_m)^{*D}(\widetilde{\Lambda}_1 \otimes \widetilde{\Lambda}_2 \otimes \dots \dots \otimes \widetilde{\Lambda}_m)$$

Hence, $(\widetilde{\Lambda}_1 \otimes \widetilde{\Lambda}_2 \otimes \dots \dots \otimes \widetilde{\Lambda}_m)$ is FS-(D ,$*$)-hyponormal operator.

Theorem 3.9. Assume $\widetilde{\Lambda}_1, \widetilde{\Lambda}_2, \widetilde{\Lambda}_3, \dots, \widetilde{\Lambda}_m$ are FS-(D ,$*$)-hyponormal operator in $\tilde{B}(\tilde{H})^D$ for all $k = 1,2,3, \dots, m$, then $(\widetilde{\Lambda}_1 \oplus \widetilde{\Lambda}_2 \oplus \dots \dots \oplus \widetilde{\Lambda}_m)$ is FS-(D ,$*$)-hyponormal operator.

Proof:

$(\widetilde{\Lambda}_1 \oplus \widetilde{\Lambda}_2 \oplus \dots \dots \oplus \widetilde{\Lambda}_m)(\widetilde{\Lambda}_1 \oplus \widetilde{\Lambda}_2 \oplus \dots . \oplus \widetilde{\Lambda}_m)^{*D} \cong (\widetilde{\Lambda}_1 \oplus \widetilde{\Lambda}_2 \oplus \dots \dots \oplus \widetilde{\Lambda}_m)(\widetilde{\Lambda}_1^{*D} \oplus \widetilde{\Lambda}_2^{*D} \oplus \dots . \oplus \widetilde{\Lambda}_m^{*D})$

$$\cong (\widetilde{\Lambda}_1\widetilde{\Lambda}_1^{*D} \oplus \widetilde{\Lambda}_2\widetilde{\Lambda}_2^{*D} \oplus \dots . \oplus \widetilde{\Lambda}_m\widetilde{\Lambda}_m^{*D})$$

$$\gneqq (\widetilde{\Lambda}_1^{*D}\widetilde{\Lambda}_1 \oplus \widetilde{\Lambda}_2^{*D}\widetilde{\Lambda}_2 \oplus \dots . \oplus \widetilde{\Lambda}_m^{*D}\widetilde{\Lambda}_m)$$

$$\cong (\widetilde{\Lambda}_1^{*D} \oplus \widetilde{\Lambda}_2^{*D} \oplus \dots . \oplus \widetilde{\Lambda}_m^{*D})(\widetilde{\Lambda}_1 \oplus \widetilde{\Lambda}_2 \oplus \dots \dots \oplus \widetilde{\Lambda}_m)$$

$$\cong (\widetilde{\Lambda}_1 \oplus \widetilde{\Lambda}_2 \oplus \dots . \oplus \widetilde{\Lambda}_m)^{*D}(\widetilde{\Lambda}_1 \oplus \widetilde{\Lambda}_2 \oplus \dots \dots \oplus \widetilde{\Lambda}_m)$$

Hence, $(\widetilde{\Lambda}_1 \oplus \widetilde{\Lambda}_2 \oplus \dots \dots \oplus \widetilde{\Lambda}_m)$ is FS-(D ,$*$)-hyponormal operator.

Corollary 3.10. If $\widetilde{\Lambda}, \tilde{k} \in$ FS-(D ,$*$)-hyponormal operator, Then $(\widetilde{\Lambda} \oplus \tilde{k}) \in$ FS-(D ,$*$)-hyponormal operator .

Proof: Assume $\widetilde{\Lambda}, \tilde{k} \in$ FS-(D ,$*$)-hyponormal operator] , then

$$(\widetilde{\Lambda} \oplus \tilde{k})\ (\widetilde{\Lambda} \oplus \tilde{k})^{*D} \cong (\widetilde{\Lambda} \oplus \tilde{k})\ (\widetilde{\Lambda}^{*D} \oplus \tilde{k}^{*D})$$

$$\cong \widetilde{\Lambda}\widetilde{\Lambda}^{*D} \oplus \tilde{k}\tilde{k}^{*D}$$

$$\gneqq \widetilde{\Lambda}^{*D}\widetilde{\Lambda} \oplus \tilde{k}^{*D}\tilde{k}$$

$$\cong (\widetilde{\Lambda}^{*D} \oplus \tilde{k}^{*D})(\widetilde{\Lambda} \oplus \tilde{k})$$

$$\cong (\widetilde{\Lambda} \oplus \tilde{k})^{*D}(\widetilde{\Lambda} \oplus \tilde{k})$$

Hence $(\widetilde{\Lambda} \oplus \tilde{k})$ is FS-(D ,$*$)-hyponormal operator .

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Proposition 3.11. Assume $\widetilde{\Lambda}, \tilde{k} \in \widetilde{B}(\widetilde{H})^D$, if $\tilde{k}$ is FS-(D, ∗)- hyponormal operator, and $\widetilde{\Lambda}$ is FS-unitary equivalent to $\tilde{k}$ Then $\widetilde{\Lambda}$ FS-(D, ∗)- hyponormal operator.

Proof: Assume $\tilde{k}$ is FS-(D, ∗)- hyponormal operator and $\widetilde{\Lambda} \in \widetilde{B}(\widetilde{H})^D$ is FS-unitary equivalent to $\tilde{k}$ there exists a FS-unitary equivalent $\tilde{V} \,\tilde{\in}\, \widetilde{B}(\widetilde{H})$, fulfilling $\widetilde{\Lambda} \cong \tilde{V}^*\tilde{k}\tilde{V}$. So $\widetilde{\Lambda}^D \cong \tilde{V}^*\tilde{k}^D\tilde{V}$.

We have:

$$\widetilde{\Lambda}\,\widetilde{\Lambda}^{*D} \cong \tilde{V}^*\tilde{k}\tilde{V}\tilde{V}^*\tilde{k}^{*D}\tilde{V}$$

$$\cong \tilde{V}^*\tilde{k}\tilde{k}^{*D}\tilde{V}$$

$$\gtrsim \tilde{V}^*\tilde{k}^{*D}\tilde{k}\tilde{V}$$

$$\gtrsim \tilde{V}^*\tilde{k}^{*D}\tilde{V}\tilde{V}^*\tilde{k}\tilde{V}$$

$$\cong \widetilde{\Lambda}^{*D}\widetilde{\Lambda}$$

Thus, $\widetilde{\Lambda}\,\widetilde{\Lambda}^{*D} \gtrsim \widetilde{\Lambda}^{*D}\widetilde{\Lambda}$.

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Conflict of interests.

The authors decelerate that there is no conflict of interest.

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