Generalization of Certain Characteristics of Multiplication Modules to Locally Multiplication Modules

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تعميم خواص محددة للمقاسات الضربية الى المقاسات الضربية محليا

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Accepted: **29/3/2026** **Published:** **31/3/2026**

ABSTRACT

Background: Consider a commutative ring R with a non-zero identity and an $R-$module M. A multiplication module is an $R-$module that satisfies the multiplication property. That is, every submodule N is presented as the product of an ideal I of the ring and the module M itself.

Materials and Methods: An $R-$module M is a locally multiplication module if M_P (or $P^{-1}M$) is a multiplication module for each prime ideal P of R. In this paper, certain properties of multiplication modules are extended to locally multiplication modules by providing conditions under which this generalization is possible.

Results: A locally multiplication module M is Noetherian whenever the ring is both semi-local and satisfies the conditions of an ascending chain on ideals with prime/semiprime property. As well as, if M is a faithful locally multiplication module and P is a prime ideal with $S_M(0) \subseteq P$, then a bijection exists between ideals and submodules of R_P and M_P. Also, if M is a locally multiplication module and N as the submodule of M is both prime and primary whenever the set of $S_M(N)$ contained in $J(R)$, along with several other results.

Conclusion: Various conditions are presented in terms of faithfulness, cyclic, prime, semi-prime, primary, and others, to illustrate the generalization of certain properties of multiplication modules to locally multiplication modules.

Keywords: Multiplication module, faithful module, locally multiplication module, Noetherian module, prime submodule, primary submodule, semiprime submodule.

ISSN: 2312-8135 | Print ISSN: 1992-0652
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1. INTRODUCTION

In 1980**,** D. D. Anderson and E. Smith **[1]**, introduced the concept of multiplicative ideals and multiplicative rings. In 2017, H. A. Tavallaee and R. Mahtabi **[2]**, studied several characteristics of modules with multiplication properties and connections between the submodules and ideals. In 2019, A. K. Jabbar and P. M. Hamaali **[3]**, studied multiplication modules in terms of locally S −prime, locally S −primary and locally S −semiprime submodules. In 2020, D. D. Anderson et al **[4]**, introduced the concept of S −multiplication modules, which is a generalization of multiplication module and Y. Tolooei **[5]**, studied characterization of finitely generated multiplication modules. In 2022, A. K. Jabbar, P. M. Hamaali, and A. I. Abid **[6]**, studied certain properties of multiplication modules satisfying both faithful and local properties. In 2024, K. Hukaew **[7]**, studied some properties of S −multiplication modules and dual notion of S −multiplication modules. The primary target of this work is to determine and analyze the extension of some characteristics related to multiplication modules to a local multiplication module.

Consider the commutative ring R with 1 not being 0 and the R −module M. A proper ideal I is an ideal that is not equal to the ring R and an ascending chain of ideals I_i of R is an increasing sequence $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$. Also, the submodule $(N: A) = \{x | x \in M$ and $Ax \subseteq N\}$ and the ideal $(L: N) = \{a | a \in R$ and $aN \subseteq L\}$ where A is an ideal of R, are said to be the residual of N by A and the residual of L by N, respectively **[8]**. When a finite set S generates a module M, the module is called finitely generated. Matter of fact, suppose that S is a n element subset of M where M is defined as the sum of R_{x_i} where $i: 1, \dots, n$ **[6]**. A module namely, M satisfies the multiplication property whenever a submodule is presented as the product of an ideal I in the ring and the module M itself $(N = IM)$ **[2 - 7], [11], [14], [19], [20]** and M is local if the localization of M at each prime ideal of R is a multiplication condition for modules **[6]**. Also, when the annihilator of a module M is 0, the module is faithful and M is cyclic if $M = Rx$, for some $x \in M$. If N is a submodule of M, then $S_M(N) = \{r \in R: rx \in N$, for some $x \in M\backslash N\}$ and if A is an ideal of R, then $S_R(A) = \{r \in R: ra \in A$, for some $a \notin A\}$ **[6], [9], [11], [10], [12]**. The intersection relation between every maximal ideal of R is denoted as $J(R)$ **[9], [14]**. A proper submodule, say N of the module M is prime if the product of $r \in R, m \in M$, belongs to N such that either $m \in N$ or $rM \subseteq N$ **[2], [6], [10], [11].** The set of all prime submodules of M is marked as $Spec(M)$ and the set of all prime ideals of R is marked as $Spec(R)$ **[2], [14].** If $S_M(N)$ forms an ideal of R, the proper submodule N is primal **[11], [12].** The nonempty subset S of ring R is considered as a multiplicative system whenever the product of every nonzero element a, b in S belongs to $S\backslash\{0\}$ and for a multiplicative system S, R_S is defined as the localization of R at S where $R_S = \left\{\frac{a}{s}: a \in R, s \in S\right\}$. Also, for P as a prime ideal, R_P indicates localization at P for the ring R where $R_p = \{\frac{a}{p}: a \in R, p \notin P\}$ and $R\backslash P$ is a multiplicative system in R **[3], [6], [9], [10], [13], [16].** A proper ideal I is called a primary ideal if for each a, b of R such that $a.b \in I$ and $a \notin I$, then there exists $k \in Z^+$ such that $b^k \in I$. Similarly, a primary submodule Q of M is a submodule when for every element $a \in R$, $x \in M$ and $x \notin Q$ such that the product ax is in Q,we obtain $a^n M \subseteq Q$ for some n in Z^+ **[8], [15]** and Q in terms of M is considered semiprime, if for every ideal I of R and every submodule K of M such that $I^2K \subseteq Q$, then $IK \subseteq Q$ and a semiprime ideal in R is an ideal I when J^2K is contained in I becoming $JK \subseteq I$ where J and K are ideals of R **[2], [6], [9], [10].** Whenever the ascending chain of ideals in are is stationary, R is called Noetherian and whenever the submodules of M are generated finitely, M is Noetherian **[13], [17], [18].**

In all that follow, the ring R be set to commutative where the identity element is not 0 and M as a module under the ring R.

2. Fundamental Lemmas in Multiplication Modules

In this section, we present fundamental lemmas and prove some results.

Lemma 2.1. Consider A and P as two ideals of R where P is considered prime and $A \nsubseteq P$. Thus, $(\frac{M}{AM})_P = 0$.

Proof. The proof is clear and straightforward.

Lemma 2.2. Every cyclic $R -$ module is a multiplication $R -$ module.

Proof. As M is cyclic, $M = Rm$ for some $m \in M$. Fix N as any submodule of M and $I = \{r \in R : rm \in N\}$. We need to demonstrate that $N = IM$. Take x in N then given that $1 \in R$, and $1.x = x$ is in N, we get $1 \in I$. Hence, $x = 1.x$ is in IM and so that $N \subseteq IM$. Next, let $x \in IM$, then $x = \sum_{i=1}^{n} a_i m_i$, where $a_i \in I$, $m_i \in M$ for $i : 1, \dots, n$. As given, $M = Rm$, for each $1 \le i \le n$, there exist $r_i \in R$ such that $m_i = r_i m$ for $1 \le i \le n$. Now, $x = \sum_{i=1}^{n} a_i m_i = \sum_{i=1}^{n} a_i r_i m = (\sum_{i=1}^{n} a_i r_i)m$. As $\sum_{i=1}^{n} a_i r_i \in I$, we have $(\sum_{i=1}^{n} a_i r_i)m \in N$, that means $x \in N$, IM is contained in N. Hence, N is equivalent to IM and so that M is a cyclic $R -$module.

Lemma 2.3. Consider I as any I of R. Then, $R \backslash I$ is viewed as a cyclic module.

Proof: The proof is clear and straightforward.

Remark 2.4. Matter of fact, every module having the multiplication property satisfies the local property. To prove this, consider M is provided to satisfy that both mentioned properties regarding modules and P considered an ideal of the ring with prime characteristic. Then, based on **[9, Cor. 2.9],** for every submodule of the localized module at the prime ideal namely, $\overline{N}$, a unique submodule N of M exists such that $\overline{N} = N_P$ and $S_M(N) \subseteq P$. Since the module M satisfy the multiplication property, N is the product of AM for certain ideal A of R and then $\overline{N} = N_P = (AM)_P = A_P M_P$, where A is viewed as the ideal of the localized ring R at P. We conclude that, the localized module M at P satisfies the local property. Therefore, the proof is achieved.

Now, we are presenting some examples in the context of multiplication modules:

Example 2.5:

(1) The set of integers $\mathbb{Z}$ is a multiplication module in terms of $\mathbb{Z} -$ module. Any submodule A belonging to $\mathbb{Z}$ is an ideal of $\mathbb{Z}$ and clearly, $A = A\mathbb{Z}$.

(2) $\mathbb{Q}$ as a $\mathbb{Z}-$ module is not a locally multiplication $\mathbb{Z}$ $-$module and hence it is not a multiplication $\mathbb{Z}-$ module. To demonstrate it, we need to know that, in terms of $\mathbb{Z}$, $\langle 0\rangle$ and $\langle p\rangle$ are ideals with prime characteristic and p is viewed as a number which is prime. Also, $\mathbb{Q}_{\langle 0\rangle} \cong \mathbb{Q}$ and $\mathbb{Q}_{\langle p\rangle} \cong \mathbb{Q}$, that means $\mathbb{Q}_{\langle p\rangle} \cong \mathbb{Q}$ for all prime ideals $\langle p\rangle$ of $\mathbb{Z}$. It is enough to clarify that $\mathbb{Q}$ is not a multiplication $\mathbb{Z}_{\langle p\rangle}$ $-$module. On the contrary, consider that $\mathbb{Q}$ in terms of $\mathbb{Z}_{\langle p\rangle}$ $-$module is a multiplication. Apparently, $\mathbb{Z}_{\langle p\rangle}$ is a submodule of $\mathbb{Q}$. Now we have, $\frac{1}{p} \in \mathbb{Q}$. If $\frac{1}{p}$ is an element in $\mathbb{Z}_{\langle p\rangle}$, then $\frac{1}{p} = \frac{m}{q}$, for certain $m \in \mathbb{Z}$ and $q \notin \langle p\rangle$. It is provided that $tq = tpm \in \langle p\rangle$ for certain $t \notin \langle p\rangle$, and given that $\langle p\rangle$ is prime, we need to be informed that t is in $\langle p\rangle$ or $q \in \langle p\rangle$, which leads us to contradiction. Hence, $\frac{1}{p}$ is not in $\mathbb{Z}_{\langle p\rangle}$. As a result, $\mathbb{Z}_{\langle p\rangle}$ is viewed as a submodule of $\mathbb{Q}$ which is proper, and given that $\mathbb{Q}$ is a multiplication in terms of $\mathbb{Z}_{\langle p\rangle}$ $-$module, we obtain that $\mathbb{Z}_{\langle p\rangle}$ is equivalent to $A\mathbb{Q}$, considering some ideal A of $\mathbb{Z}_{\langle p\rangle}$. Nevertheless, $A\mathbb{Q} = \mathbb{Q}$, this means $\mathbb{Z}_{\langle p\rangle} = \mathbb{Q}$, and it leads to a contradiction. Hence, $\mathbb{Q}$ is not a multiplication in terms of $\mathbb{Z}_{\langle p\rangle}$ $-$module and as $\mathbb{Q}_{\langle p\rangle} \cong \mathbb{Q}$, so that $\mathbb{Q}_{\langle p\rangle}$ is not a multiplication $\mathbb{Z}_{\langle p\rangle}$ $-$module, so we get that $\mathbb{Q}$ as a $\mathbb{Z}-$ module is not a locally multiplication $\mathbb{Z}$ $-$module and hence it also is not a multiplication $\mathbb{Z}$ $-$module.

(3) It is known that, every multiplication module is locally multiplication module but in general the converse is not true and we present the following example. That means, a locally multiplication module is a generalization of multiplication module.

Consider the polynomial ring $K = F[x, y]$, where F is considered as a field. Subsequently, K is a K $-$module and as $\langle x\rangle$ and $\langle y\rangle$ are ideals of K. Moreover, they are maximal ideals of K and hence prime ideals. Since, we have taken the ring K as a module over itself, any element namely x belonging to K, becomes an ideal of the ring where denoted as $\langle x\rangle$ if and only if $\langle x\rangle$ in the K $-$module K assumed to be a submodule. To avoid any confusion while verifying our assumption, considering the ring K, and as $\langle x\rangle$ is implemented in the process as the ideal of the ring K generated by x in order to denote the submodule of the K $-$module K by $\langle x\rangle K$. It refers to the multiplication between $\langle x\rangle$ as the ideal belong to K and K as the module. In fact, assuming that I is any ideal of K, then the product $IK = I$. If $a \in I$, then as $1 \in K$, we have $a = a.1 \in IK$, so that $I \subseteq IK$. Let $x \in IK$, then $x = \sum_{finite} a_i k_i \in I$, so that $IK \subseteq I$. Hence, $IK = I$. Regarding that in the equivalence relation between $\langle x\rangle$ and $\langle x\rangle K$ for every element x belong to K, (the notation of $\langle x\rangle$ indicates an ideal from the ring K formed by an element namely, x. However, the notation $\langle x\rangle K$ refers to the submodule in the $K-$ module K.) Thereby, if M is $\frac{K}{\langle x\rangle K} \oplus \frac{K}{\langle y\rangle K}$, which refers to a K $-$module, where $\langle x\rangle$ and $\langle y\rangle$ are maximal ideals of K. As a result, it leads to both $\langle x\rangle$ and $\langle y\rangle$ are prime in terms of K. First, our task is mainly to verify that M satisfies the condition regarding locally multiplication module. For that purpose, we localize M at prime ideals of K. In K, if we take any prime ideal P, we come across three cases:

(**1**) If $P = \langle x\rangle$, then we get $M_{\langle x\rangle} = (\frac{K}{\langle x\rangle K})_{\langle x\rangle} \oplus (\frac{K}{\langle y\rangle K})_{\langle x\rangle}$. As $y \in \langle y\rangle$ and $y \notin \langle x\rangle$, we get $\langle y\rangle \nsubseteq \langle x\rangle$, based on **Lemma 2.1**, $(\frac{K}{\langle y\rangle K})_{\langle x\rangle}$ is 0. Hence, $M_{\langle x\rangle} = (\frac{K}{\langle x\rangle K})_{\langle x\rangle} \oplus 0 = \frac{K_{\langle x\rangle}}{(\langle x\rangle K)_{\langle x\rangle}} = \frac{K_{\langle x\rangle}}{\langle x\rangle_{\langle x\rangle} K_{\langle x\rangle}}$, which is cyclic. Now, based on **Lemma 2.3**, $M_{\langle x\rangle}$ satisfies the multiplication property.

(**2**) If $P = \langle y\rangle$, then as $x \in \langle x\rangle$ and $x \notin \langle y\rangle$, we get $\langle x\rangle \nsubseteq \langle y\rangle$, using Lemma 2.1, $(\frac{K}{\langle x\rangle K})_{\langle y\rangle} = 0$. Hence, $M_{\langle y\rangle} = (\frac{K}{\langle x\rangle K})_{\langle y\rangle} \oplus \left(\frac{K}{\langle y\rangle K}\right)_{\langle y\rangle} = 0 \oplus (\frac{K}{\langle y\rangle K})_{\langle y\rangle} = \frac{K_{\langle y\rangle}}{(\langle y\rangle K)_{\langle y\rangle}} = \frac{K_{\langle y\rangle}}{\langle y\rangle_{\langle y\rangle} K\langle y\rangle_{\langle y\rangle}}$, which is cyclic. Now, based on Lemma 2.3, $M_{\langle x\rangle}$ satisfies the multiplication property.

(**3**) Suppose that, P is neither $\langle x\rangle$ nor $\langle y\rangle$, as $x \in \langle x\rangle$ and $x \notin P$, we get $\langle x\rangle \nsubseteq P$. Similarly, $y \in \langle y\rangle$ and $y \notin P$, so we get $\langle y\rangle \nsubseteq P$. Next, according to **Lemma 2.1**, $(\frac{K}{\langle x\rangle K})_P$ is 0 and $(\frac{K}{\langle y\rangle K})_P = 0$. Hence, $M_P = (\frac{K}{\langle x\rangle K})_P \oplus (\frac{K}{\langle y\rangle K})_P = 0 \oplus 0$, which is trivially a multiplication module. It indicates the case where in K, M_P is a module that satisfies the multiplication property at any ideal P with prime characteristic. Hence, that M satisfies the condition regarding the local multiplication module. Following that, we indicate the situation where the module M does not satisfy the condition regarding multiplication property. Now, consider the submodule $N = \frac{K}{\langle x\rangle K} \oplus 0$ of M. Consider I to be any ideal from the module K, then the product of $IM = I\left(\frac{K}{\langle x\rangle K} \oplus \frac{K}{\langle y\rangle K}\right) = I\left(\frac{K}{\langle x\rangle K}\right) \oplus I\left(\frac{K}{\langle y\rangle K}\right) = \frac{IK+\langle x\rangle K}{\langle x\rangle K} \oplus \frac{IK+\langle y\rangle K}{\langle y\rangle K} = \frac{IK+\langle x\rangle K}{\langle x\rangle K} \oplus \frac{IK+\langle y\rangle K}{\langle y\rangle K}$, which is not equivalent to $N = \frac{K}{\langle x\rangle K} \oplus 0$. Since, if $\frac{K}{\langle x\rangle K} \oplus 0 = \frac{IK+\langle x\rangle K}{\langle x\rangle K} \oplus \frac{IK+\langle y\rangle K}{\langle y\rangle K}$, we obtain these two cases:

Case 1: $\frac{K}{\langle x\rangle K} = \frac{IK+\langle x\rangle K}{\langle x\rangle K}$, so that $\frac{K}{\langle x\rangle} = \frac{IK+\langle x\rangle K}{\langle x\rangle K} = \frac{I+\langle x\rangle}{\langle x\rangle}$.

Case 2: $\frac{IK+\langle y\rangle K}{\langle y\rangle K} = 0$, that means $\frac{I+\langle y\rangle}{\langle y\rangle} = 0$.

From **case 1**, we have $1 + \langle x\rangle \in \frac{K}{\langle x\rangle} = \frac{I+\langle x\rangle}{\langle x\rangle}$, then $1 + \langle x\rangle = f + \langle x\rangle$, for some $f \in I + \langle x\rangle$. Hence, $1 - f \in \langle x\rangle$, so that $1 = 1 - f + f \in I + \langle x\rangle + \langle x\rangle = I + \langle x\rangle$, so that $I + \langle x\rangle = K$. From **case 2**, we get $I + \langle y\rangle = \langle y\rangle$, so that $I \subseteq \langle y\rangle$. Hence, we get that $K = I + \langle x\rangle \subseteq \langle y\rangle + \langle x\rangle = \langle x\rangle + \langle y\rangle = \langle x, y\rangle$, which leads to a contradiction because in K, $\langle x, y\rangle$ is maximal. Now, we obtain that $N = \frac{K}{\langle x\rangle} \oplus 0 \neq IM = \frac{I+\langle x\rangle}{\langle x\rangle} \oplus \frac{I+\langle y\rangle}{\langle y\rangle}$ if we take any ideal I from K. It indicates that the submodule N of M cannot be displayed as IM if any ideal I is taken from K, so that M is not a multiplication module. Hence, we get $M = \frac{K}{\langle x\rangle} \oplus \frac{K}{\langle y\rangle}$ over $K = k[x, y]$ which satisfies the condition regarding locally multiplication however it is not determined as a module with multiplication property.

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3. EXTENDING PROPERTIES TO LOCALLY MULTIPLICATION MODULES

In this section, some properties regarding multiplication modules are extended to locally multiplication module. Subsequently, we determine proper submodules of locally multiplication modules.

Proposition 3.1. Consider the non-trivial ring R and the module M that fulfill the local multiplication condition. The product of any proper ideal I of the ring with the module M is not equivalent to the module M with $I \subseteq J(R)$.

Proof. On the contrary, it's considered that the product of any proper ideal I of the ring with the module M is equivalent to the module M along with I contained in $J(R)$. Given that, the ring is a commutative and equipped with identity, this result with R contains more than a maximal ideal, P for instance. Now, $M_P = (IM)_P = I_P M_P$. Since P is maximal, it is prime and M is considered to be locally multiplication, then M_P is a multiplication R_P −module. At this point, we need to demonstrate that $R_P \neq 0$. If possible, suppose that $R_P = 0$. Then, $\frac{1}{1} = 0 \Rightarrow r.1 = 0$ for certain r not in P. Then r is equivalent to be 0, and it gives that 0 is not an element in P, which is a contradiction. Hence, $R_P \neq 0$, and R_P is a non-trivial ring. Next, suppose that $I_P = R_P$, then by **[8, Prop. 3.5]**, we reach the result that the intersection between I and $(R \backslash P)$ is not empty. Therefore, the intersection includes x where $x \in I$ and $x \notin P$, and given that $J(R)$ is contained in P, it implies that x is not an element in $J(R)$. It shows that I is not contained in $J(R)$, and it leads to the converse of our assumption. Thereby, $I_P \neq R_P$. Now, based on **[9, Lemma 2.1]**, $I_P M_P \neq M_P$ must be obtained. Similarly, it leads to the converse of our assumption. Therefore, we have reached the peak and the original statement is fulfilled.

Furthermore, we present conditions which convert the product of ideals of R with prime characteristics by M into a submodule satisfying prime condition in locally multiplication modules.

Proposition 3.2. Take M as the local multiplication module and $(0) \neq P \in Spec(R)$. If $P \cap S_R(0) = \emptyset$ and $S_M(PM) \subseteq P$, then $PM \in Spec(M)$ and $(PM{:}M) = P$.

Proof. Given that, P is an element belongs to $Spec(R)$. It indicates that, M_P satisfies the multiplication property under R_P −module and P_P considered the unique ideal of the local ring R_P with maximal characteristic. Let $P_P = 0$. If $p \in P$ is any element, $\frac{p}{1}$ in P_P is 0, so that $rp = 0 \in P$ for certain $r \notin P$, then $r \neq 0$, so that $p \in S_R(0)$. Hence, $P \cap S_R(0) \neq \emptyset$, which is a contradiction, so that $P_P \neq 0$. Hence, $0 \neq P_P \in Spec(R_P)$, so that by **[2, Lemma 2.2]**, $(P_P M_P{:}M_P)$ is viewed as P_P. As a result, $((PM)_P{:}M_P)$ is also viewed as P_P. Given that, $S_M(PM)$ is contained in P, from **[3, Cor. 2.5]**, there is an equivalence relation between $(PM{:}M)_P$, $((PM)_P{:}M_P)$ and P_P. Now, assume that PM is equivalent to M. As a result, $(PM{:}M) = (M{:}M) = R$.Thus, there is an equivalence relation between P_P, $(PM{:}M)_P$ and R_P, and it leads to the converse of our assumption, (P_P considered as the unique ideal of the local ring R_P with maximal characteristic). Hence, $PM \neq M$, and PM is determined as a proper submodule of M . Now, take $r \in R$ and $x \in M$ where the product

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ISSN: 2312-8135 | Print ISSN: 1992-0652
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rx in PM. If $x \notin PM$, then r is in the containment of $S_M(PM)$ in P. Hence, $rM \subseteq PM$, so that PM is a submodule of M with prime characteristic. As a result, $PM \in Spec(M)$. Next, take $r \in (PM{:}M)$, then $\frac{r}{1} \in (PM{:}M)_P = P_P$, then $ra \in P$ for some $a \notin P$ and as P is a submodule with prime characteristic. It's derived that, r is in P and $(PM{:}M)$ is included in P. Now, take $x \in P$, then xM is contained in PM. We acquire that, x is in $(PM{:}M)$, that gives P is contained in $(PM{:}M)$. Therefore, we obtain that $(PM{:}M)$ is P.

The following remark indicates a module which is faithful locally multiplication.

Remark 3.3. Based on **[6, Prop. 3.10]**, for a faithful module M and an ideal P with prime characteristic of R where $S_M(0)$ contained in P, the annihilator of the localized module works as the localization of the annihilator of the module. As given, the module satisfies the faithful property, and it shows that the annihilator of the module is 0. Thus, $ann(M_P) = (ann(M))_P = 0_P = 0$. Hence, the localized module at a prime ideal is a module satisfying the faithful multiplication property.

Proposition 3.4. Take M as the multiplication module which fulfills the faithful local condition and as P an ideal with prime characteristic of the ring R with $S_M(0)$ included in P. Then, every proper direct summand N of M with $S_M(N)$ contained in P is prime.

Proof. Given that, the module M fulfills local multiplication condition. This implies that, M_P fulfils the multiplication condition. Also, from **Remark 3.3**, M_P satisfies the faithful condition. Thus, M_P meets both conditions. Consider N as any proper direct summand of M with $S_M(0)$ contained in P. This implies that, $N \oplus L$ can be presented as M for certain submodules of M namely, L. As a result, M is the sum of N and L with the intersection between N and L which is $\{0\}$. Next, we have $M_P = (N + L)_P = N_P + L_p$ and $N_P \cap L_P = (N \cap L)_p = 0_P = 0$, so that $M_P = N_P \oplus L_P$. Let $N_P = M_P$ and $m \in M$ be any element, then we have $\frac{m}{1} \in N_P$, and as $S_M(N) \subseteq P$, based on **[11, Lemma 2.1]**, an element x exists in N such that $M = N$, which is the opposite of our assumption. Hence, $N_P \neq M_P$ and as a result, N_P viewed as a proper submodule of the faithful multiplication R_P −module and from **[2, Prop. 2.4]**, N_P becomes a submodule of M_P with prime characteristic. Now, take rx from N, where $r \in R$ and $x \in M$ with $x \notin N$. Then, $\frac{r}{1}\frac{x}{1} = \frac{rx}{1} N_P$. If $\frac{x}{1} \in N_P$ and as $S_M(N) \subseteq P$, we get $x \in N$ which is the opposite of our assumption. Now, $\frac{x}{1} \notin N_P$ and as N_P is prime, we get $\frac{r}{1} M_P \subseteq N_P$ and then, by **[10, Cor. 2.9]**, we get $\frac{r}{1} M_P = (rM)_P$, so that $(rM)_P \subseteq N_P$. Let $m \in M$, then $\frac{rm}{1} \in (rM)_P \subseteq N_P$, and based on **[11, Lemma 2.1]**, $S_M(N)$ is a subset of P. It implies that, rm belongs to N, so that rM contained in N. Hence, the submodule N is prime in M.

Corollary 3.5. Take M as the multiplication module which fulfills the faithful local condition with $S_M(0)$ contained in $J(R)$. Then, the prime submodule of M is a primal direct summand N of M.

Proof. Given that, the primal direct summand of M is N. It means that $N \neq M$ and $S_M(N)$ is considered to be proper ideal of R. It implies that a maximal ideal P exists in R that result in the containment of $S_M(N)$ in P and as $J(R)$ contained in P. We obtain that $S_M(0)$ is also included in

P. In addition, since the direct summand N is not equivalent to M. Hereby, based on Proposition 3.4, N as a direct summand is prime.

Proposition 3.6. Take M as the multiplication module which fulfills the local condition and for any ideal I from R, whenever IM contained in PM,with $P \in Spec(R)$, then I contained in P.

Proof. Given that, M fulfills the local condition. If any submodule from the localization M at P was taken, namely $\overline{N}$. Then, $\overline{N}$ is the product of $\overline{I}M_P$, where $\overline{I}$ is an ideal of R_P. It means, $\overline{I}$ is equivalent to I_P for certain ideal I of R and then IM is a submodule of M. By the given hypothesis we have $IM \subseteq PM$, so that $I_P M_P = (IM)_P \subseteq (PM)_P = P_P M_P$. Now, as M_P is a multiplication R_P −module and $P_P \in Spec(R_P)$, by **[2, Prop. 2.5]**, we get I_P contained in P_P. If I_P is equivalent to R_P, then $R_P = I_P \subseteq P_P \subseteq R_P$, so that $P_P = R_P$, which is a contradiction. Hence, we get $I_P \neq R_P$ and based on **[8, Prop. 3.5]**, the intersection between I, $R - p$ is empty. Now, assume that x belongs to I. It implies that, x is not an element in $R - p$. It indicates that, in terms of P, x exists. Hence, I contained in P.

Proposition 3.7. Take M as the multiplication module which fulfills the faithful local condition and A as an ideal of R. If both $S_M(0)$ and $S_M(AM)$ contained in $J(R)$.Then, $(AM: M)$ is equivalent to A.

Proof. Consider an ideal from the ring R with maximal characteristic, namely P. Obviously, P is prime and M satisfies the local condition. It implies that M_P also satisfies multiplication condition. In addition, M satisfies the faithful condition and $S_M(0)$ contained in $J(R)$. It implies that $J(R)$ is contained in P. According to **Remark 3.3**, M_P fulfills the faithful condition. Thus, M_P is a faithful multiplication R_P −module and A_P is an ideal of R_P and based on **[2, Cor. 2.7]**, $(A_P M_P : M_P)$ is equivalent to A_P. As a result, $((AM)_P : M_P) = A_P$. Since $S_M(AM)$ contained in $J(R)$. It implies that $J(R)$ is also contained in P. According to **[9, Cor. 2.5]**, $(AM: M)_P = ((AM)_P : M_P) = A_P$. Hence, by **[10, Cor. 2.2]**, $(AM: M) = A$.

In the following result, we give some conditions under which the annihilator of a module is contained in non-zero prime ideal in terms of locally multiplication modules.

Proposition 3.8. Consider the module M which satisfies the local multiplication condition under the integral domain R. Then, for each $0 \neq P \in Spec(R)$, for which $S_M(0) \subseteq P$, we have $ann(M) \subseteq P$.

Proof. Let $0 \neq P \in Spec(R)$, such that $S_M(0)$ is contained in P. Then, from [6, Prop. 3.10], $ann(M_P) = (ann(M))_P$. In contrast, M_P is considered as a module satisfying the multiplication property. If $P_P = 0$. Then, for every $p \in P$, we have $\frac{p}{1} = 0$. Hence, $qp = 0$, for some $q \notin P$, so that $q \neq 0$. Now, since R is viewed as an integral domain. As a result, $p = 0 \Rightarrow P = 0$, and it leads to the converse of our assumption. Hence, we get that $P_P \neq 0$ and then $0 \neq P_P \in Spec(R_P)$. Now, as M_P is a multiplication R_P −module, by **[2, Cor. 2.10]**, we get $ann(M_P) \subseteq P_P$. Hence, $(ann(M))_P$ is contained in P_P. Suppose that, r belongs to $ann(M) \Rightarrow \frac{r}{1} \in (ann(M))_P \subseteq P_P \Rightarrow$ $rp \in P$ for certain $p \notin P$ and P is considered as an ideal with prime characteristic. As a result, r belongs to P. Therefore, $ann(M) \subseteq P$.

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ISSN: 2312-8135 | Print ISSN: 1992-0652
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Proposition 3.9. Consider the module M which satisfies the local multiplication condition under a semi-local ring R. If M is Noetherian. Then, R fulfills the conditions of ascending chain on ideals with prime characteristic.

Proof. Consider $P_1 \subseteq P_2 \subseteq P_3 \subseteq \cdots$ as any increasing sequence of ideals P_i of R with prime characteristic and take the ideals $Q_1, Q_2, \dots, Q_k$ of R with maximal characteristic. It implies that, these ideals are also characterized as prime. As M is Noetherian, M_{Q_i} is also Noetherian $\forall i = 1, \dots, k$. Given that, M is locally multiplication, M_{Q_i} is also multiplication $\forall i = 1, \dots, k$. As a result, $\forall j = 1, \dots, k$, $(P_1)_{Q_j} \subseteq (P_2)_{Q_j} \subseteq (P_3)_{Q_j} \subseteq \cdots$. Now, either $P_i \nsubseteq Q_j$ for some i, or P_i is contained in Q_j $\forall i$. If $P_i \nsubseteq Q_j$ for some i, then an element x exits in $P_i \backslash Q_j$ such that $\frac{x}{x} \in (P_i)_{Q_j}$ and that gives, $(P_i)_{Q_j} = R_{Q_j}$, so that $(P_{i+t})_{Q_j} = R_{Q_j}$, for all $t \geq 0$. Hence, we get $(P_i)_{Q_j} = (P_{i+1})_{Q_j} = (P_{i+2})_{Q_j} = \cdots$ and if P_i is contained in Q_j $\forall i$, then by **[8, Cor. 3.11]**, there is a 1 to 1 relation among the ideals of R_{Q_j} and the ideals of R with prime characteristic which contained in Q_j, that means each $(P_i)_{Q_j}$ is a prime ideal of R_{Q_j}, so that $(P_1)_{Q_j} \subseteq (P_2)_{Q_j} \subseteq (P_3)_{Q_j} \subseteq \cdots$ set as an increasing sequence of prime ideals in R_{Q_j}. As M_{Q_j} is both Noetherian and multiplication, by **[2, Cor. 2.9]**, we get that $(P_1)_{Q_j} \subseteq (P_2)_{Q_j} \subseteq (P_3)_{Q_j} \subseteq \cdots$ satisfies the ascending chain condition, so that $(P_s)_{Q_j} = (P_{s+1})_{Q_j} = (P_{s+2})_{Q_j} = \cdots$ for some s. Now, take $t_j = \max\{i, s\}$, then we get $(P_{t_j})_{Q_j} = (P_{t_j+1})_{Q_j} = (P_{t_j+2})_{Q_j} = \cdots$. That means, for each $1 \leq j \leq k$, there exists t_j for which $(P_{t_j})_{Q_j} = (P_{t_j+1})_{Q_j} = (P_{t_j+2})_{Q_j} = \cdots$. Now, if we let $u = max\{t_j\}_{j=1}^{j=k}$, then we get $(P_u)_{Q_j} = (P_{u+1})_{Q_j} = (P_{u+2})_{Q_j} = \cdots$ for all $1 \leq j \leq k$, that means $(P_u)_{Q_j} = (P_{u+1})_{Q_j} = (P_{u+2})_{Q_j} = \cdots$ for all maximal Q_j of R. Hence, based on **[8, Cor. 3.13]**, $P_u = P_{u+1} = P_{u+2} = \cdots$, obtained.

Remark 3.10: In **[8, Cor. 3.11]**, it is proved that:

Consider the commutative ring R equipped with 1 as an identity element and S as a multiplicative system in R. There is a 1 to 1 correspondence $P \leftrightarrow P_S$, among the prime ideals of R which do not meet S and the proper prime ideals of R_S. This correspondence is order preserving. Now, a special case occurs when P is viewed as an ideal of R with prime characteristic. As a result, $R \backslash P$ is considered as a multiplicative system in R with the intersection between P and $(R \backslash P)$ is equivalent to $\emptyset$. For that reason, we can rephrase the stated corollary in the following manner:

For a ring R with commutative property equipped with a non-zero identity element and an ideal P with prime characteristic. There is a 1 to 1 relation among the prime ideals of R which is contained in P and the proper prime ideals of R_P. This correspondence is order preserving. We conclude that, P is equivalent to 0. The intersection between $P \cap S$ and $\{0\} \cap S$ is equivalent to $\emptyset$ (since, $0 \in S$). Hence, it results in $0 \leftrightarrow 0_S$. Then, if we exclude the prime ideal $P = 0$ from the set of $spec(R)$ that do not meet S with excluding the prime ideal $P_S = 0_S$ from $Spec(R_S)$. Now, take T as $\{P : P \in Spec(R) \backslash \{0\}$ and $P \cap S = \emptyset\}$. Then, the aforementioned statement can be reduced to:

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Consider R and S as the ring satisfying commutative property with an identity element not 0 and a multiplicative system in R, respectively. Then, there is a 1 to 1 relation among $Spec(R_S)\backslash\{0\}$ and T. A special case, when P is prime, $S = R\backslash P$ as a system in R which is multiplicative and $P\cap S = P\cap(R\backslash P) = \emptyset$, and there is a one-to-one correspondence, between $Spec(R_P)\backslash\{0\}$ and T as $\{P: P \in Spec(R)\backslash\{0\}$ and $P\cap S = \emptyset\}$.

In **[2, Cor. 2.3]** and **[2, Lemma 2.8]**, it is proved that:

(1) Consider the commutative ring R equipped with identity and the module M which satisfy the multiplication condition. Then, there exists a bijection between $Spec(R)\backslash\{0\}$ and $Spec(M)\backslash\{0\}$.

(2) Consider the commutative ring R equipped with identity and the module M which satisfy the faithful multiplication condition. Then, a bijection between R and M exists in terms of their ideals and submodules. Now, by combining the results mentioned, we get the following corollaries.

Corollary 3.11. Consider the commutative ring R with identity and M which satisfies the local multiplication condition. If P is an ideal of R with prime characteristic, then there exists a bijection between $Spec(M_P)\backslash\{0\}$ and T.

Proof. Provided that M is locally multiplication. It indicate that, M_P also satisfies the multiplication condition, by **[2, Cor. 2.3]**, there exists a bijection $f: Spec(M_P)\backslash\{0\} \to Spec(R_P)\backslash\{0\}$ and by Remark 3.10, there is a bijection $g: Spec(R_P)\backslash\{0\} \to T$. As the composition of two bijections is a bijection, we get $g \circ f: Spec(M_P)\backslash\{0\} \to T$ is considered a bijection which is defined to be the set of $\{P: P \in Spec(R)\backslash\{0\}$ and $P\cap S = \emptyset\}$.

Corollary 3.12. Consider the commutative ring R with identity and M which satisfies the local multiplication condition. If M is faithful and P is an ideal of R with prime characteristic where $S_M(0)$ is contained in P. Then, a bijection exists between ideals and submodules of R_P and M_P.

Proof. Provided that M is locally multiplication. It leads to M_P being a multiplication and since $S_M(0)$ is contained in P, from **Remark 3.3**, M_P is also faithful. Thus, M_P is a faithful multiplication module. Hereby, based on **[2, Lemma 2.8],** there exists a bijection relation between R_P and M_P in terms of ideals and submodules, respectively.

Proposition 3.13. Consider the commutative ring R with identity and M which satisfies the local multiplication condition. If R is a semi-local, then $\bigcap_{i=1}^{n}(P_iM) = (\bigcap_{i=1}^{n}P_i)M$ for any non-empty finite set of non-zero prime ideals P_i of R with $S_R(0)\cap P_i = \emptyset$ and $P_i \subseteq J(R)$ for all i.

Proof. Consider the ideals $Q_1, Q_2, \dots, Q_k$ of the semi-local ring with maximal characteristic, then they are also prime and let $\{P_i\}_{i=1}^{n} \neq \emptyset$ be a finite set of non-zero prime ideals P_i of R. For each $1 \leq j \leq k$, M_{Q_j} is a multiplication R_{Q_j} −module. Now, if $(P_i)_{Q_j} = 0$, for some i, then for each $p \in P_i$, we get $\frac{p}{1} = 0 \to rp = 0$ for some $r \notin Q_j$, then r is nonzero. Hence, we have p as an element of $S_R(0)$. It implies that, p exists in $S_R(0)\cap P_i$, this gives $S_R(0)\cap P_i \neq \emptyset$, which is a contradiction. Hence, $(P_i)_{Q_j} \neq 0$, for all i. Now, as $P_i \subseteq J(R) \subseteq Q_j$, $\forall\ i$. Now, in terms of R_{Q_j} and based on **[8, Cor. 3.11]**, $(P_i)_{Q_j}$ is a prime ideal for all i. As a result, $\{(P_i)_{Q_j}\}_{i=1}^{n} \neq \emptyset$ is a finite set of non-zero prime ideals $(P_i)_{Q_j}$ of R_{Q_j}. Hence, based on **[2, Lemma 2.11],** we get that

$\bigcap_{i=1}^{n}((P_i)_{Q_j}M_{Q_j}) = (\bigcap_{i=1}^{n}(P_i)_{Q_j})M_{Q_j}$. Next, $(\bigcap_{i=1}^{n}(P_iM))_{Q_j} = (P_1M\cap P_2M\cap \dots \cap P_nM)_{Q_j} = (P_1M)_{Q_j}\cap(P_2M)_{Q_j}\cap \dots \cap(P_nM)_{Q_j} = ((P_1)_{Q_j}M_{Q_j})\cap((P_2)_{Q_j}M_{Q_j})\cap \dots \cap((P_n)_{Q_j}M_{Q_j}) = \bigcap_{i=1}^{n}((P_i)_{Q_j}M_{Q_j}) = (\bigcap_{i=1}^{n}(P_i)_{Q_j})M_{Q_j} = (\bigcap_{i=1}^{n}P_i)_{Q_j})M_{Q_j} = ((\bigcap_{i=1}^{n}P_i)M)_{Q_j}$. The last part of the equality holds for Q_j of R_j where Q_j is maximal. Hence, based on **[10, Cor. 2.2],** we get that $\bigcap_{i=1}^{n}(P_iM) = (\bigcap_{i=1}^{n}P_i)M$.

The following remark and lemma demonstrate that a primary submodule is a prime submodule under a certain condition.

Remark 3.14. A prime submodule N is also primary of M. Matter of fact, N is not equivalent to M. Also, for any element r, m belongs to R and M, respectively. The product of both elements belongs to N. In addition, if $m \notin N$ and N is a prime submodule. Then, $rM \subseteq N$. Thus, the submodule N is primary.

Example 3.15. Taking M as a module, every submodule with prime characteristic is primary. Nevertheless, the converse of the statement in general is not true. For instance, in the set of integers, $\mathbb{Z}$ as a $\mathbb{Z}-$ module, the submodule $8\mathbb{Z}$ is primary but does not satisfy the prime condition.

Lemma 3.16 Consider the commutative ring R equipped with identity and M as a module. Then, the submodule N is prime $\leftrightarrow$ N is primary and $(N{:}M)$ is prime of R.

Proof. Consider N to be prime. Then, by **Remark 3.14**, it implies that N is a primary. Next, to show the residual ideal is prime. If $(N{:}M)$ is equivalent to R, then 1 is an element in $(N{:}M)$, so that $M = 1.M$ is contained in N and as N is contained in M, we get N is equivalent to M, which is a contradiction. Thereby, the residual of N and M is not equivalent to R. Now, for $a, b \in R$, we get that $ab \in (N{:}M)$ and $b \notin (N{:}M)$. Now, if $abM \subseteq N$ and $bM \nsubseteq N$ then $\exists\ m \in M$ such that $bm \notin N$. And if $abm \in N$ and N is prime, aM is as subset of N and $a \in (N{:}M)$. Hence, the residual of N and M is an ideal of R with prime characteristic. Next, suppose that N is primary and $(N{:}M)$ is prime. We need to verify that N is prime. As N is primary, N is proper submodule. Let $rm \in N$, where $r \in R, m \in M$, but $m \notin N$, since N is primary, $r^nM \subseteq N$ for some elements n of $\mathbb{Z}_+$ so that $r^n \in (N{:}M)$ and due to $(N{:}M)$ being prime, r belongs to $(N{:}M)$, and it implies that $rM \subseteq N$. Hence, the submodule N of M is prime.

Lemma 3.17. Consider the commutative ring R equipped with identity and M as a module. Then:

(1) If N is a primary submodule of $M \rightarrow$ $(N{:}M)$ is a primary ideal of R.

(2) If N is a semiprime submodule of $M \rightarrow$ $(N{:}M)$ is a semiprime ideal of R.

Proof. **(1)** Take $(N{:}M) = R$, then 1 belongs to $(N{:}M)$. It implies that, $M = 1.M$ is contained in N and given that N is contained in M, we reach that N is equivalent to M, which leads to a contradiction. As a result, $(N{:}M)$ is not equivalent to R. Now, take a, b as elements of R, then the product of ab belongs to $(N{:}M)$ except b, then abM contained in N and $bM \nsubseteq N$. It implies that an element $m\ exists\ in$ M such that $bm \notin N$. Now, if $abm \in N$ and N is a submodule with primary characteristic. Then the product of a^nM is contained in N, so that $a^n \in (N{:}M)$. Thus, the primary ideal of R is $(N{:}M)$.

(2) If $(N:M) = R$, then $1 \in (N:M)$, so that $M = 1.M$ is contained in N and as N is contained in M, we get $N = M$, which lead to a contradiction. Hence, $(N:M) \neq R$. Now, for $a \in R$ we have $a^2 \in (N:M)$, then $a^2M \subseteq N$, and as N is a submodule with semiprime characteristic, we get $aM \subseteq N$, so that $a \in (N:M)$. Hence, $(N:M)$ is an ideal with semiprime characteristic of R.

Proposition 3.18. Consider the commutative ring R equipped with identity and M is as a module. a prime ideal P and a submodule N, $S_M(N)$ is contained in P, we have:

(1) N is primary $\leftrightarrow$ N_P is a primary submodule of M_P.

(2) N is semiprime $\leftrightarrow$ if N_P is a semiprime submodule of M_P.

Proof. (1) Let N be a submodule of M with primary characteristic. If $M_P = N_P$, then $M_P \subseteq N_P$ and as $S_M(N) \subseteq P$, based on **[2, Lemma 4.8]**, M is contained in N and as N being contained in M. As a result, N is equivalent to M, and it leads to the converse of our assumption. Hence, we get $N_P \neq M_P$. Let, $\frac{r}{p}\frac{m}{q} \in N_P$, where $\frac{r}{p} \in R_P$ and $\frac{m}{q} \in M_P$. Then $\frac{rm}{pq} \in N_P$, for certain $s \notin P$ srm is in N . If $rm \notin N$, then the element s belongs to $S_M(N)$ is contained in P. Similarly, it leads to the converse of our assumption. Hence, we get $rm \in N$. Now, if $\frac{m}{q} \notin N_P$, then $m \notin N$ and as N considered to be primary, for certain positive integer $n \in \mathbb{Z}_+$, r^nM contained in N . Then, $(r^nM)_P \subseteq N_P$. As $p \notin P$, we get $p^n \notin P$, so by **[10, Cor. 2.9]**, we get $\frac{r^n}{p^n}M_P = (r^nM)_P$, and so that $(\frac{r}{p})^nM_P = \frac{r^n}{p^n}M_P = (r^nM)_P$ which is contained in N_P. Hence, N_P considered a submodule of M_P with primary characteristic. Next, take N_P as a submodule of M_P with primary characteristic. To demonstrate that, N is a submodule of M with primary characteristic. If N is equivalent to M, then N_P also equivalent to M_P. Similarly, it leads to the converse of our assumption. Hence, $N \neq M$. Let the product of $rm \in N$, where $r \in R$ and $m \in M$, but $m \notin N$. Then, $\frac{r}{1}\frac{m}{1} = \frac{rm}{1} \in N_P$. Now, if $\frac{m}{1} \in N_P$, then $pm \in N$, for some $p \notin P$ and as $m \notin N$. As a result, p belongs to $S_M(N)$ which is contained in P, it leads to the converse of our assumption. Hence, $\frac{m}{1} \notin N_P$ and as N_P a submodule primary characteristic. Hence, $(\frac{r}{1})^nM_P \subseteq N_P$, that is $(r^nM)_P \subseteq N_P$. Then, as $S_M(N)$ contained in P, from **[12, Lemma 4.8]**, r^nM is contained in M. Hence, N a submodule of M with a prime characteristic.

(2) Let N be a submodule of M with semiprime characteristic. If $M_P = N_P$, then $M_P \subseteq N_P$ and as $S_M(N) \subseteq P$, from **[12, Lemma 4.8]**. We need to demonstrate that, M is contained in N and as N is contained in M. As a result, N is equivalent to M, it leads to the converse of our assumption. Hence, we get $N_P \neq M_P$. Let, $(\frac{r}{p})^2\frac{m}{q} \in N_P$, where $\frac{r}{p}$ is in R_P and $\frac{m}{q} \in M_P$. Then $\frac{r^2m}{pq} \in N_P$. It implies that, $sr^2m \in N$ for certain $s \notin P$. If $r^2m \notin N$, then the element s belongs to $S_M(N)$ is contained in P. Similarly, it leads to the converse of our assumption. Hence, we get $r^2m \in N$. Now, if $\frac{m}{q} \notin N_P$, then $m \notin N$ and N is semiprime, the product of rm belongs to N. Thus, $\frac{r}{p}\frac{m}{q} = \frac{rm}{pq} \in N_P$. Hence, N_P is a submodule of M_P with a semiprime characteristic. Next, take N_P as a a submodule of M_P with semiprime characteristic. To demonstrate that, N is a submodule of M with semiprime characteristic. If N is equivalent to M, then N_P is also equivalent to M_P. Similarly, it leads to the converse of our assumption. Thus, $N \neq M$. Take $r^2 m \in N$, where $r \in R$ and $m \in M$, but $m \notin N$. Then, $(\frac{r}{1})^2 \frac{m}{1} = \frac{r^2}{1}\frac{m}{1} = \frac{r^2 m}{1} \in N_P$. Now, if $\frac{m}{1} \in N_P$, then $pm \in N$, for certain $p \notin P$ and as $m \notin N$, p is in $S_M(N)$ which is contained in P. Similarly, it leads to the converse of our assumption. Hence, $\frac{m}{1} \notin N_P$ and as N_P is semiprime, we get $\frac{r}{1}\frac{m}{1} \in N_P$, that is $\frac{rm}{1} \in N_P$. Then, $srm \in N$ for certain $s \notin P$. If $rm \notin N$, then the element s belongs to $S_M(N)$ is contained in P. Similarly, it leads to the converse of our assumption. As a result, rm belongs to N. Hence, the submodule of M is a semiprime.

Proposition 3.19. Consider the commutative ring R equipped with identity and $A \neq R$ be an ideal. A is semiprime $\leftrightarrow$ A_P is a semiprime ideal of R_P, whenever $S_R(A)$ is contained in P and P is prime.

Proof. Consider A to be a semiprime ideal. If A_P is equivalent to R_P, then $\frac{1}{1}$ belongs to A_P, then $a.1 \in A$ for certain $a \notin P$, and as $1 \notin A$, we get $a \in S_R(A)$ is contained in P, which contradict to our assumption. Hence, $A_P \neq R_P$. Now, let $(\frac{a}{p})^2 \in A_P$, where $\frac{a}{p} \in R_P$. Then, $\frac{a^2}{p^2} \in A_P$ implies that $sa^2 \in A$ for certain $s \notin P$. Next, $(sa)^2 = sasa = ssaa = s(sa^2) \in A$ and as A is semiprime, $sa \in A$ and then $\frac{a}{p} = \frac{s}{s}\frac{a}{p} = \frac{sa}{sp} \in A_P$. Hence, A_P is a semiprime ideal. Now, consider A_P to be a semiprime. If $A = R$, then $A_P = R_P$, which contradicts to our assumption. Hence, $A \neq R$. Let $a^2 \in A$, where $a \in R$. Then, $(\frac{a}{1})^2 = \frac{a}{1}\frac{a}{1} = \frac{a^2}{1} \in A_P$ and as A_P is semiprime, we get $\frac{a}{1}$ belongs to A_P *then* $pa \in A$ for certain $p \notin P$. If $a \notin A$, then p belongs to $S_R(A)$ is contained in P. It leads to a contradiction. Thus, a belongs to A. Hence, A is semiprime.

It is known that, every prime submodule is primary. In the upcoming proposition, the condition to make sure the converse is hold, is being provided recently .

Proposition 3.20. Consider the commutative ring R equipped with identity and the module M which satisfies the local multiplication condition. A submodule N of M is both prime and primary whenever the set of $S_M(N)$ is contained in $J(R)$.

Proof. To demonstrate that N is primary. Let N be prime. Given that, R is commutative, it means that there exists a maximal ideal, namely P. As a result, P is also prime and since M is locally multiplication. We obtain that M_P also a multiplication module. Now, $S_M(N) \subseteq J(R) \subseteq P$, and based on **[6, Prop. 3.15]**, N_P is prime. In addition, by **[2, Th. 2.18]**, N_P is considered to be primary. Hence, by **Proposition 3.18**, N is a primary. On the other hand, to show N is prime, we follow the same technique and we obtain that N is prime.

Proposition 3.21. Consider the commutative ring R equipped with identity and P as the ideal with prime characteristic. If R fulfills the conditions of ascending chain on ideals with semiprime characteristics, then R_p also satisfies the conditions of ascending chain on ideals with semiprime characteristic.

Proof. Take the chain of ideals $\overline{A_1} \subseteq \overline{A_2} \subseteq \overline{A_3} \subseteq \cdots$ as any increasing sequence in R_P with semiprime characteristic. Next, by **[4, Cor. 3.18]**, for each ideal $\overline{A_i}$ of R_P, a unique ideal B_i of R exists such that $S_R(B_i)$ is contained in P and $\overline{A_i} = (B_i)_P$. It implies that, $(B_1)_P \subseteq (B_2)_P \subseteq (B_2)_P \subseteq \cdots$ is an ascending chain for ideals in R_p and as $S_R(B_i)$ is contained in P $\forall\, i$, from **[6, Lemma 3.8]**, we acquire $B_1 \subseteq B_2 \subseteq B_3 \subseteq \cdots$. As $(B_i)_P$ is considered as an ideal of R_P with semiprime characteristic $\forall\, i$ and $S_R(B_i)$ is contained in P, based on **Proposition 3.19**, it is clear that the ideal B_i of R is semiprime for all i, that means $B_1 \subseteq B_2 \subseteq B_3 \subseteq \cdots$ is an increasing sequence of B_i, and as R satisfies the condition, $\exists\, k \in \mathbb{Z}_+$ which $B_k = B_{k+1} = \cdots$. As a result, $(B_k)_P = (B_{k+1})_P = (B_{k+3})_P = \cdots$ and that means $\overline{A_k} = \overline{A_{k+1}} = \overline{A_{k+2}} = \cdots$. Hence, R_P satisfies the chain condition.

Proposition 3.22. Consider the commutative ring R equipped with the identity element and M as a locally multiplication R −module. Then, the module M is Noetherian whenever the ring is both semi-local and satisfies the conditions of ascending chain on ideals with semiprime characteristic.

Proof. Take the chain of submodules $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$ as any increasing sequence in M and take $P_1, P_2, \dots, P_n$ as ideals of R with maximal characteristic. Now, for $i = 1, \dots, n$, we have M_{P_i} is a multiplication R_{P_i} −module and $(N_1)_{P_i} \subseteq (N_2)_{P_i} \subseteq (N_3)_{P_i} \subseteq \cdots$. Given that R satisfies the conditions on semiprime ideals, based on Proposition 3.21, R_{P_i} satisfies the same condition for all $1 \leq i \leq n$. Hence, by **[2, Cor. 2.19]**, M_{P_i} is a Noetherian R_{P_i} −module for all $1 \leq i \leq n$. In addition, for $i = 1, \dots, n$, $\exists\, k_i \in \mathbb{Z}_+$ which $(N_{k_i})_{P_i} = (N_{k_i+1})_{P_i} = (N_{k_i+2})_{P_i} = \cdots$. Next, take $k = \{k_i\}_{i=1}^{n}$, then, we get $(N_k)_{P_i} = (N_{k+1})_{P_i} = (N_{k+2})_{P_i} = \cdots$, for $i = 1, \dots, n$ and by **[10, Cor. 2.2],** we get $N_1 = N_2 = N_3 = \cdots$. Hence, M is Noetherian.

4. CONCLUSION

Multiplication module is considered as one of the essential topics in module theory and many authors had contributed in this literature by investigating and analyzing global to local properties. In this paper, we have studied certain properties of multiplication modules and extended them to locally multiplication module by providing conditions in terms of many crucial concepts such as faithfulness, cyclic, prime, semi-prime, primary, and others, under which this generalization was possible. The results of this paper may also serve as a foundation for further development in module theory and help with exploring the extension of these finding to a non-commutative ring or investigating other properties pf multiplication module under a commutative ring with identity.

Acknowledgments:

The primary author of this work would like to thank his supervisors for their advice and assistance, without which this work would not have been completed.

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

Conflict of interest.

There are no conflicts of interest.

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الخلاصة

المقدمة:

لتكن (R) حلقة ابدالية ذات عنصر محايد غير صفري، و (M) مقاس على (R). المقاس الضربى هو المقاس الذي كل مقاس جزئي منه يتكون من حاصل ضرب مثالي من الحلقة بالمقاس نفسه.

المواد والأساليب:

نقول ان المقاس (M) هو مقاس ضربى محليا (او موضعيا) اذا كان المقاس $(M_P(P^{-1}M))$ هو مقاس ضربي لكل مثالي اولي (P) في (R) والذي يكون اعماما للمقاس الضربي. تم تعميم خصائص محددة للمقاسات الضربية الى المقاسات الضربية محليا في هذه الورقة من خلال توفير الشروط التي يكون فيها هذا التعميم ممكنًا.

النتائج:

المقاس الضربى محليا يكون مقاسا نويثيريا اذا كانت الحلقة (R) هي شبه موضعية و تحقق شرط السلسلة التصاعدية على المثاليات شبه الاولية. المقاس النويترى و الضربي محليا على حلقة شبه موضعية تحقق شرط السلسلة التصاعدية على المثاليات الاولية بالاضافة الى نتائج عديدة اخرى.

الاستنتاجات:

يُقدّم هذا العمل العديد من الشروط المتعلقة بالدورية، والأولية، وشبه الأولية، والأبتدائية، وغيرها، والتي بموجبها تُعمّم بعض خصائص المقاسات الضربية الى المقاسات الضربية محليا.

الكلمات المفتاحية:

المقاس الضربي، المقاس التابع (او الولي), المقاس الضربي محليا, المقاس النويتري, المقاس الجزئي الاولي، المقاس الجزئي الابتدائي , المقاس الجزئي شبه الاولي.