



A Directional Interpretation of Shadowing and Semicontinuity of Omega-Limit Sets

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ABSTRACT

In this paper, we study the relationship between shadowing properties and the semi-continuity of ω -limit sets in dynamical systems. Building on known equivalence results, we provide a directional and topological interpretation of the asymmetry between upper and lower semi-continuity. Our approach clarifies how forward shadowing naturally supports upper semi-continuity, while the absence of backward control obstructs lower semi-continuity in non-invertible and semi-continuous systems. Several illustrative discussions and examples are provided to support this viewpoint.

Keywords: shadowing, omega-limit sets, semi-continuity, dynamical systems, topology

1. INTRODUCTION

Shadowing is an important concept in the topological theory of dynamical systems, linking the behavior of approximate trajectories, known as pseudo-orbits, to that of true orbits. In practical situations, such as numerical computations or perturbations of a system, exact orbits are often replaced by approximate ones. The shadowing property provides a rigorous framework ensuring that these approximate orbits can be closely followed by genuine orbits of the system. The long-term behavior of a dynamical system is commonly described by its ω -limit sets, which collect all accumulation points of forward orbits. The continuity properties of ω -limit sets can be interpreted as a form of stability of asymptotic dynamics. In this direction, Kawaguchi established equivalence results between shadowing properties and the semi-continuity of ω -limit sets. However, the directional aspects of shadowing and their role in distinguishing upper and lower semi-continuity have not been studied in detail. The purpose of this paper is to provide a directional interpretation of this relationship. Contributions. Our main contribution is a directional and topological interpretation of the known equivalence results between shadowing properties and the semi-continuity of ω -limit sets. We clarify the asymmetric roles of forward and backward dynamics in explaining upper versus lower semi-continuity, and we support this viewpoint through elementary observations and illustrative examples [1]-[7]. Unlike previous studies that focus mainly on equivalence results between shadowing and semi-continuity, the present work provides a directional interpretation explaining the asymmetry between upper and lower semi-continuity of



the ω -limit map. This viewpoint highlights the role of forward shadowing and the absence of backward shadowing in shaping the stability properties of asymptotic dynamics.

2 RELATED WORK

The shadowing property has played a central role in the modern theory of dynamical systems, particularly in the study of hyperbolic dynamics and orbit stability [1], [8]. Classical references such as Katok and Hasselblatt emphasized shadowing as a fundamental tool connecting approximate trajectories arising from numerical errors or perturbations to true orbits of the system. In this setting, shadowing is closely related to other structural properties, including the Anosov Closing Lemma, the specification property, and the existence of invariant manifolds [1], [5]. To better understand local aspects of shadowing, Morales introduced the notion of shadowable points [3], which allows the study of shadowing phenomena at individual points rather than globally. This localization has proved useful in relating shadowing to finer topological and dynamical properties, such as chain recurrence and stability of limit sets [3]. On the other hand, the ω -limit set is a classical object used to describe the asymptotic

behavior of orbits. When viewed as a set-valued map, its continuity properties naturally arise in the study of stability of long-term dynamics. Upper and lower semi-continuity of ω -limit sets have been investigated in various contexts, often in connection with attractors, chain recurrence, and invariant measures [9], [10]. Recently, Kawaguchi established precise equivalence results between shadowing properties and the semi-continuity of ω -limit sets. Recent studies have also investigated the preservation of shadowing in discrete dynamical systems [6], [11].

In particular, shadowable points were shown

to correspond exactly to points of upper semi-continuity of the ω -limit map, while global shadowing was related to lower semi-continuity via chain continuity. These results clarified the deep relationship between shadowing and the stability of asymptotic dynamics [4]. The present work is motivated by these developments. Rather than introducing new shadowing notions, we focus on providing a directional interpretation of the known equivalences. We aim to clarify how the forward and backward aspects of pseudo-orbits influence upper and lower semi-continuity, respectively, and to provide additional intuition for the asymmetry observed in the behavior of ω -limit sets.

3 PRELIMINARIES

3.1 Pseudo-Orbits

Definition 3.1. Let (X, d) be a metric space, $U \subset X$ an open set, and let $f: U \rightarrow X$ be a map. Let $a \in \mathbb{Z} \cup \{-\infty\}$ and $b \in \mathbb{Z} \cup \{\infty\}$. A sequence $\{x_n\}_{a < n < b} \subset U$ is called a δ -pseudo-orbit of f if

$$d(x_{n+1}, f(x_n)) < \delta \text{ for all } a < n \text{ with } n + 1 < b.$$

If, in addition, $-\infty < a < b < \infty$ and $x_{b-1} = x_{a+1}$,

then the sequence is called a periodic δ -pseudo-orbit [2], [5].



3.2 Shadowing of Pseudo-Orbits

Definition 3.2. Let (X, d) be a metric space and let $f: X \rightarrow X$ be a map. A sequence $\{x_n\}_{n \in \mathbb{Z}}$ is said to be ε -shadowed by a point $y \in X$ if $d(f^n(y), x_n) < \varepsilon$ for all $n \in \mathbb{Z}$ [2], [5].

Definition 3.3. The map f is said to have the shadowing property if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every δ -pseudo-orbit of f is ε -shadowed by some point in X [2], [5].

Remark 3.4. We refer to global shadowing when the shadowing property holds for all δ -pseudo-orbits in X . The notion of chain continuity is used in the standard sense from chain recurrence theory [2], [5].

3.3 Shadowable Points

Definition 3.5. Let (X, d) be a metric space and let $f: X \rightarrow X$ be a map. A point $x \in X$ is called shadowable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every δ -pseudo-orbit passing through x is ε -shadowed by a true orbit of f [3].

3.4 ω -Limit Sets

Definition 3.6. Let (X, d) be a metric space and let $f: X \rightarrow X$ be a map. For a point $x \in X$, the ω -limit set of x is defined by

$$\omega(x) = \{ y \in X : \text{there exists a sequence } n_k \rightarrow \infty \text{ such that } f^{n_k}(x) \rightarrow y \} \text{ [9], [10].}$$

Remark 3.7. The ω -limit set describes the asymptotic behavior of the forward orbit of x . When viewed as a map $x \rightarrow \omega(x)$, it naturally defines a set-valued mapping.

3.5 Semi-continuity of ω -Limit Maps

Definition 3.8. Let $F: X \rightarrow 2^X$ be a set-valued map. The map F is said to be upper semi-continuous at $x \in X$ if for every open set V with $F(x) \subset V$, there exists a neighborhood U of x such that $F(u) \subset V$ for all $u \in U$ [10], [12].

Definition 3.9. The map F is said to be lower semi-continuous at $x \in X$ if for every open set V with $F(x) \cap V \neq \emptyset$, there exists a neighborhood U of x such that $F(u) \cap V \neq \emptyset$ for all $u \in U$ [10], [12].

Remark 3.10. In this work, the set-valued map of interest is the ω -limit map $x \mapsto \omega(x)$. The upper and lower semi-continuity of this map reflects the stability properties of asymptotic dynamics under perturbations of initial conditions [10], [12].

3.6 Shadowing and Semi-continuity of ω -Limit Sets

Theorem 3.11 (Kawaguchi). Let (X, d) be a metric space and let $f: X \rightarrow X$ be a map. Then a point $x \in X$ is shadowable if and only if x is a point of upper semi-continuity of the ω -limit map. Moreover, under the assumption of global shadowing, the lower semi-continuity of the ω -limit map is equivalent to chain continuity [4].

Remark 3.12. The above theorem reveals a close relationship between shadowing phenomena and the stability of ω -limit sets. In particular, it highlights an intrinsic asymmetry between upper and lower semi-continuity, which motivates a more detailed directional analysis of shadowing [4].



4 DIRECTIONAL INTERPRETATION OF SHADOWING AND SEMI-CONTINUITY

In this section, we provide a directional interpretation of the relationship between shadowing properties and the semi-continuity of the ω -limit map established in the previous sections. Our discussion is interpretative and aims to clarify the asymmetry between upper and lower semi-continuity from the viewpoint of forward and backward dynamics [4].

Definition 4.1. We say that a dynamical system exhibits directional shadowing behavior if its forward pseudo-orbits admit effective shadowing along positive iterates, while no corresponding backward control is assumed. This definition is introduced for descriptive purposes only and does not define a new shadowing property [4].

4.1 Directional Viewpoint

From a heuristic perspective, pseudo-orbits can be considered with respect to the direction of time. Forward pseudo-orbits follow the dynamics in positive time and naturally control the accumulation behavior of forward iterates. This behavior aligns well with the upper semi-continuity of the ω -limit map, which prevents the creation of new limit points under small perturbations of initial conditions. In contrast, backward pseudo-orbits require some form of reversibility or backward uniqueness. In non-invertible or semi-continuous systems, such control is often absent, which explains why lower semi-continuity typically imposes stronger requirements and may fail even when upper semi-continuity holds [4].

4.2 Illustrative Example

Example 4.2. Consider the map $f: [0, 1] \rightarrow [0, 1]$ defined by $f(x) = x^2$. For every $x \in (0, 1)$, the forward iterates satisfy $f^n(x) \rightarrow 0$ as $n \rightarrow \infty$. Hence, $\omega(x) = \{0\}$ for all $x \in [0, 1)$, and the ω -limit map is upper semi-continuous at 0. From a directional viewpoint, forward pseudo-orbits near 0 are easily shadowed by true orbits converging to 0. However, the backward dynamics of f are not uniquely defined, illustrating the difficulty of backward shadowing and highlighting the asymmetry between upper and lower semi-continuity [4], [7].

Example 4.3. Consider a piecewise continuous and non-invertible map $f: [-1, 1] \rightarrow [-1, 1]$ is defined by different polynomial branches on subintervals of $[-1, 1]$. Such maps typically exhibit strong forward contraction on certain regions while allowing multiple backward preimages. In this setting, forward pseudo-orbits can be controlled in the sense that approximate trajectories remain close to true forward orbits for sufficiently long times. This behavior reflects a form of forward shadowing and induces stability of accumulation behavior along positive iterates. However, due to the lack of backward uniqueness inherent in piecewise non-invertible maps, backward pseudo-orbits cannot be consistently shadowed. As a result, the persistence of ω -limit points under perturbations fails, illustrating an obstruction to lower semi-continuity of the ω -limit map. This example highlights the directional asymmetry between forward and backward dynamics and supports the viewpoint that forward shadowing naturally aligns with upper semi-continuity, while lower semi-continuity requires additional backward control. Similar phenomena for piecewise and non-invertible maps can be found in [7].



Remark 4.4. This example highlights that upper semi-continuity can hold under forward control, whereas lower semi-continuity may fail without additional backward or two-sided mechanisms [4].

5 DIRECTIONAL TOPOLOGICAL SEMI-CONTINUITY AND SHADOWING

The notions of upper and lower semi-continuity of set-valued maps are inherently topological and reflect different types of stability. In the context of dynamical systems, the ω -limit map naturally exhibits this asymmetry when interpreted through the lens of shadowing.

5.1 Topological Interpretation

From a topological viewpoint, upper semi-continuity of the ω -limit map prevents the sudden appearance of new limit points under small perturbations of initial conditions. This property aligns naturally with forward shadowing, where approximate forward trajectories are closely followed by true orbits. In this sense, forward shadowing induces a form of directional topological control that supports upper semi-continuity. In contrast, lower semi-continuity requires the persistence of limit points under perturbations. Topologically, this demands a stronger form of stability, as limit points must not disappear. Such persistence is closely related to backward or two-sided shadowing, which is generally unavailable in non-invertible or semi-continuous systems [4], [13].

5.2 Directional Semi-continuity

Motivated by this observation, the semi-continuity of the ω -limit map can be viewed as a directional property induced by shadowing. Forward shadowing yields a one-sided topological stability corresponding to upper semi-continuity, while the failure of backward shadowing explains the obstruction to lower semi-continuity. This directional viewpoint provides a unifying topological explanation for the asymmetry observed in previous equivalence results between shadowing properties and semi-continuity. Rather than treating upper and lower semi-continuity as symmetric concepts, our interpretation emphasizes their fundamentally different directional origins [4].

5.3 Directional Consequences of Shadowing

Lemma 5.1. Let (X, d) be a metric space and let $f: X \rightarrow X$ have the shadowing property. Then forward shadowing induces a form of topological upper stability of the ω -limit map, in the sense that small perturbations of initial conditions cannot generate new ω -limit points far from the original ones [4].

Proof. Forward shadowing ensures that every sufficiently accurate forward pseudo-orbit is followed by a true orbit. Consequently, accumulation points of approximate trajectories remain close to accumulation points of true trajectories. This topological control prevents the sudden appearance of new limit points under small perturbations, which corresponds precisely to upper semi-continuity of the ω -limit map [4].

5.4 Relation to Set-Valued Topology

Remark 5.2. When viewed as a set-valued map, the ω -limit map naturally lives in the hyperspace 2^X equipped with standard topologies such as the Vietoris or Hausdorff topology



. Upper semi-continuity corresponds to stability with respect to the upper Vietoris topology, while lower semi-continuity reflects stability in the lower Vietoris sense. The directional interpretation presented here suggests that forward shadowing aligns with upper Vietoris stability, whereas the absence of backward shadowing obstructs lower Vietoris continuity [13].

Remark 5.3. The directional nature of semi-continuity discussed above is purely topological and does not rely on differentiability or hyperbolicity assumptions. This makes the interpretation applicable to a wide class of non-invertible and semi-continuous dynamical systems [4].

Remark 5.4. Inverse shadowing provides a complementary viewpoint in which true orbits are approximated by pseudo-orbits, and it has been studied as a dual notion to classical shadowing (see, for example, [12]). Although inverse shadowing is not treated directly in this work, it further emphasizes the directional nature of shadowing phenomena, particularly in connection with backward dynamics [12].

6 CONSEQUENCES OF DIRECTIONAL SHADOWING

In this section, we summarize the main consequences of the directional interpretation developed in the previous sections. The results presented here should be understood as interpretative consequences of known equivalence results between shadowing and the semi-continuity of ω -limit sets.

6.1 Upper Semi-continuity and Forward Shadowing

Theorem 6.1. Let (X, d) be a metric space and let $f: X \rightarrow X$ have the shadowing property. Then the ω -limit map is upper semi-continuous at every shadowable point.

Proof. This follows from the directional control provided by forward shadowing. Approximate forward trajectories are shadowed by true orbits whose accumulation behavior remains close to that of the original orbit. As a result, no new ω -limit points can appear under small perturbations of initial conditions, which characterizes upper semi-continuity [4].

6.2 Global Shadowing

Corollary 6.2. If $f: X \rightarrow X$ has the global shadowing property, then the ω -limit map is upper semi-continuous on X .

Proof. Global shadowing implies that every point of X is shadowable. The conclusion follows immediately from the previous theorem [4].

6.3 Directional Obstruction to Lower Semi-continuity

Proposition 6.3. In general, lower semi-continuity of the ω -limit map cannot be guaranteed by forward shadowing alone. The lack of backward shadowing provides a directional obstruction to lower semi-continuity.

Proof sketch. Lower semi-continuity requires persistence of limit points under perturbations. Such persistence typically demands backward or two-sided control of trajectories. In non-invertible or semi-continuous systems, backward shadowing is often unavailable, which explains the failure of lower semi-continuity despite the presence of forward shadowing [4].



Remark 6.4. The above results emphasize that upper and lower semi-continuity of the ω -limit maps arise from fundamentally different directional mechanisms. This asymmetry is topological in nature and does not rely on differentiability assumptions [4].

7 DISCUSSION

The above discussion suggests that the difference between upper and lower semi-continuity of the ω -limit map can be understood through the directional behavior of shadowing. Forward shadowing naturally supports upper semi-continuity, while the absence or weakness of backward shadowing mechanisms explains the challenges associated with lower semi-continuity. We begin with several elementary observations that follow from the shadowing property and are consistent with known results in the literature. These observations are included to clarify the relationship between shadowing and the semi-continuity of ω -limit sets.

Lemma 7.1. Let (X, d) be a metric space and let $f: X \rightarrow X$ have the shadowing property.

If a δ -pseudo-orbit $\{x_n\}_{n \geq 0}$ is ε -shadowed by a point $y \in X$, then $\omega(\{x_n\}) \subset B_\varepsilon(\omega(y))$.

Proof. Since $\{x_n\}$ is ε -shadowed by y , we have $d(x_n, f^n(y)) < \varepsilon$ for all $n \geq 0$.

Let $z \in \omega(\{x_n\})$. Then there exists a sequence $n_k \rightarrow \infty$ such that $x_{n_k} \rightarrow z$.

By the triangle inequality, $d(z, f^{n_k}(y)) \leq d(z, x_{n_k}) + d(x_{n_k}, f^{n_k}(y)) < \varepsilon$ for k large enough. Hence, $z \in B_\varepsilon(\omega(y))$.

Remark 7.2. The above lemmas and corollary should be viewed as interpretative consequences of the equivalence results established by Kawaguchi. They are included to emphasize the directional intuition behind upper semi-continuity and do not aim to introduce new independent results.

Lemma 7.3. Let $f: X \rightarrow X$ be a map with the shadowing property. If $\{x_n\}$ is a family of points converging to x , then no new ω -limit points can appear outside arbitrarily small neighborhoods of $\omega(x)$.

Proof. Assume by contradiction that there exists $\varepsilon > 0$ and a sequence $x_n \rightarrow x$ such that $\omega(x_n)$ contains a point z_n with $d(z_n, \omega(x)) > \varepsilon$. Using shadowing, one can construct pseudo-orbits starting near x whose ω -limits remain close to $\omega(x)$, contradicting the existence of such z_n . The remarks presented in this appendix reinforce the idea that semi-continuity of ω -limit sets should be viewed as a directional topological phenomenon. Forward shadowing supports upper semi-continuity through control of positive-time dynamics, while lower semi-continuity remains sensitive to backward behavior. These considerations suggest that directional shadowing provides a unifying conceptual framework for understanding asymmetries in the stability of limit sets beyond the classical settings of invertible or hyperbolic systems.

8 CONCLUSION

In this paper, we investigated the relationship between shadowing properties and the semi-continuity of ω -limit sets from a directional and topological perspective. Building on known equivalence results in the literature, particularly those established by Kawaguchi, our aim was not to introduce new shadowing notions, but rather to clarify the intrinsic asymmetry between upper



and lower semi-continuity. Our analysis shows that forward shadowing naturally induces a form of topological upper stability of the ω -limit map, which explains the close connection between shadowable points and upper semi-continuity. In contrast, lower semi-continuity requires stronger persistence properties that are closely related to backward or two-sided control of pseudo-orbits. The absence of such control in many non-invertible or semi-continuous systems provides a natural obstruction to lower semi-continuity. The lemmas, propositions, and examples presented in this work should be understood as interpretative consequences of existing results. Their role is to highlight the directional mechanisms underlying semi-continuity phenomena and to provide additional topological intuition for the behavior of ω -limit sets. We believe that the directional viewpoint proposed here offers a unifying framework for understanding stability properties of asymptotic dynamics beyond classical invertible or hyperbolic settings. Possible directions for future research include a deeper investigation of backward shadowing mechanisms, extensions to other types of limit sets, and applications to broader classes of semi-continuous dynamical systems.

Additional Remarks on Directional Shadowing

In this appendix, we collect several supplementary remarks that further clarify the directional viewpoint of shadowing and its relationship with the semi-continuity of ω -limit sets. These observations are intended to provide additional topological intuition and do not introduce new independent results [4].

A.1 Forward and Backward Pseudo-Orbits

The shadowing property is inherently directional. In many dynamical systems, especially non-invertible or semi-continuous ones, forward pseudo-orbits can be naturally defined and controlled, while backward pseudo-orbits may fail to exist or lack uniqueness. This asymmetry plays a crucial role in understanding the different behaviors of upper and lower semi-continuity of ω -limit sets. Forward shadowing ensures that approximate trajectories evolving in positive time remain close to true orbits. As a consequence, accumulation points of such trajectories cannot escape far from the original ω -limit sets, providing a topological explanation for upper semi-continuity [4].

A.2 Obstructions to Lower Semi-continuity

Lower semi-continuity requires that limit points persist under small perturbations of initial conditions. From a directional perspective, this persistence often demands some form of backward control. In the absence of backward shadowing, approximate backward trajectories may diverge significantly, causing limit points to disappear. This observation highlights why lower semi-continuity typically imposes stronger conditions than upper semi-continuity and why it is closely related to additional structures such as chain continuity or invertibility assumptions [4].

A.3 Topological Considerations

From the viewpoint of set-valued topology, the ω -limit map can be regarded as a mapping into a hyperspace equipped with Vietoris-type topologies. In this framework, upper semi-continuity corresponds to stability with respect to upper Vietoris neighborhoods, which naturally aligns with forward shadowing. The lack of backward shadowing, on the other hand, obstructs stability in



lower Vietoris neighborhoods, providing a topological interpretation for the failure of lower semi-continuity in many semi-continuous systems [4].

Conflict of interest.

There are no conflicts of interest.

References

- [1] A. Katok and B. Hasselblatt, Introduction to the Modern Theory of Dynamical Systems, Cambridge University Press, 1995.
- [2] S. Yu. Pilyugin, Shadowing in Dynamical Systems, Lecture Notes in Mathematics, Vol. 1706, Springer, 1999.
- [3] C. Morales, "Shadowable points and stability," Dynamical Systems, 2005.
- [4] N. Kawaguchi, "Shadowing and the continuity of ω -limit sets," arXiv:2601.08407, 2026.
- [5] K. Palmer, Shadowing in Dynamical Systems, Springer, 2000.
- [6] J. Meddaugh, "Shadowing as a structural property of the space of dynamical systems," Discrete and Continuous Dynamical Systems, 2021.
- [7] C. Good, P. Oprocha, and M. Puljiz, "Shadowing, asymptotic shadowing, and s-limit shadowing," Fundamenta Mathematicae, 2016.
- [8] N. Aoki and K. Hiraide, Topological Theory of Dynamical Systems, North-Holland, 1994.
- [9] E. Akin, The General Topology of Dynamical Systems, American Mathematical Society, 1993.
- [10] C. Conley, Isolated Invariant Sets and the Morse Index, American Mathematical Society, 1978.
- [11] C. Good, J. Mitchell, and J. Thomas, "Preservation of shadowing in discrete dynamical systems," Discrete and Continuous Dynamical Systems, 2019.
- [12] S. Yu. Pilyugin, "Inverse shadowing by continuous methods," Discrete and Continuous Dynamical Systems, 1999.
- [13] J.-P. Aubin and H. Frankowska, Set-Valued Analysis, Birkhäuser, 1990.