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An Approach to Generalized High-Order Fredholm Integro-Differential Equations via Fractional Cubic Spline

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ABSTRACT

This paper presents an efficient numerical technique for the approximate solution of multi-term integro-differential equations using a new form of fractional cubic spline (FCS). The proposed method is derived from a fractional boundary condition and a fractional continuity condition on a fractional spline, expressed in matrix form. A detailed convergence analysis is provided, and sufficient conditions for stability and error bounds of the method are established. Moreover, several numerical examples are solved, and the results are compared with exact solutions and methods from previous papers, presented in tables and figures. These comparisons demonstrate the efficacy and suitability of the suggested fractional spline scheme for solving integro-differential problems, achieving higher accuracy with fewer grid points. All computational results were obtained using Python 3.12.4.

Keywords: fractional calculus, integro-differential equations, fractional spline function, collocation method, convergence analysis.

1. INTRODUCTION

A fundamental concept in mathematical physics, integral and integro-differential equations are employed to express a range of ideas. Although it is impossible to cover all their applications, this paper will emphasize their importance by focusing on certain instances. Many works of literature have been dedicated to these equations since they are crucial to almost every area of mathematical physics and applied mathematics. Usually, they belong to one of three categories: Volterra, Fredholm, or Fredholm-Volterra integral equations. These equations represent a wide range of physical processes, but they can sometimes be extremely difficult to solve analytically. Fractional calculus is an extension of classical calculus that broadens the principles of differentiation and integration to encompass non-integer (fractional) orders. In contrast to integer-order operators, fractional derivatives inherently integrate memory and nonlocal

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effects, rendering them particularly suitable for modelling complex processes in physics, engineering, biology, and finance. Consequently, fractional calculus has emerged as a powerful mathematical tool for precisely characterizing systems exhibiting hereditary and anomalous dynamic behavior. [1]-[8].

Because many complicated issues in a variety of domains lack analytical answers, numerical approaches are essential for their analysis. As a result, scientists have developed and refined a range of numerical methods. The Legendre wavelet method [4], the Tau method [5], Taylor-series expansion [9], the hybrid function method [10], the modified Laplace Adomian decomposition [11], the finite difference method [6],[12], the Haar function method [13],[14] the Homotopy perturbation [15], and the differential transform [16] are a few interesting techniques. The linear Fredholm and Volterra integro-differential equations have been effectively solved using various numerical methods [7],[15],[17]-[21]. Researchers continue to work on improving these techniques to enhance accuracy and efficiency. Hilmi et al. [22], for example, have created a technique that offers accurate numerical solutions for fractional differential equations.

Other research has concentrated on developing numerical techniques for solving Fredholm integro-differential equations of both first and second order using non-polynomial cubic spline functions and cubic splines [23],[24].

This work is structured as follows: in Sections 2 and 3, the researchers proposed a fractional spline method (FSM) to solve integro-differential equations using a system of equations; in Section 4, the error boundedness, convergence, and stability analysis of the fractional spline model have been investigated; and in Section 5, numerical examples are presented to demonstrate the applications and efficacy of the approach. A conclusion is provided in Section 6.

We will investigate the integro-differential equations, on the interval $[a, b]$, as follows:

$$\sum_{\alpha=0}^{m} \zeta_\alpha y^\alpha(x) = v(x) + \int_a^b k(x,z)y(z)dz, \tag{1.1}$$

With a general initial condition.

$$\sum_{i=0}^{m-1} \eta_i \frac{\partial^i y(a)}{\partial x^i} = \mu_i. \tag{1.2}$$

Where ζ, v and k are known functions, $\alpha \in N, u \in C^2[J, R]$, and $J = [a, b], y(x)$ needs to be determined, which is an unknown solution function, while $\zeta \in Z^+, u(x) \in C([a, b], R)$, and $K(x,z) \in C(D, R), D = (x,z): a \leq x \leq z \leq b$, are known, with η and μ are real appropriate constants $\forall j = 0,1,2, \dots, m-1$, and m is of arbitrary integer order ($m \in Z^+$).

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2. FRACTIONAL SPLINE MODEL AND MATRIX FORM

This section consists of a new fractional spline model to solve equation (1.1) approximately by dividing the domain J into $M-1$ sub-intervals, and M is an equally spaced mesh of points $z_1, z_2, \ldots, z_M$ where $z_i = a_i + h$ (h is the step size), $i = 0,1, \ldots, M-1$, as follows:

$$F_i(x) = \sum_{k=0}^{\frac{5}{2}} a_k (x - x_i)^k, k = 0, \frac{1}{2}, \frac{3}{2}, \text{ and }, \frac{5}{2}. \tag{2.1}$$

With the following boundary condition:

$$F(x_i) = a_0(x_i - x_i)^0 + a_{\frac{1}{2}}(x_i - x_i)^{\frac{1}{2}} + a_{\frac{3}{2}}(x_i - x_i)^{\frac{3}{2}} + a_{\frac{5}{2}}(x_i - x_i)^{\frac{5}{2}} = y_i,$$

$$F(x_{i+1}) = a_0(x_{i+1} - x_i)^0 + a_{\frac{1}{2}}(x_{i+1} - x_i)^{\frac{1}{2}} + a_{\frac{3}{2}}(x_{i+1} - x_i)^{\frac{3}{2}} + a_{\frac{5}{2}}(x_{i+1} - x_i)^{\frac{5}{2}} = y_{i+1}, \tag{2.2}$$

$$F^{\left(\frac{1}{2}\right)}(x_i) = \frac{\sqrt{\pi}}{2} a_{\frac{1}{2}}(x_i - x_i)^0 + \frac{3\sqrt{\pi}}{4} a_{\frac{3}{2}}(x_i - x_i) + \frac{15\sqrt{\pi}}{16} a_{\frac{5}{2}}(x_i - x_i)^2 = U_i,$$

$$F^{\left(\frac{1}{2}\right)}(x_{i+1}) = \frac{\sqrt{\pi}}{2} a_{\frac{1}{2}}(x_{i+1} - x_i)^0 + \frac{3\sqrt{\pi}}{4} a_{\frac{3}{2}}(x_{i+1} - x_i) + \frac{15\sqrt{\pi}}{16} a_{\frac{5}{2}}(x_{i+1} - x_i)^2 = U_{i+1}.$$

After solving the above system, we obtained the following:

$$\begin{aligned} a_0 &= y_i, \\ a_{\frac{1}{2}} &= \frac{2}{\sqrt{\pi}} U_i, \\ a_{\frac{3}{2}} &= \frac{5}{h^{\frac{3}{2}}}[y_{i+1} - y_i] - \frac{2}{3\sqrt{\pi}h}[7U_i + 8U_{i+1}], \\ a_{\frac{5}{2}} &= \frac{4}{h^{\frac{5}{2}}}[y_i - y_{i+1}] + \frac{8}{3\sqrt{\pi}h^2}[U_i + 2U_{i+1}]. \end{aligned} \tag{2.3}$$

Using the continuity condition $F_i^{\left(\frac{3}{2}\right)}(x_i) = F_{i-1}^{\left(\frac{3}{2}\right)}(x_i)$, we obtained the following:

$$3U_{i-1} + 19U_i + 8U_{i+1} = \frac{15\sqrt{\pi}}{2\sqrt{h}}[y_{i+1} - y_{i-1}]. \tag{2.4}$$

From equation (2.4), we obtain $M-2$ equations; therefore, to obtain a unique solution, we need two additional equations. For this purpose, the fractional Taylor series was used:

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The Fractional Taylor Series

$$y^{(m)}(z)=\sum_{k=-\infty}^{\infty}\frac{y^{(k)}(a)z^{k-m}}{\Gamma(k-m+1)},$$

$$3U_{i-1}+8U_{i+1}=\frac{\sqrt{\pi}}{2\sqrt{h}}[8y_{i+2}-8y_{i+1}+3y_i-3y_{i-1}], \tag{2.5}$$

$$19U_i+8U_{i+1}=\frac{\sqrt{\pi}}{2\sqrt{h}}[8y_{i+2}+11y_{i+1}-19y_i].$$

Equation (2.4) can converted to a matrix form in the following:

$$\begin{bmatrix}19 & 8 & 0 & \cdots & 0\\ 3 & 19 & 8 & \ddots & \vdots\\ 0 & 3 & 19 & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & 8\\ 0 & \cdots & 0 & 3 & 19\end{bmatrix}\begin{bmatrix}U_1\\ U_2\\ U_3\\ \vdots\\ U_M\end{bmatrix}=r\begin{bmatrix}-1 & 0 & 1 & 0 & 0 & \cdots & 0\\ 0 & -1 & 0 & 1 & 0 & \cdots & 0\\ \vdots & & \ddots & \ddots & \ddots & & \vdots\\ 0 & \cdots & 0 & -1 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\\ \vdots\\ y_{M+1}\end{bmatrix}$$

$$\mathcal{A}U=r\mathcal{B}\mathcal{Y}.$$

where $r=\frac{15\sqrt{\pi}}{2\sqrt{h}}$. Hence,

$$U=r\mathcal{Z}\mathcal{Y},\ \text{where}\ \mathcal{Z}=\mathcal{A}^{-1}\mathcal{B}, \tag{2.6}$$

3. METHODOLOGY

In many scientific studies, fractional spline functions are employed to investigate Volterra, Fredholm, and mixed Volterra-Fredholm integral equations, as well as integro-differential equations (1; 2; 3; 4; 5). The results of these studies have been quite positive. In this section, the suggested techniques have been applied to approximate integro-differential equations. By substituting equation (2.3) in equation (2.1), we obtained:

$$\begin{aligned}F_i(x)&=y_i+\frac{2}{\sqrt{\pi}}U_i(x-x_i)^{\frac{1}{2}}+\left[\frac{5}{h^{\frac{3}{2}}}[y_{i+1}-y_i]-\frac{2}{3\sqrt{\pi}h}[7U_i+8U_{i+1}]\right](x-x_i)^{\frac{3}{2}}\\&+\left[\frac{4}{h^{\frac{5}{2}}}[y_i-y_{i+1}]+\frac{8}{3\sqrt{\pi}h^2}[U_i+2U_{i+1}]\right](x-x_i)^{\frac{5}{2}}+O(h^4),i=0,1,\dots,M\end{aligned} \tag{3.1}$$

From substituting equation (3.1) in equation (1.1) we obtained:

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$$\sum_{\alpha=0}^{m} \zeta_\alpha y^\alpha(x_j) = v(x_j) + \int_a^b k(x_j,z)y(z)dz \cong u(x_j) + \sum_{i=0}^{M-1} \int_a^b k(x_j,z)F_i(z)dz$$

$$= v(x_j) + \int_a^b k(x_j,z)\left[y_i + \frac{2}{\sqrt{\pi}}U_i(z-z_i)^{\frac{1}{2}} + \left[\frac{5}{h^{\frac{3}{2}}}[y_{i+1}-y_i] - \frac{2}{3\sqrt{\pi}h}[7U_i + 8U_{i+1}]\right](z-z_i)^{\frac{3}{2}}\right.$$

$$\left. + \left[\frac{4}{h^{\frac{5}{2}}}[y_i - y_{i+1}] + \frac{8}{3\sqrt{\pi}h^2}[U_i + 2U_{i+1}]\right](z-z_i)^{\frac{5}{2}}\right]dz + O(h^4)$$

$$= v(x_j) + \sum_{i=0}^{M-1}[y_i]\int_a^b k(x_j,z)dz + \sum_{i=0}^{M-1}\left[\frac{2}{\sqrt{\pi}}U_i\right]\int_a^b k(x_j,z)(z-z_i)^{\frac{1}{2}}dz$$

$$+ \sum_{i=0}^{M-1}\left[\frac{5}{h^{\frac{3}{2}}}[y_{i+1}-y_i] - \frac{2}{3\sqrt{\pi}h}[7U_i + 8U_{i+1}]\right]\int_a^b k(x_j,z)(z-z_i)^{\frac{3}{2}}dz$$

$$+ \sum_{i=0}^{M-1}\left[\frac{4}{h^{\frac{5}{2}}}[y_i - y_{i+1}] + \frac{8}{3\sqrt{\pi}h^2}[U_i + 2U_{i+1}]\right]\int_a^b k(x_j,z)(z-z_i)^{\frac{5}{2}}dz + O(h^4)$$

$$\sum_{\alpha=0}^{m} \zeta_\alpha y^\alpha(x_j) = v(x_j) + \sum_{i=0}^{M-1}[y_i]\mathrm{a}_{j,i} + \sum_{i=0}^{M-1}\left[\frac{2}{\sqrt{\pi}}U_i\right]\mathrm{b}_{j,i} - \sum_{i=0}^{M-1}\left[\frac{5}{h^{\frac{3}{2}}}y_i + \frac{14}{3\sqrt{\pi}h}U_i\right]\mathrm{c}_{j,i}$$

$$+ \sum_{i=0}^{M-1}\left[\frac{5}{h^{\frac{3}{2}}}y_{i+1} + \frac{16}{3\sqrt{\pi}h}U_{i+1}\right]\mathrm{d}_{j,i} + \sum_{i=0}^{M-1}\left[\frac{4}{h^{\frac{5}{2}}}y_i + \frac{8}{3\sqrt{\pi}h^2}U_i\right]\mathrm{e}_{j,i}$$

$$+ \sum_{i=0}^{M-1}\left[\frac{16}{3\sqrt{\pi}h^2}U_{i+1} - \frac{4}{h^{\frac{5}{2}}}y_{i+1}\right]\mathrm{f}_{j,i} + O(h^4).$$

Where

$$\mathrm{a}_{j,i} = \int_a^b k(x_j,z)dz$$

$$\mathrm{b}_{j,i} = \int_a^b k(x_j,z)(z-z_i)^{\frac{1}{2}}dz$$

$$\mathrm{c}_{j,i} = \int_a^b k(x_j,z)(z-z_i)^{\frac{3}{2}}dz = \mathrm{d}_{j,i+1}$$

$$\mathrm{e}_{j,i} = \int_a^b k(x_j,z)(z-z_i)^{\frac{5}{2}}dz = \mathrm{f}_{j,i+1}$$

Let the matrix form of integral parts be:

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$$\mathcal{A} = \mathrm{a}_{j,i}, \mathcal{B} = \mathrm{b}_{j,i}, \mathcal{C} = \mathrm{c}_{j,i}, \mathcal{D} = \mathrm{d}_{j,i}, \mathcal{E} = \mathrm{e}_{j,i}, \text{ and } \mathcal{F} = \mathrm{f}_{j,i}, \mathcal{V} = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_M \end{bmatrix}, \mathcal{U} = \begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ U_M \end{bmatrix}, \mathcal{Y} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_M \end{bmatrix},$$

$$\text{and } \mathcal{Y} = \begin{bmatrix} \sum_{\alpha=1}^{M} \zeta_\alpha y^\alpha(x_0) \\ \sum_{\alpha=1}^{M} \zeta_\alpha y^\alpha(x_1) \\ \vdots \\ \sum_{\alpha=1}^{M} \zeta_\alpha y^\alpha(x_M) \end{bmatrix} = \begin{bmatrix} y^0(x_0) & y^1(x_0) & \cdots & y^M(x_0) \\ y^0(x_1) & y^1(x_1) & \cdots & y^M(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ y^0(x_M) & y^1(x_M) & \cdots & y^M(x_M) \end{bmatrix} \begin{bmatrix} \zeta_0 \\ \zeta_1 \\ \vdots \\ \zeta_M \end{bmatrix}.$$

So, we obtain the following:

$$\mathcal{Y} = \mathcal{V} + \left[\mathcal{A} + \frac{5}{h^{\frac{3}{2}}}(\mathcal{D} - \mathcal{C}) + \frac{4}{h^{\frac{5}{2}}}(\mathcal{E} - \mathcal{F})\right]\mathcal{Y} + \left[\frac{2}{\sqrt{\pi}}\mathcal{B} + \frac{2}{3\sqrt{\pi h}}(8\mathcal{D} - 7\mathcal{C}) + \frac{8}{3\sqrt{\pi h^2}}(\mathcal{E} + 2\mathcal{F})\right]\mathcal{U},$$

$$\mathcal{Y} = [I - \mathcal{P} - \mathcal{Q}_0\mathcal{Q}]^{-1}\mathcal{V}. \tag{3.2}$$

where $\mathcal{Q} = r\mathcal{Z}\mathcal{Y}$, the exact solution of equation (3.1) is considered as follows:

$$\begin{aligned} \mathbb{F}_i(x) = y_i + \frac{2}{\sqrt{\pi}}\mathbb{U}_i(x - x_i)^{\frac{1}{2}} + \left[\frac{5}{h^{\frac{3}{2}}}[\mathbb{Y}_{i+1} - \mathbb{Y}_i] - \frac{2}{3\sqrt{\pi}h}[7\mathbb{U}_i + 8\mathbb{U}_{i+1}]\right](x - x_i)^{\frac{3}{2}} \\ + \left[\frac{4}{h^{\frac{5}{2}}}[\mathbb{Y}_i - \mathbb{Y}_{i+1}] + \frac{8}{3\sqrt{\pi}h^2}[\mathbb{U}_i + 2\mathbb{U}_{i+1}]\right](x - x_i)^{\frac{5}{2}} + O(h^4), i = 0,1, \dots, M \end{aligned} \tag{3.3}$$

where $\mathbb{F}, \mathbb{Y}$ and $\mathbb{U}$ are the exact solutions of F, y and U respectively.

4. ERROR BOUNDED, CONVERGENCE, AND STABILITY ANALYSIS

The suggested fractional cubic spline approach for solving generalized high-order Fredholm integro-differential equations is examined in this section for error boundedness, convergence, and stability. The discretization of the integral term and the truncation error of the spline interpolation are used to generate an upper bound for the approximation error. It is demonstrated that when the mesh size decreases, the numerical solution converges to the exact solution under appropriate smoothness requirements on the exact solution. Additionally, the stability of the resulting algebraic system is investigated, proving that the suggested approach yields accurate and stable numerical results.

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Consider T_i, which denotes the local truncation error linked to the j-th step in the numerical approach equation (3.3). This error essentially measures the deviation between the exact solution of the differential equation at that specific stage and the approximation generated by the spline scheme fractional cubic spline. By doing so, we can make necessary adjustments to enhance precision. To provide reliable and precise numerical responses, our main goal is to minimize T_j, especially when decreasing the step size. From equations (3.1) and (3.3), we obtain:

$$e = F(x) - \mathbb{F}(x) = \mathbb{Y}_i - y_i + \frac{2}{\sqrt{\pi}}(\mathbb{U}_i - U_i)(x - x_i)^{\frac{1}{2}}$$
$$+ \left[\frac{5}{h^{\frac{3}{2}}}[(\mathbb{Y}_{i+1} - y_i) - (\mathbb{Y}_i - y_i)] - \frac{2}{3\sqrt{\pi}h}[7(\mathbb{U}_i - U_i) + 8(\mathbb{U}_{i+1} - U_{i+1}]\right](x - x_i)^{\frac{3}{2}}$$
$$+ \left[\frac{4}{h^{\frac{5}{2}}}\left[(\mathbb{Y}_i - y_i) - (\mathbb{Y}_{i+1} - y_{i+1}] + \frac{8}{3\sqrt{\pi}h^2}[(\mathbb{U}_i - U_i) + 2(\mathbb{U}_{i+1} - U_{i+1})]\right]\right](x - x_i)^{\frac{5}{2}} + O(h^4)$$
$$i = 0,1, \dots, M$$

where λ is constant, e is the error function, and

$$|e| \equiv \lambda h^4 \tag{4.1}$$

The following Truncation error is obtained by using the fractional Taylor series for x_i :

$$T_i = \left[\frac{14}{\sqrt{\pi}}\right] h^{\frac{3}{2}} x_i^{(2)} + \left[\frac{11}{2}\right] h^2 x_i^{\left(\frac{5}{2}\right)} + \left[\frac{88}{15}\right] h^{\frac{5}{2}} x_i^{(3)} + \cdots \tag{4.2}$$

Lemma 4.1. [26] Assume $\boldsymbol{\mathcal{W}}$ be a square matrix with $\|\boldsymbol{\mathcal{W}}\| < \mathbf{1}$ then the matrix $(\boldsymbol{I} - \boldsymbol{\mathcal{W}})$ is invertible, in addition to $\left\|(\boldsymbol{I} - \boldsymbol{\mathcal{W}})^{-1}\right\|_\infty \leq \frac{1}{1 - \|\boldsymbol{\mathcal{W}}\|_\infty}$.

Lemma 4.2. The matrix $[\boldsymbol{I} - \boldsymbol{\mathcal{P}} - \boldsymbol{\mathcal{Q}_0 \mathcal{Q}}]$ is invertible if $[\boldsymbol{\mathcal{P}} + \boldsymbol{\mathcal{Q}_0 \mathcal{Q}}] \leq \mathbf{1}$.

Proof. Let $\mathcal{W} := \mathcal{P} + \mathcal{Q}_0\mathcal{Q}$, from Lemma 4$\mathcal{W}$ be invertible, and

$$\|(I - \mathcal{P} - \mathcal{Q}_0\mathcal{Q})^{-1}\|_\infty \leq \frac{1}{1 - \|\mathcal{P} + \mathcal{Q}_0\mathcal{Q}\|_\infty}, \text{ if } \|\mathcal{W}\|_\infty < 1$$

Assume that $k(x,z)$ is bounded on $[a,b]$, the integral blocks $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$ and $\mathcal{F}$ are $O\left(h^{\frac{1}{2}}\right), O\left(h^{\frac{3}{2}}\right)$ and $O\left(h^{\frac{5}{2}}\right)$ as given by their definitions, so

$$\|\mathcal{P} + \mathcal{Q}_0\mathcal{Q}\|_\infty \leq \|\mathcal{P}\|_\infty + \|\mathcal{Q}_0\|_\infty \|\mathcal{Q}\|_\infty \leq 1$$

So, $[I - \mathcal{P} - \mathcal{Q}_0\mathcal{Q}]$ is invertible and equation (3.2) has a unique solution.

Theorem 4.3. Assume that $v(x) \in C^4[J]$ and $k(x,z) \in C^4[J \times J], J = [a,b]$, and

$$\|e\|_\infty \leq \frac{14}{\sqrt{\pi}} h^{\frac{3}{2}} x_i^{(2)}.$$

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Proof. From equation (4.1), we obtained:

$$\|F(x) - \mathbb{F}(x)\|_\infty \leq \lambda h^4 \tag{4.3}$$

By using equations (3.2), (3.3), and (4.2), we obtained the following:

$$[I - \mathcal{P} - \mathcal{Q}_0\mathcal{Q}]\hat{\mathcal{Y}} = \mathcal{V} + T_i, \tag{4.4}$$

where $\hat{\mathcal{Y}}$ is the exact solution, by subtracting equation (4.4) from equation (3.2), we obtained:

$e = [I - \mathcal{P} - \mathcal{Q}_0\mathcal{Q}]^{-1}T_i$, By using equation (4.2) we obtained:

$$\|e\|_\infty \leq \|[I - \mathcal{P} - \mathcal{Q}_0\mathcal{Q}]^{-1}\|_\infty \|T_i\|_\infty. \tag{4.5}$$

Then by using Lemma 4 and Lemma 4, we obtained:

$$\|e\|_\infty \leq \frac{\|T_i\|_\infty}{1 - \|[I - \mathcal{P} - \mathcal{Q}_0\mathcal{Q}]^{-1}\|_\infty}, \tag{4.6}$$

From equations (4.3) and (4.6), we obtained:

$$\|\hat{\mathcal{Y}}(x) - \mathbb{F}(x)\|_\infty \leq \|\hat{\mathcal{Y}}(x) - F(x)\|_\infty + \|F(x) - \mathbb{F}(x)\|_\infty \leq \frac{14}{\sqrt{\pi}} h^{\frac{3}{2}} x_i^{(2)}.$$

Hence, the one-and-a-half-order convergence orders were proved.

Theorem 4.4. The proposed spline scheme is unconditionally stable.

Proof. The Fourier stability principle is thought to dictate the structure of the solution to equation (2.4), suppose a Fourier mode:

$$U_i = \hat{U}_i e^{i\eta hi}, y_i = \hat{y}_i e^{i\eta hi},$$

then

$$\hat{U}_i = \frac{\omega_0(\eta)}{\omega_1(\eta)} \hat{y}_i, \tag{4.7}$$

where $i = \sqrt{-1}$ and and η is spatial wave constant, $\omega_1(\eta) \neq 0$, and after some simplification

$$\omega_0(\eta) = \frac{15\sqrt{\pi}}{h^{\frac{3}{2}}}\left(-4\sin^2\left(\frac{\eta h}{2}\right)\right)$$
$$\omega_1(\eta) = 19 + 11\cos(\eta h) + 5i\sin(\eta h)$$

So, equation (4.7) becomes:

$$\hat{U} = \omega\hat{Y}$$

then

$$|\hat{U}| \leq \omega|\hat{Y}|$$

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Thus, the proposed method is unconditionally stable.

5. NUMERICAL IMPLEMENTATION

This section applies the presented technique to solve integro-Fredholm differential equation problems. By comparing graphs and tables of the proposed FCS method with the exact solution, its accuracy and efficiency are demonstrated. The previously obtained values for maximum
absolute errors are compared to the newly calculated values.

Example 5.1. Consider the integro-differential equation:

$$y^{(3)}(x) = \sin x + x - \int_0^{\frac{\pi}{2}} (xz)y'(z)dz$$

with initial conditions

$$y(0) = 1, y'(0) = 0, y''(0) = 1.$$

where $y(x) = \cos x$ is the exact solution, and this integro-differential equation is solved for $m = 3$, $\zeta_0 = \zeta_1 = \zeta_2 = 0, \zeta_3 = 1$ and $a = 0$, and $b = \frac{\pi}{2}$. The numerical results of this example in Table 1 present a comparison between the exact solution, the results obtained using the FCS methods, and errors for $M = 10$, and Figure 1 includes the comparison between exact and FSM solutions for $M = 100$ of Example 5.1.

Table 1: Approximation solutions by FCS method for Example 5.1 with $M = 10$.

Grid points	Exact	FCS Method	Absolute Error
0.0	1.0	1.0	0.0
0.1	0.9950041652780258	0.9982604944109351	3.3×10^{-3}
0.2	0.9800665778412416	0.9927645657489346	1.3×10^{-2}
0.3	0.955336489125606	0.9832253441018365	2.8×10^{-2}
0.4	0.9210609940028851	0.9693493935248303	4.8×10^{-2}
0.5	0.8775825618903728	0.9513431519575047	7.4×10^{-2}
0.6	0.8253356149096783	0.7962738942672349	2.9×10^{-2}
0.7	0.7648421872844885	0.6774695312814130	8.7×10^{-2}
0.8	0.6967067093471655	0.6554938670177460	4.1×10^{-2}
0.9	0.6216099682706645	0.6323294441273820	1.1×10^{-2}
1.0	0.5403023058681398	0.6076373366584500	6.7×10^{-2}
L.S.E.			2.4×10^{-2}

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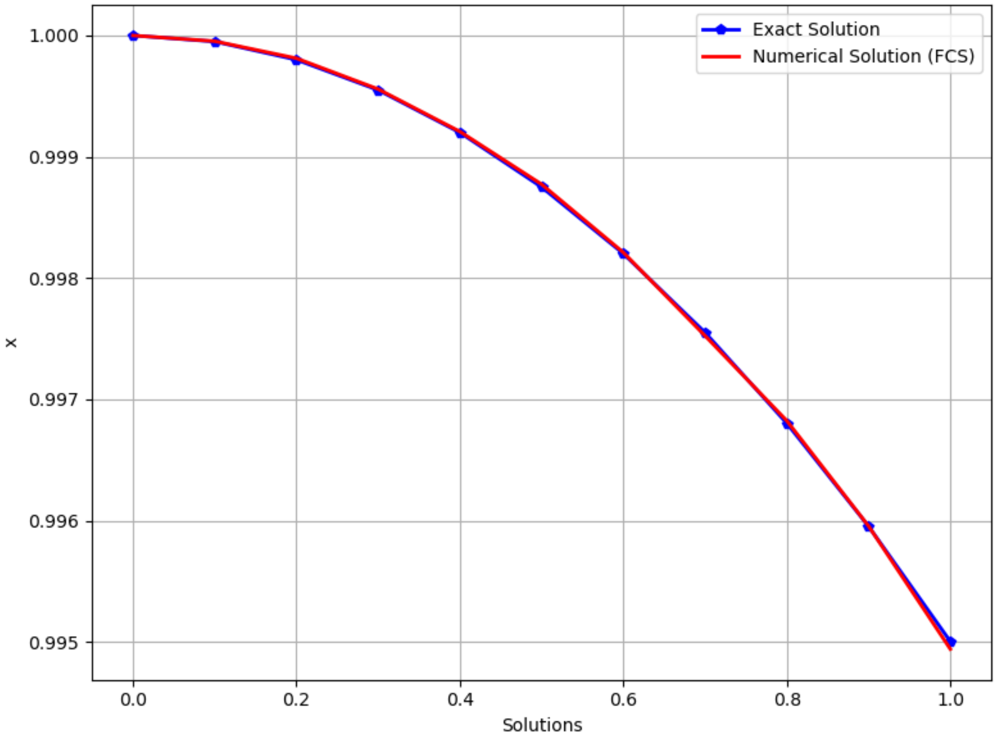

Figure 1:Comparison between exact and FSM solutions for M=100 of Example 5.1.

Example 5.2. [13] Consider the integro-differential equation:

$$y'(x) = xe^x + e^x - x + \int_0^1 (xz)y(z)dz$$

with initial condition

$$y(0) = 0.$$

where $y(x) = xe^x$ is the exact solution, and this integro-differential equation is solved for $m = 1$, $\zeta_0 = 0, \zeta_1 = 1$ and $a = 0$, and $b = 1$. The numerical results of this example in Table 2 present a comparison between the exact solution, the results obtained using the FCS methods, the errors, and the best results reported in [6] for $M = 10$, and Figure 2 includes the comparison between exact and FSM solutions for $M = 100$ of Example 5.2.

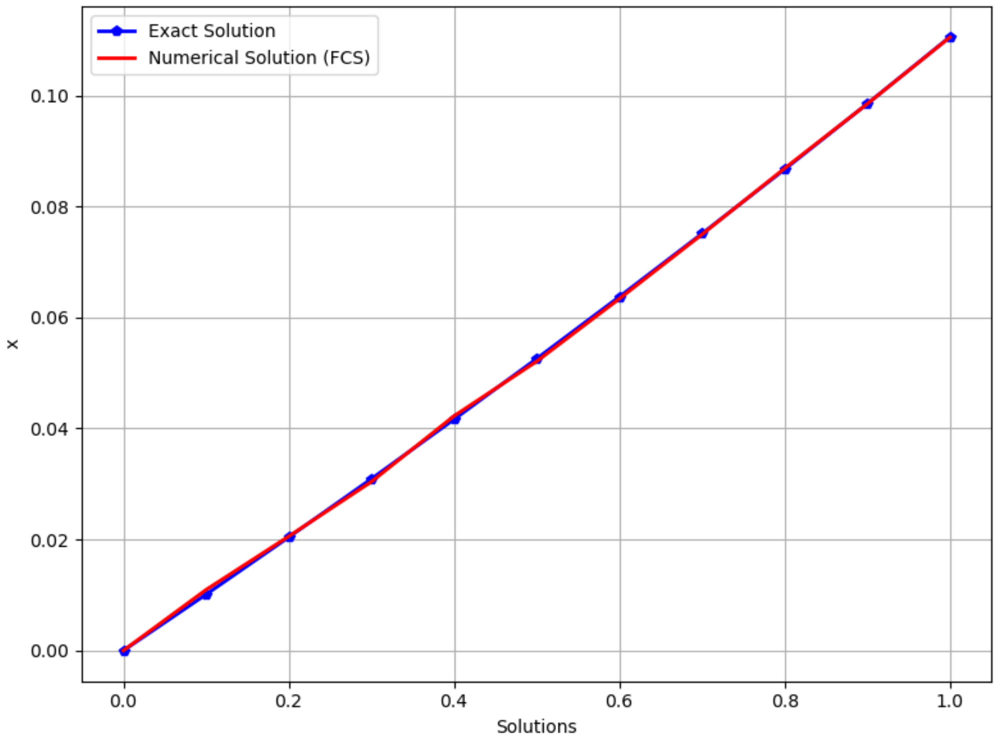

Figure 2:Comparison between exact and FSM solutions for M=100 of Example 5.2.

Table 2: Approximation solutions by FCS method for Example 5.2 with $M = 10$.

Grid points	Exact	FCS Method	Absolute Error	Best in [13]
0.0	0.0	0.0	0.0	$> \times 10^{-5}$
0.1	0.11051709180756478	0.11043063019798612	8.6×10^{-5}	1.0×10^{-2}
0.2	0.244280551632034	0.24419639118293102	8.4×10^{-5}	2.8×10^{-2}
0.3	0.404957642272801	0.4017193245112236	3.2×10^{-3}	5.1×10^{-2}
0.4	0.5967298790565082	0.5969277062872085	1.9×10^{-4}	7.6×10^{-2}
0.5	0.8243606353500641	0.829945279548355	5.6×10^{-3}	9.7×10^{-2}
0.6	1.0932712802343052	1.09289978451074	3.7×10^{-4}	1.1×10^{-1}
0.7	1.4096268952293336	1.4154647182771574	5.8×10^{-3}	1.0×10^{-1}
0.8	1.780432742793974	1.7805771254468958	1.4×10^{-4}	7.0×10^{-2}
0.9	2.2136428000412542	2.2128115861087827	8.3×10^{-4}	1.0×10^{-2}
1.0	2.7182818284590446	2.719203978671292	9.2×10^{-4}	1.6×10^{-1}
L.S.E.			7.6×10^{-5}	

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Example 5.3. [25] Consider the integro-differential equation:

$$y'(x) = 1 - \frac{x}{3} + \int_0^1 xty(t)dt$$

with initial condition

$$y(0) = 0.$$

where $y(x) = x$ is the exact solution, and this integro-differential equation is solved for $m = 1$, $\zeta_0 = 0, \zeta_1 = 1$ and $a = 0$, and $b = 1$. The numerical results of this example in Table 3 present a comparison between the exact solution, the results obtained using the FCS methods, errors, and the best results reported in [25] for $M = 10$, and Figure 3 includes the comparison between exact and FSM solutions for M = 10 of Example 5.3. Also, figure 4 includes the comparison of absolute errors with M = 10 for all examples.

Table 3: Approximation solutions by FCS method for Example 5.3 with $M = 10$.

Grid points	Exact	FCS Method	Absolute Error	Best in [25]
0.0	0.0	0.0	0.0	0.0
0.1	0.1	0.10094759248725661	9.5×10^{-4}	1.7×10^{-3}
0.2	0.2	0.2004329808837825	4.3×10^{-4}	6.1×10^{-3}
0.3	0.3	0.299847030694684	1.5×10^{-4}	1.3×10^{-2}
0.4	0.4	0.4005806470822501	5.8×10^{-4}	2.3×10^{-2}
0.5	0.5	0.4998379236449176	1.6×10^{-4}	3.5×10^{-2}
0.6	0.6	0.5990187125521584	9.8×10^{-4}	6.7×10^{-2}
0.7	0.7	0.6995159166929457	4.8×10^{-4}	$> 10^{-3}$
0.8	0.8	0.7999361193393252	6.4×10^{-5}	8.6×10^{-2}
0.9	0.9	0.8992329260779701	7.7×10^{-4}	1.1×10^{-1}
1.0	1.0	1.0005432866379858	5.4×10^{-4}	1.3×10^{-1}
L.S.E			3.6×10^{-6}	

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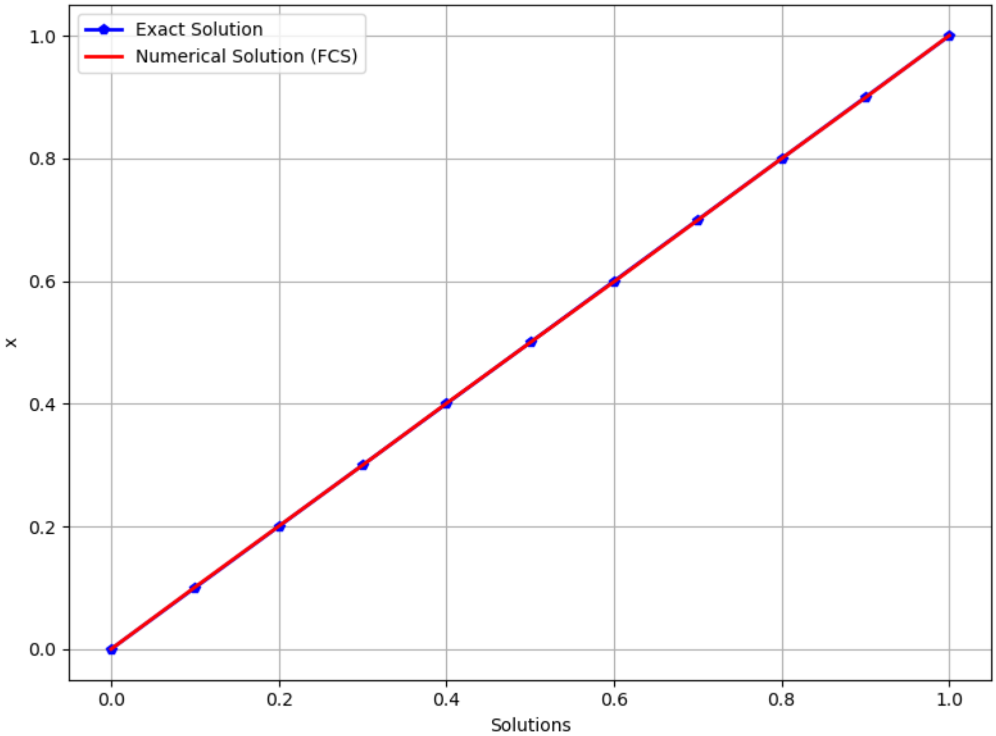

Figure 3: Comparison between exact and FSM solutions for M = 10 of Example 5.3.

6. CONCLUSION

In this work, a new numerical scheme is proposed to solve multi-term integro-differential equations using a new form of spline function with fractional order in matrix form. Some helpful lemmas and theorems have been established to facilitate the investigation of convergence analysis. The researchers designed a Python program to precisely describe the procedure, and the results presented in tables and figures demonstrate that the suggested strategies are comparable to the previous approaches. Future work will concentrate on expanding the proposed fractional cubic spline framework to encompass higher-dimensional challenges and adaptive mesh techniques. Furthermore, its applicability to a broader category of fractional differential equations and practical models will be examined, accompanied by a comprehensive investigation of convergence and stability, making it appropriate for complex engineering and scientific modeling.

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Conflict of interests.

The authors decelerate that there is no conflict of interest.

References

[1] M. Abbas, S. Aslam, FA. Abdullah, MB. Riaz, and KA. Gepreel, "An efficient spline technique for solving time-fractional integro-differential equations", *Heliyon, vol.* 9, no. 9, pp. 1-19 , Sep. 2023.

[2] F. Pitolli. "A fractional B-spline collocation method for the numerical solution of fractional predator-prey models", *Fractal and Fractional,* vol. 2, no. 1, pp. 13, Feb. 2018.

[3] F. Hamasalih, and R. Qadir. "On the numerical solution of Volterra and Fredholm integral equations using the fractional spline function method", *Journal of numerical analysis and approximation theory,* vol. 51, no.2, pp. 167-180, Dec. 2022.

[4] M. Riahi. "Legendre wavelet method combined with the Gauss quadrature rule for numerical solution of fractional integro-differential equations", *Iranian Journal of Numerical Analysis and Optimization,* vol. 12, no. 1, pp. 229-249, Mar. 2022.

[5] M. Masoud. "Numerical solution of systems of fractional order integro-differential equations with a Tau method based on monic Laguerre polynomials", *Journal of Mathematical Analysis and Modeling,* vol. 3, no. 3, pp. 1-13, Dec. 2022.

[6] D. Xu, Q. Wenlin, and G. Jing. "A compact finite difference scheme for the fourth-order time-fractional integro-differential equation with a weakly singular kernel", *Numerical Methods for Partial Differential Equations,* vol. 36, no. 2, pp. 439-458, Mar. 2020.

[7] N. E. D. Ndidiamaka, T. Oyedepo, and A. E. Adenipekun, "Fourth-kind Chebyshev Computational Approach for Integro-Differential Equations," *Mathematics and Computational Sciences*, vol. 5, no. 1, pp. 55–64, Mar. 2024.

[8] S. J. Mohammedfaeq and R. J. Qadir, "An Efficient Fractional-Order Shifted Legendre Function for Solving Multiterm High-Order Fractional-Differential Equations of Caputo-Type With Convergence Analysis," *Mathematical Methods in the Applied Sciences*, vol. 48, no. 12, pp. 1-26, Dec. 2025.

[9] S. Aggarwal, K. Bhatnagar, and A. Dua, "Method of Taylor's Series for the Primitive of Linear Second Kind Non-Homogeneous Volterra Integral Equations," *International Journal of Research and Innovation in Applied Science*, vol. 5, no. 5, pp. 32–35, 2020.

[10] A. Hosry, R. Nakad, and S. Bhalekar, "A Hybrid Function Approach to Solving a Class of Fredholm and Volterra Integro-Differential Equations," *Mathematical and Computational Applications*, vol. 25, no. 2, p. 30, May. 2020.

[11] L. Dawood, A. Hamoud, and N. Mohammed, "Laplace Discrete Decomposition Method for Solving Nonlinear Volterra-Fredholm Integro-Differential Equations," *Journal of Mathematics and Computer Science*, vol. 21, no. 2, pp. 158–163, Apr. 2020.

[12] S. R. Saber, Y. Sabawi, and M. S. Hasso, "Numerical Solution of the Fredholm Integro-Differential Equations Using High-Order Compact Finite Difference Method," *Journal of Education and Science*, vol. 32, no. 3, pp. 9–10, Sep. 2023.

[13] M. H. Reihani and Z. Abadi, "Rationalized Haar Functions Method for Solving Fredholm and Volterra Integral Equations," *Journal of Computational and Applied Mathematics*, vol. 200, no. 1, pp. 12–20, Mar. 2007.

[14] M. Erfanian and H. Zeidabadi, "Solving Two-Dimensional Nonlinear Volterra Integral Equations Using Rationalized Haar Functions," *International Journal of Nonlinear Analysis and Applications*, vol. 14, no. 8, pp. 95–105, Aug. 2023.

[15] A. Afzal, A. R. Kaladgi, R. D. Jilte, M. Ibrahim, R. Kumar, M. A. Mujtaba, S. Alshahrani, and C. A. Saleel, "Thermal Modelling and Characteristic Evaluation of Electric Vehicle Battery System," *Case Studies in Thermal Engineering*, vol. 26, p. 101058, Aug. 2021.

[16] P. Darania and A. Ebadian, "A Method for the Numerical Solution of the Integro Differential Equations," *Applied Mathematics and Computation*, vol. 188, no. 1, pp. 657–668, May. 2007.

[17] P. Assari and S. Cuomo, "The numerical solution of fractional differential equations using the Volterra integral equation method based on thin plate splines", *Engineering with Computers*, vol. 35, no. 4, pp. 1391–1408, Oct. 2019.

[18] S. H. Salim, R. K. Saeed, and K. H. F. Jwamer, "Solving Volterra-Fredholm integral equations by quadratic spline function," *Journal of Al-Qadisiyah for Computer Science and Mathematics*, vol. 14, no. 4, pp. 1–10, Dec. 2022.

[19] G. Ajileye and S. A. Amoo, "Numerical solution to Volterra integro-differential equations using collocation approximation," *Mathematics and Computational Sciences*, vol. 4, no. 1, pp. 1–8, Mar. 2023.

[20] S. J. Mohammedfaeq and S. S. Ahmed, "Operational matrix of generalized block pulse functions for solving fractional Volterra-Fredholm integro-differential equations," *Journal of Southwest Jiaotong University*, vol. 57, no. 3, pp. 39–59, Jun. 2022.

[21] S. Mohammadizadeh, J. Rashidinia, and R. Ezzati, "C3-spline methods for solving fractional integro-differential equations," *Mathematics and Computational Sciences*, vol. 5, no. 1, pp. 30–42, Mar. 2024.

[22] H. Hilmi, S. J. MohammedFaeq, and S. S. Fatah, "Exact and approximate solution of multi-higher order fractional differential equations via Sawi transform and sequential approximation method," *J. Univ. Babylon Pure Appl. Sci.*, vol. 32, no. 1, pp. 311–334, Mar. 2024.

[23] O. M. Ogunlaran and M. O. Oke, "A numerical approach for solving first order integro-differential equations," 2013.

[24] T. Tahernezhad and R. Jalilian, "Exponential spline for the numerical solutions of linear Fredholm integro-differential equations," *Advances in Difference Equations*, vol. 2020, no. 1, p. 141, Mar. 2020.

[25] P. Darania and A. Ebadian, "A method for the numerical solution of the integro-differential equations," *Applied Mathematics and Computation*, vol. 188, no. 1, pp. 657–668, May. 2007.

[26] R. Jaza and F. Hamasalh, "Non-polynomial fractional spline method for solving Fredholm integral equations," *Journal of Innovative Applied Mathematics and Computational Sciences*, vol. 2, no. 3, pp. 1–14, Dec. 2022.

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