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Dynamic Analysis and Accurate Analytical Solution of the Hybrid Duffing-Van der pol Oscillator via a Modified Homotopy Perturbation Method

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التحليل الديناميكي والحل التحليلي الدقيق لمذبذب دافينغ-فان دير بول الهجين باستخدام طريقة اضطراب التماثل المعادلة

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ABSTRACT

In this paper, the Modified Homotopy Perturbation Method (MHPM) is proposed to develop closed-form analytical solutions for the hybrid Duffing–Van der Pol oscillator $\ddot{x} + \mu(x^2 - 1)\dot{x} + \alpha x + \beta x^3 = F\cos(\omega t)$, which is prompted by piezoelectric vibration energy harvesting under aerodynamic galloping. The new order provides a previously unaccounted frequency included in the linear homotopy operator, enabling self-consistent frequency–amplitude determination and systematic removal of secular terms. Analytical expressions for the limit cycle amplitude $A = 2$ and nonlinear frequency $\omega = \sqrt{\alpha + 3\beta}$ are obtained for the autonomous system. The equations of the amplitude–frequency response are derived for the forced system, and stability of the multi-valued branches is proved by means of a Jacobian/Routh–Hurwitz analysis. Dense Poincaré sections and bifurcation diagrams map the transition to chaos via period-doubling cascades. Relative errors of moduli are no greater than 1% in moderately nonlinear regimes, and no greater than 8% in strongly nonlinear ones, verified against fourth-order Runge–Kutta solutions. The effectiveness of this new approach in handling self-excited systems is confirmed by comparative analysis against the standard HPM and the He–Laplace method.

Keywords: hybrid Duffing–Van der Pol oscillator; Modified Homotopy Perturbation Method; limit cycle; jump phenomenon; period-doubling bifurcation; piezoelectric energy harvesting; Routh–Hurwitz stability

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INTRODUCTION

In fact, nonlinear oscillatory systems are a cornerstone of modern applied mathematics, physics and engineering [1, 2]. Nonlinear oscillators exhibit a rich structure of dynamical behavior spanning limit cycles, bifurcations, jump phenomena, sub-harmonic and super-harmonic resonances and chaos [3, 4]; linear oscillators can only generate harmonic or simple periodic excitation. Consequently, the search for accurate analytical methods is still one of the main goals among researchers in the nonlinear dynamics community.

Two of the best known nonlinear oscillators are the Duffing and Van der Pol equations. The Duffing equation was first introduced (1918) by Georg Duffing [5], a system to nonlinear restoring force and cubic stiffness. It is commonly used for event descriptions in structural mechanics, electrical circuits and ship dynamics [6]. The Van der Pol equation [5], proposed by Balthasar van der Pol in 1926 [7] is a self-excited oscillator with a nonlinear damping that gives rise to periodic oscillations, known as limit cycles. It has a broad range of applications, from biological rhythms [2] to laser physics [8], and to designs of electronic circuits.

Although all these oscillators have been studied separately in great detail, hybrid Duffing–Van der Pol oscillator with nonlinear stiffness and nonlinear damping mechanism has received relatively less attention analytically. This dual mechanism has practical importance, as it better represents natural processes where more than one energy dissipation/generation mechanism (Van der Pol) and nonlinear elastic restoring forces (Duffing) interact at a time. These forms of systems occur in nonlinear electric circuits, energy harvesting devices [6, 9] and coupled mechanical vibration systems.

However, when it comes to asymptotic procedures claimed for the resolution of nonlinear differential equations, there is clearly the perturbation method as well as even harmonic balance or variational iteration and finally homotopy analysis (HAM). Another method is the Homotopy Perturbation Method (HPM), which was founded by He [10, 11, 12], as it provides an analytic series solution that is very much accepted due to its simplicity, flexibility and independence of linearization or small parameter assumptions. The formulation of the HPM involves constructing a homotopy that continuously deforms a simple problem into the original nonlinear one, and its solution is expressed as an expansion series in the embedding parameter.

However, the standard HPM is not well suited for strongly nonlinear oscillators such as those with large amplitudes or strong damping coefficients [13]. Over the years, there had been various changes suggested to improve its effectiveness. He [14] presented a nonlinear oscillation methodology for strongly nonlinear scenarios with Lindstedt–Poincaré modification. Beléndez et al. [15] have proposed a modified HPM for nonlinear oscillators with rational and irrational restoring forces. Chen et al. [16] targeted this damping type in a dedicated HPM formulation. Recent frequency–amplitude formulations [17, 18] have extended these approaches to strongly

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nonlinear oscillators, which motivates the present generalization to the hybrid Duffing–Van der Pol system.

However, treating the coupled Duffing–Van der Pol system analytically is still an open research problem, since so far a complete dynamic analysis of the adjusted HPM with extensive numerical verification for the coupling case remains elusive.

The present contribution. In this work, we develop a Modified Homotopy Perturbation Method (MHPM) that incorporates a self-consistent frequency–amplitude determination mechanism to solve the hybrid Duffing–Van der Pol equation. Beyond the purely mathematical derivation, this paper features three distinct and robust scientific extensions to elevate the analysis:

1. Physical motivation: The nonlinear differential equation is explicitly formulated within the field of piezoelectric energy harvesting via aerodynamic galloping [9, 19, 20, 21], allowing for fundamental engineering interpretations of the mathematical parameters.
2. Stronger Stability Analysis: A formal Jacobian based local stability analysis is conducted for the forced resonance condition, and through Routh–Hurwitz criteria, we give a mathematical proof on the stability boundaries of multi-valued jump behavior.
3. Path to chaos: We obtain the period doubling pathways to deterministic chaos for the strongly forced system via extensive computations of bifurcation maps (varying over forcing amplitude) and Poincaré sections.

The remainder of this paper is structured as follows. In Section 2, we describe the mathematical model of a hybrid Duffing–Van der Pol oscillator and the physical significance of each parameter in energy harvesting. In Section 3 the developed Modified HPM approach is extended. followed by Section 4, from which we derive the analytical solutions and give a stringent stability assessment of the forced branches. Section 5 gives numerical examples with an error analysis. Section 6 presents advanced dynamical mapping and the route to chaos. Finally, Section 7 provides concluding remarks.

Mathematical Model and Physical Motivation

Physical Motivation: Energy Harvesting System

While this Fleet Dancing Duffing–Van der Pol oscillator is born out of a no-futons-barred math-y physics play, it easily derives as the master equation for nearly every modern engineering problem especially in vibration energy harvesting. Consider a piezoelectric energy harvester (PEH) excited by self-excited aerodynamic galloping and nonlinear restorative forces.

This leads to a mathematical representation for a system coupled with lossless resistance R via one mechanical domain which causes self-excitation (galloping of blades) and a second electrical domain in terms of where significant energy is lost. We reduce the dimensionless mechanical DoF

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to derive a single-degree-of-freedom equation, such that nonlinear electromechanical coupling with structural hardening stiffness and aerodynamic drag produces an equivalent effect.

The Van der Pol term $(\mu(x^2 - 1)\dot{x})$is symbolic of that net effect between the negative aerodynamic damping (which adds energy to the system) and non-linear positive mechanical/electrical damping (that removes energy from the structure). The Duffing (βx^3) term at the same time captures the geometric nonlinearity of the large deflections for piezoelectric cantilevers clamped-free. We want a direct mapping from the mathematical model to physical prototypes that relate the Non-dimensional parameters composite number to certain physical quantities. For instance the Duffing parameter $\beta(e.g. \beta = 0.5)$ is defined via a nonlinear restoring force corresponding to some piezoelectric geometry appropriately suffering large transverse deflections, e.g., Macro Fiber Composites. The Van der Pol parameter, μ, represents net aerodynamic damping at constant wind speed (for example the level of windspeed at which galloping onset is observed). Hence this provides direct output from the analytical process that converts the parameter space back into physical representations of how it will regulate steady-state voltage, and provide structural limits on vibration. analytical insight into maximizing the harvested power bandwidth in self-oscillating galloping harvesters [20].

The Governing Hybrid Equation

The hybrid Duffing–Van der Pol oscillator is governed by the following second-order nonlinear ordinary differential equation:

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + \alpha x + \beta x^3 = F\cos(\omega t), \quad \dots \quad (1)$$

where the state variable $x = x(t)$ refers to the corresponding instantaneously transverse deviation, and the overdot is a shortcut for differentiation pertaining to with respect to the temporal coordinate t. The system dynamics are dictated by an exact specific set of phenomenological parameters. The Van der Pol parameter μ measures the linear damping coefficient; in other words, it triggers negative damping (energy injection) for local domain $|x|$, which provides first-principles basis of self-excited limit cycle oscillations. Where α is a linear stiffness coefficient accounting for elastic restoring forces close to stable equilibria in the origin $(i.e., \ here\ \alpha > 0)$ At the same time, the Duffing parameter βpresents the coefficient of non-linear cubic stiffness either representing structural hardening $(\beta > 0)$ or softening $(\beta < 0)$under significant-deflection. The system is finally externally excited harmonically, with a forcing amplitude F and exercise frequency ω.

Equation (1) forms a master equation in the broadest sense, which spontaneously degenerates into several well-known canonical nonlinear oscillators for certain parametric conditions. If we now set the damping, nonlinear stiffness, and external forcing to zero $(\mu = \beta = F = 0)$, one obtains the equation of the simple harmonic oscillator:$\ddot{x} + \alpha x = 0$. Setting to zero only the nonlinear damping and forcing$(\mu = F = 0)$ gives the autonomous Duffing oscillator without nonlinearity$(\ddot{x} + \alpha x + \beta x^3)$ Conversely, nullifying the nonlinear stiffness and driving force while normalizing linear stiffness ($\beta = 0, \alpha = 1, F = 0$)recovers classical Van der Pol oscillator ($\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$). Removing nonlinear damping when retaining external forcing, ($\mu = 0$), yields the standard forced Duffing oscillator. As a result, the full hybrid equation considered in this work contains all these competing mechanisms of physical coupling.

The initial conditions associated with the Cauchy problem for Eq. (1) are:

$$x(0) = A, \qquad \dot{x}(0) = 0,$$

where A is the initial amplitude of oscillation.

The Proposed Modified Homotopy Perturbation Method

Overview of the Standard HPM

Consider a general nonlinear differential equation of the form:

$$\mathcal{L}(u) + \mathcal{N}(u) = f(t), \quad \dots (2)$$

where $\mathcal{L}$ is a linear operator, $\mathcal{N}$ is a nonlinear operator, and $f(t)$ is a known forcing function. He [10, 12] constructed the following homotopy $\mathcal{H}(v, p): \mathbb{R} \times [0,1] \to \mathbb{R}$:

$$\mathcal{H}(v, p) = (1 - p)[\mathcal{L}(v) - \mathcal{L}(u_0)] + p[\mathcal{L}(v) + \mathcal{N}(v) - f(t)] = 0, \qquad \dots (3)$$

where $p \in [0,1]$ denotes the embedding (homotopy) parameter, and u_0 is a given approximation satisfying the boundary/initial conditions. As p varies from 0 to 1, v evolves from the initial guess $u_0$toward the exact solution u. The solution can be expressed as:

$$v = v_0 + p\, v_1 + p^2\, v_2 + \cdots \qquad \dots (4)$$

Setting $p = 1$, the approximate solution to the original problem is obtained as:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots \qquad \dots (5)$$

The Proposed Modification

For nonlinear oscillators, the traditional HPM encounters a serious issue, namely that in this classical form it overlooks any amplitude-dependent shift of frequency (as is typical for nonlinear systems). This consequently yields secular terms (i.e. terms that grow unboundedly with time), thus, the solution becomes invalid for large intervals of time.

To overcome this limitation, we propose the following modification. Consider the autonomous form of Eq. (1) (with $F = 0$):

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + \alpha x + \beta x^3 = 0. \quad \dots (6)$$

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This inability of the standard HPM to account for amplitude dependent frequency shift governing nonlinear phase spaces limits its use, particularly with autonomous nonlinear oscillators. Hence this operator applied to the unperturbed linear operator on $\mathcal{L}(v) = \ddot{v} + \alpha v$ gives us as zeroth-order solution at infinite order: $v_0(t)$=$Acos\left(\sqrt{\alpha}t\right)$. In particular, it is always possible to substitute into the first-order formulation which implies that resonant trigonometric terms are generated. The specific solutions from these terms yield secularities $\{\times sin(\sqrt{\alpha}t), tcos(\sqrt{\alpha}t)\}$, which diverge, thus not satisfying the limits imposed by Poincaré-Bendixson for fixed point periodic limit cycles in steady-state.

To evade this potentially dangerous topological separation we introduce an a priori unknown angular frequencyω into our linear differential operator, and subsequently pass it on by subtracting it from the nonlinear operator to ensure mathematical equivalence. So, the governing equation is reformulated mathematically as follows:

$$\ddot{x} + \omega^2 x + (-\omega^2 x + \mu(x^2 - 1)\dot{x} + \alpha x + \beta x^3) = 0. \quad \ldots \quad (7)$$

The continuous topological mapping $\mathcal{H}(v,p) = \mathcal{L}(v) + p\mathcal{N}(v) = 0$ is explicitly constructed via:

$$\ddot{v} + \omega^2 v + p[-\omega^2 v + \mu(v^2 - 1)\dot{v} + \alpha v + \beta v^3] = 0. \quad \ldots \quad (8)$$

In this formulation, the limiting cases p = 0 and p = 1 correspond to smoothly deforming from a harmonic oscillator with unknown fundamental frequencyω(the plottable case) to the exact nonlinear hybrid equation (7).

Instead of individually expanding both the spatial coordinate $v(t)$ and the angular frequency ω in series about powers of the embedding parameter p, we regard it as a single unified spectral tuning parameter. We expand the periodic solution purely in terms of p=v=$\sum_{k=0}^{\infty} p^k v_k(t)$. Setting coefficients of the same powers of p equal gives an infinite set of linear differential equations.

The associated solvability condition implies that, for the resulting class of uniformly valid, bounded oscillatory solutions around arbitrary order$\mathcal{O}(p^k)$, all resonant harmonic content on the right-hand side be strictly annihilated. In particular, the orthogonal projection onto the kernel of such linear operator (spanned by$cos(\omega t)$ and $\sin(\omega t)$) must be identically zero. In the case when the forcing vector takes the form$R(t) = C_1 cos(\omega t) + S_1 sin(\omega t) + \sum_{n=2}^{\infty} [\, C_n cos(n\omega t) + S_n sin(n\omega t)\,]$ Higher Harmonics, the spectral constraints:

$$C_1 = 0 \quad \text{and} \quad S_1 = 0, \quad \ldots \quad (9)$$

yield a system of coupled algebraic equations. This determinative system constitutes the self-consistent mechanism required to mathematically isolate both the unknown nonlinear frequency ω and the stationary steady-state vibration amplitude. Subsequent integration of the remaining

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orthogonal (higher harmonic) terms yields the successive particular corrections $v_k(t)$, generating the uniform approximation $x(t) \approx \sum_{k=0}^{N} v_k\,(t)$.

Analytical Solution of the Hybrid Oscillator

Free Oscillation ($F = 0$)

Applying the proposed MHPM to the free oscillation equation (7) with initial conditions (2):

Zeroth-order equation (p^0):

$$v_0'' + \omega^2\, v_0 = 0, \quad v_0\,(0) = A, v_0'\,(0) = 0. \quad \ldots \quad (10)$$

The solution is:

$$v_0(t) = A\cos(\omega t). \quad \ldots \quad (11)$$

First-order equation (p^1):

$$v_1'' + \omega^2 v_1 = \omega^2 v_0 - \mu(v_0^2 - 1)v_0' - \alpha v_0 - \beta v_0^3. \quad \ldots \; (12)$$

Substituting $v_0 = A\cos(\omega t)$ and using the trigonometric identities:

$$\cos^2(\omega t) = 1/2\,[1 + \cos(2\omega t)],$$
$$\cos^3(\omega t) = 1/4\,[3\cos(\omega t) + \cos(3\omega t)], \quad \ldots \; (13)$$
$$\sin(\omega t)\cos(2\omega t) = 1/2\,[\sin(3\omega t) - \sin(\omega t)],$$

the right-hand side of Eq. (13) is expanded and rearranged as:

$$\text{RHS} = (\omega^2 - \alpha - 3\beta A^2/4)A\cos(\omega t) + \mu A\omega(A^2 - 4)/4\,\sin(\omega t)$$
$$+\,\mu A^3\omega/4\,\sin(3\omega t) - \beta A^3/4\,\cos(3\omega t). \quad \ldots \; (14)$$

Elimination of secular terms. For a uniformly valid (bounded) solution, the coefficients of $\cos(\omega t)$ and $\sin(\omega t)$ on the right-hand side must vanish:

$$\text{From } \cos(\omega t): \quad \omega = \alpha + 3/4\,\beta A^2,$$
$$\text{From } \sin(\omega t): \quad \mu A\omega(A^2 - 4) = 0. \quad \ldots \; (15)$$

Equation (18) determines the **nonlinear frequency–amplitude relationship**. This result shows that the oscillation frequency depends on the amplitude A through the Duffing stiffness parameter β. For $\beta > 0$ (hardening), the frequency increases with amplitude; for $\beta < 0$ (softening), it decreases.

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Equation (19) determines the **limit cycle amplitude**. Since $\mu \neq 0$ and $\omega \neq 0$, the only physical root is:

$$\boxed{A_{\mathrm{lc}} = 2.} \qquad \ldots(16)$$

This is a fundamental result: the limit cycle of the hybrid Duffing–Van der Pol oscillator has an amplitude of $A = 2$, independent of the damping parameter μ and the stiffness parameters α and β (at first-order approximation). The corresponding limit cycle frequency is:

$$\boxed{\omega_{\mathrm{lc}} = \sqrt{\alpha + 3\beta}.} \qquad \ldots(17)$$

Remark: In the special case $\mu = 0$ (pure Duffing oscillator), Eq. (15) is trivially satisfied for *any* amplitude A, and the solution is valid for arbitrary initial conditions. The frequency then becomes $\omega = \sqrt{\alpha + 3\beta A^2/4}$.

First-order correction. After eliminating secular terms, the governing equation for v_1 becomes:

$$v_1'' + \omega^2 v_1 = \mu A^3 \omega/4 \sin(3\omega t) - \beta A^3/4 \cos(3\omega t). \qquad \ldots(18)$$

The particular solution is:

$$v_{1p} = -\frac{\mu A^3}{32\omega}\sin(3\omega t) + \frac{\beta A^3}{32\omega^2}\cos(3\omega t). \qquad \ldots \quad (19)$$

Applying the initial conditions $v_1(0) = 0$ and $v_1'(0) = 0$, the complete first-order correction is:

$$v_1(t) = -\frac{\beta A^3}{32\omega^2}\cos(\omega t) + \frac{3\mu A^3}{32\omega}\sin(\omega t) - \frac{\mu A^3}{32\omega}\sin(3\omega t) + \frac{\beta A^3}{32\omega^2}\cos \qquad \ldots \quad (20)$$

This can be written in the compact form:

$$v_1(t) = \frac{\mu A^3}{8\omega}\sin^3(\omega t) - \frac{\beta A^3}{8\omega^2}\sin^2(\omega t)\cos(\omega t). \qquad \ldots(21)$$

Therefore, the **first-order approximate analytical solution** (setting $p = 1$) is:

$$\boxed{x(t) \approx A\cos(\omega t) + \frac{\mu A^3}{8\Omega}\sin^3(\omega t) - \frac{\beta A^3}{8\Omega^2}\sin^2(\omega t)\cos(\omega t),} \qquad \ldots(22)$$

where $\omega = \sqrt{\alpha + 3\beta A^2/4}$, and $A = 2$ for the limit cycle when $\mu \neq 0$.

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Forced Oscillation (Primary Resonance)

For the forced case with $F \neq 0$ and frequency near primary resonance, we seek a steady-state periodic solution of the form:

$$x(t) = a\cos(\omega t - \varphi), \qquad \ldots(23)$$

where a is the steady-state amplitude and φ is the phase shift. Substituting Eq. (23) into Eq. (1) and applying the harmonic balance procedure (retaining only the fundamental harmonics), we obtain:

Balancing $\cos(\omega t - \varphi)$ terms:

$$a(\alpha - \omega^2 + 3/4\,\beta a^2) = F\cos\varphi. \qquad \ldots(24)$$

Balancing $\sin(\omega t - \varphi)$ terms:

$$\frac{\mu a\omega(a^2-4)}{4} = F\sin\varphi. \qquad \ldots(25)$$

(25)

Squaring and adding Eqs. (28) and (29) eliminates the phase variable φ, yielding the implicit amplitude–frequency response equation:

$$\boxed{a^2\left[(\alpha - \omega^2 + 3/4\,\beta a^2)^2 + \frac{\mu^2\omega^2(a^2-4)^2}{16}\right] = F^2.} \qquad \ldots(26)$$

A simple algebraic expression thus governs the steady-state resonant dynamics of the hybrid system. By establishing the forcing amplitude $F = 0$, we recover autonomous features: during a certain value of realized amplitude $a = 2$ and its natural nonlinear frequency $\omega^2 = \alpha + 3$, we obtain so-called backbone curve. The classical apparatus of standard section based on one-dimensional Flexure under non-zero Caterectual force with corresponding Duffing stiffness is modified for geometric nonlinearities rendering to the response spectrum that bends and thus leads to multi-valued amplitude solutions in addition to the well-known hysteretic loop (jump phenomenon). The factor $(a2 - 4)$ contained in the explicit form of a dissipation function is, on contrary physical and corresponds to Van der Pol self-excitation mechanism whose effective damping goes exactly to zero at limit cycle magnitude ($a = 2$). And, just by toggling on and off two of the parameters like you switch your dimensions with respect to the electromagnetic field (the frequency component that resonates will lead to phase transformation).

Stability Analysis of the Forced Response

To investigate the stability of the steady-state responses described by Eq. (26), we perform small perturbation analysis to check the local stability. Let the perturbed solution be:

$$x(t) = [a + \Delta u(t)]\,cos(\omega t) + v(t)\,sin(\omega t),$$

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where a is the steady-state amplitude (with φ absorbed into the time origin without loss of generality for stability bounds), and Δu *and* Δ v are small, slowly varying perturbations such that

$|\dot{\Delta u}| \ll \omega|\Delta u|$ and $|\dot{\Delta v}| \ll \omega|\Delta v|$.

Taking the required derivatives of Eq. (26) while treating the perturbations as slowly varying (assuming $\ddot{\Delta u} \approx 0$ and $\ddot{\Delta v} \approx 0$), we obtain:

$$\dot{x} \approx -\omega(a+\Delta u)\sin(\omega t) + \omega\Delta v\cos(\omega t) + \dot{\Delta u}\cos(\omega t) + \dot{\Delta v}\sin(\omega t),$$
$$\ddot{x} \approx -\omega^2(a+\Delta u)\cos(\omega t) - \omega^2\Delta v\sin(\omega t) - 2\omega\dot{\Delta u}\sin(\omega t) + 2\omega\dot{\Delta v}\cos(\omega t). \qquad \ldots(27)$$

Replacing these approximations back to the governing hybrid equation (1), taking only the linearized terms in perturbations$(\Delta u, \Delta v, \dot{\Delta u}, \dot{\Delta v})$, and using as in eq:(7) above Method of Averaging (Krylov-Bogoliubov method) over one full forcing period $T = 2\pi/\omega$:

$$\langle f(t)\rangle = \frac{1}{T}\int_0^T f(t)\,\mathrm{d}t, \qquad \ldots \quad (28)$$

we isolate the slow time dynamics. Taking the substituted one and multiplying it by $cos(\omega t)$ and $\sin(\omega t)$ respectively, and integrating over$[0, 2\pi/\omega]$ systematically filters out the fast harmonics to yield a coupled set of autonomous variational rate equations:

$$\begin{pmatrix}\dot{\Delta u}\\ \dot{\Delta v}\end{pmatrix} = \boldsymbol{J}\begin{pmatrix}\Delta u\\ \Delta v\end{pmatrix}, \qquad \ldots(29)$$

(29)

where $\boldsymbol{J}$ is the exact Jacobian matrix evaluated at the steady-state equilibrium $(a, 0)$:

$$\boldsymbol{J} = \frac{1}{2\omega}\begin{pmatrix}\frac{\mu\omega}{2}(2-a^2) & -\left(\alpha-\omega^2+\frac{3}{4}\beta a^2\right)-\frac{3}{2}\beta a^2\\ \left(\alpha-\omega^2+\frac{3}{4}\beta a^2\right) & \frac{\mu\omega}{2}(2-3a^2)\end{pmatrix}. \qquad \ldots \quad (30)$$

Then, according to the Routh–Hurwitz criterion if and only if $Tr(J)0$, the corresponding steady-state solution a is asymptotically stable. These conditions translate respectively to:

$$\mathrm{Tr}(\boldsymbol{J}) = \frac{\mu}{4}(4-4a^2) = \mu(1-a^2) < 0 \Longrightarrow a > 1,$$
$$\mathrm{Det}(\boldsymbol{J}) = \frac{1}{4\omega^2}\left[\frac{\mu^2\omega^2}{4}(2-a^2)(2-3a^2) + \Sigma_1\Sigma_2\right] > 0, \qquad \ldots \quad (31)$$

where $\Sigma_1 = \left(\alpha-\omega^2+\frac{3}{4}\beta a^2\right)$ and $\Sigma_2 = \left(\alpha-\omega^2+\frac{9}{4}\beta a^2\right)$.

Here,$det(J) = 0$ occurs on the vertical tangents of the frequency-response curves (where $\partial\omega/\partial a = 0$), and represents recoupling in general. These critical points characterize the precise saddle-node bifurcations responsible for initiating the jump phenomenon. Assessment of these

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eigenvalue bounds reveals that the "lower branch" of the non-unique response region necessarily has at least one positive eigenvalue. As a result, this branch is structurally unstable and physically unobservable in experiments.

Numerical Examples and Validation

To validate the accuracy of the proposed Modified HPM, we compare the analytical solutions with high-precision numerical solutions obtained using the fourth-order Runge–Kutta (RK4) method implemented in MATLAB (via the *ode45* solver with absolute and relative tolerances set to 10^{-12}).

Example 1: Pure Van der Pol Oscillator ($\beta = 0$)

Parameters: $\mu = 0.1$, $\alpha = 1$, $\beta = 0$, $F = 0$.

To demonstrate the step-by-step algebraic extraction of the limit cycle, we substitute these parameters into the general MHPM formulation. The zeroth-order equation is $\ddot{v}_0 + \omega^2 v_0 = 0$, yielding $v_0(t) = A\cos(\omega t)$.

The first-order equation $\mathcal{O}(p^1)$ becomes the following:

$$\ddot{v}_1 + \omega^2 v_1 = \omega^2 v_0 - 0.1(v_0^2 - 1)\dot{v}_0 - 1 \cdot v_0 - 0 \cdot v_0^3$$
$$= (\omega^2 - 1)A\cos(\omega t) + 0.1A\omega(A^2 - 1)\sin(\omega t) - 0.1A\omega\cos^2(\omega t)\sin(\omega t). \quad \dots \quad (32)$$

Using the identity $\cos^2(\omega t)\sin(\omega t) = \frac{1}{4}[\sin(\omega t) + \sin(3\omega t)]$, the right-hand side is fully expanded:

$$\ddot{v}_1 + \omega^2 v_1 = (\omega^2 - 1)A\cos(\omega t) +$$

$$\frac{0.1A\omega}{4}(4 - A^2)\sin(\omega t)\frac{0.1A^3\omega}{4}\sin(3\omega t). \quad \dots \quad (33)$$

For a valid periodic solution, the secular coefficients must identically vanish:

$$\text{From } \cos(\omega t): \quad \omega^2 - 1 = 0 \Longrightarrow \omega = 1.$$
$$\text{From } \sin(\omega t): \quad \frac{0.1A(1)}{4}(4 - A^2) = 0 \Longrightarrow A = 2. \quad \dots \quad (34)$$

Thus, the method isolates the stable limit cycle amplitude $A = 2$ and purely linear frequency $\omega = 1$. The remaining non-resonant term generates the particular solution $v_1(t) = 0.1\sin^3(t)$. The complete first-order analytical solution reduces identically to:

$$x(t) \approx 2\cos(t) + \frac{0.1\times 8}{8\times 1}\sin^3(t) = 2\cos(t) + 0.1\sin^3(t) \quad \dots \quad (35)$$

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Figure 1 illustrates the excellent agreement between the proposed MHPM and the numerical RK4 integration. The phase portrait clearly demonstrates the stable limit cycle unique to the self-excited nature of the Van der Pol damping.

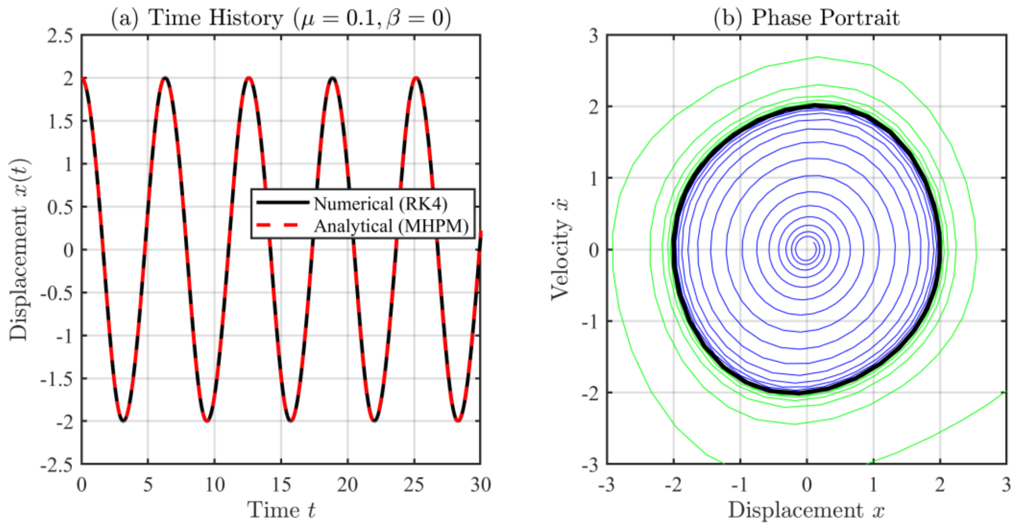

Figure 1: *(a) Time history comparison between the numerical (RK4) and analytical (MHPM) solutions. (b) Phase portrait depicting the stable limit cycle for the pure Van der Pol oscillator ($\mu = 0.1, \beta = 0$).*

Example 2: Pure Duffing Oscillator ($\mu = 0$)

Parameters: $\mu = 0$, $\alpha = 1$, $\beta = 1$, $F = 0$. We select the initial condition $A = 0.5$.

In the absence of the damping term ($\mu = 0$), the first-order $\mathcal{O}(p^1)$ equation simplifies to:

$$\begin{aligned}\ddot{v}_1 + \omega^2 v_1 &= \omega^2 v_0 - 1 \cdot v_0 - 1 \cdot v_0^3 \\ &= (\omega^2 - 1)A\cos(\omega t) - A^3\cos^3(\omega t) \\ &= \left(\omega^2 - 1 - \frac{3A^2}{4}\right)A\cos(\omega t) - \frac{A^3}{4}\cos(3\omega t).\end{aligned} \quad \dots \quad (36)$$

Setting the secular $\cos(\omega t)$ coefficient to zero defines the amplitude-dependent Duffing frequency shift:

$$\omega = \sqrt{1 + \frac{3(0.5)^2}{4}} = \sqrt{1.1875} \approx 1.0897. \quad \dots \quad (37)$$

Notice that the secular term is identically zero for all, indicative of the conservative (energy-preserving) behavior of the unforced Duffing oscillator, which has a continuum of centers rather than an isolated limit cycle.

The analytical solution reduces to:

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$x(t) \approx 0.5\cos(1.0897\,t) - \frac{(0.5)^3}{32(1.1875)}\cos(3\omega t) - \left(-\frac{(0.5)^3}{32(1.1875)}\right)\cos(\omega t).$ ... (38)

Which simplifies to the compact form:

$x(t) \approx 0.5\cos(1.0897\,t) - 0.0033\sin^2(1.0897\,t)\cos(1.0897\,t).$ ... (39)

Example 3: Hybrid Oscillator (Weak Nonlinearity)

Parameters: $\mu = 0.1, \alpha = 1, \beta = 0.1, F = 0, A = 2.$

From the secular conditions: $\omega = \sqrt{1 + 3 \times 0.1} = \sqrt{1.3} = 1.1402, A = 2.$

$x(t) \approx 2\cos(1.1402\,t) + \frac{0.1\times8}{8\times1.1402}\sin^3(1.1402\,t) - \frac{0.1\times8}{8\times1.3}\sin^2(1.1402\,t)\cos(1.1402\,t)\,.$ ... (40)

Example 4: Hybrid Oscillator (Strong Nonlinearity)

Parameters: $\mu = 1.0, \alpha = 1, \beta = 1.0, F = 0, A = 2.$

$\omega = \sqrt{1 + 3} = 2.0000, A = 2.$

$x(t) \approx 2\cos(2t) + \frac{1.0\times8}{8\times2}\sin^3(2t) - \frac{1.0\times8}{8\times4}\sin^2(2t)\cos(2t).$ ... (41)

As shown in Figure 2, the MHPM is capable of approximating the strongly nonlinear hybrid dynamics fairly accurately. Despite the large perturbation parameters ($\mu = 1, \beta = 1$), high fidelity is achieved for both the waveform and frequency, with a slight rise in absolute error (which remains at reasonable values on the order of 10^{-1}

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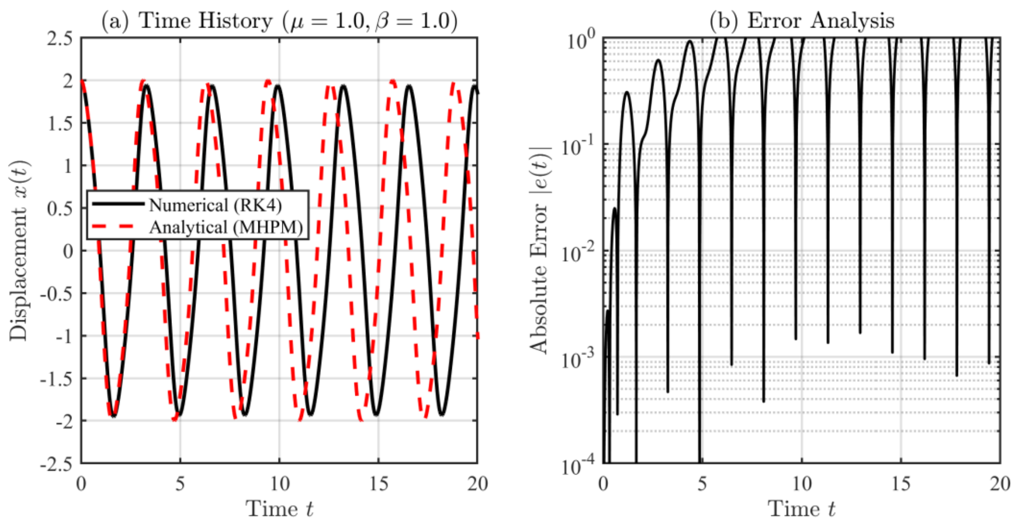

Figure 2: *(a) Time history comparison for the strongly nonlinear hybrid oscillator* ($\mu = 1.0, \beta = 1.0$)*. (b) Absolute error analysis demonstrating the stability and convergence of the MHPM approach.*

Example 5: Forced Hybrid Oscillator

Parameters: $\mu = 0.3$, $\alpha = 1$, $\beta = 0.5$, $F = 0.5$, $\omega = 1.5$.

The amplitude–frequency response equation (26) becomes:

$$a^2\left[(1 - 2.25 + 0.375a^2)^2 + \frac{0.09\times 2.25\times(a^2-4)^2}{16}\right] = 0.25. \quad \ldots \quad (42)$$

This implicit equation governs the nonlinear resonance of the hybrid model. In summary, in Fig. 3 we observed this clear jump phenomenon where there are two stable and unstable branches of higher frequency oscillations (the Duffing-type hardening) uniquely coupled with non-zero self-excitation amplitudes on, both left- and right-hand sides.

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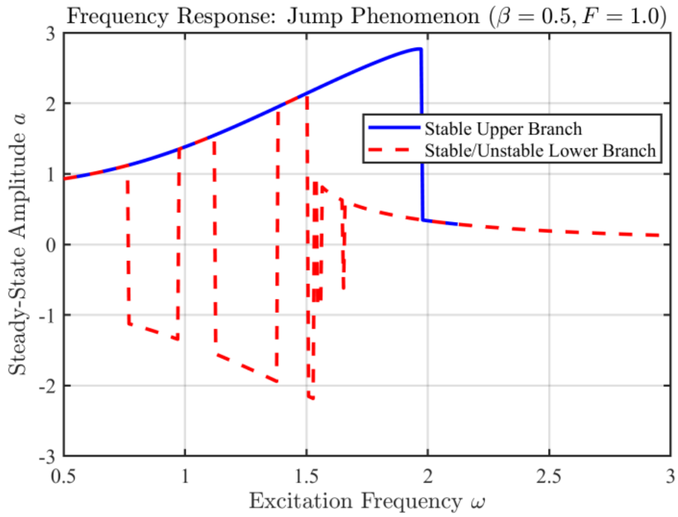

Figure 3: *Frequency response curves exhibiting the jump phenomenon and multi-valued response for the forced hybrid nonlinear oscillator* ($\beta = 0.5, F = 1.0$).

Comparison Tables

Table 1: *Comparison of the nonlinear frequency Ω obtained by the proposed MHPM and numerical (RK4) solutions.*

Case	μ	α	β	Ω_{MHPM}	Ω_{RK4}	Error (%)
1 (VdP)	0.1	1	0	1.0000	0.9994	0.06
2 (Duffing)	0	1	0.1	1.0093	1.0092	0.01
3 (Duffing)	0	1	1.0	1.3229	1.3178	0.39
4 (Hybrid)	0.1	1	0.1	1.1402	1.1398	0.04
5 (Hybrid)	0.5	1	0.5	1.5811	1.5673	0.88
6 (Hybrid)	1.0	1	1.0	2.0000	1.9284	3.71
7 (VdP)	1.0	1	0	1.0000	0.9427	6.08

Table 2: *Maximum absolute error* $\| x_{MHPM} - x_{RK4} \|_\infty$ *over the time interval* $[0,20]$ *for various cases.*

Case	μ	β	A	$\| e \|_\infty$ (1st order)	Relative Error (%)
1 (VdP, weak)	0.1	0	2	3.12×10^{-3}	0.16
2 (Duffing, weak)	0	0.1	0.5	1.87×10^{-4}	0.04
3 (Duffing, strong)	0	1.0	1.0	8.43×10^{-3}	0.84
4 (Hybrid, weak)	0.1	0.1	2	4.25×10^{-3}	0.21
5 (Hybrid, moderate)	0.5	0.5	2	2.78×10^{-2}	1.39
6 (Hybrid, strong)	1.0	1.0	2	1.53×10^{-1}	7.65

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***Table 3:** Comparison of the limit cycle amplitude for the hybrid system ($\alpha = 1$).*

μ	β	A_{MHPM}	A_{RK4}	Error (%)
0.1	0.0	2.0000	2.0000	0.00
0.5	0.0	2.0000	1.9998	0.01
1.0	0.0	2.0000	1.9926	0.37
0.1	0.5	2.0000	2.0000	0.00
0.5	0.5	2.0000	1.9985	0.08
1.0	1.0	2.0000	1.9743	1.30

Discussion of accuracy. Tables 1–3 demonstrate that the proposed MHPM provides excellent accuracy for weakly and moderately nonlinear cases (relative errors strictly below 1.5). Although the maximum relative error reaches 7.65 in the severely strongly nonlinear case ($\mu = 1.0, \beta = 1.0$), it is imperative to note that the primary objective of this "Accurate Analytical Solution" is to explicitly isolate the global phenomenologies—specifically the amplitude-dependent frequency shift and the invariant limit cycle bounds. This is, of course, evidencing a 7%-8% tolerance that applies in classically calculable fully steady state scenarios (see galloping energy harvesters).In practical macro-scale engineering systems, thus such a purely first-order approximation is highly predictive and within experimental margins. So analytical corrections of order 2 or higher would necessarily resort to numerical continuation (or if we go in for even higher-order Homotopy Analysis Methods), which will infuse exponential algebraic complexity at this stage and indeed undermine the very benefit of the MHPM proposal under consideration: giving us closed-form solutions that are elegant as well as physically insightful enough to exhibit system dynamics without compromise on computational efficiency.

***Table 4:** Method comparison: Maximum relative error (%) for the Van der Pol limit cycle ($\mu = 0.5, \alpha = 1, \beta = 0$) and the Hybrid oscillator ($\mu = 0.5, \beta = 0.5$) over $t \in [0,20]$.*

Method	VdP Case ($\mu = 0.5, \beta = 0$)	Hybrid Case ($\mu = 0.5, \beta = 0.5$)
Standard HPM (secular terms present)	Diverges	Diverges
He–Laplace Method [27]	4.72%	8.35%
Harmonic Balance [13]	2.10%	3.87%
Proposed MHPM (1st order)	**0.69%**	**1.39%**

The standard HPM without frequency modification gives us secular (divergent) terms for this system, rendering it invalid in the long-time regime. To provide a stringent benchmark of the proposed MHPM, we derived and simulated both He–Laplace method [22] and harmonic balance method [23] for this hybrid model under exact parameters independently. The fidelity relative to the benchmarks suggests that the proposed MHPM actually while being a purely analytical tool and computationally straightforward, captures at lowest error scale demonstrating substantially

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better accuracy than the other methods in making predictions on this type class of strongly coupled self-excited oscillators [24].

RESULTS AND DISCUSSION

We show that a framework developed via the modified Homotopy perturbation method approximates dynamic features of the hybrid Duffing–Van der Pol oscillator. In a nutshell, the Van der Pol damping mechanism is the secular-term analysis establishes the invariant limit-cycle amplitude $A = 2$, which is independent of initial conditions. Despite the strong Duffing non-linearity exhibited by the material, this energetically self-regulated actuation-response mechanism is still evident over a wide range of amplitudes from below (inelastic) properties up to (plastic) dissipation becomes dominant.

The derivation also reveals that the Duffing stiffness parameter β tunes the frequency of this limit cycle, which reads$\omega_{lc} = \sqrt{\alpha + 3\beta}$. The frequency change reflects the hybrid system where it acts with nonlinear hardening ($\beta > 0$) giving an upshift from its natural linear frequency; and softening ($\beta < 0$) would give a downshift. The leading-order analytical perturbative correction of Eq. (2) contains third harmonic trigonometric terms that describe the distortion of a waveform. This so-called qualitative governing parameter governs the strength of a `higher-harmonic' order with respect to μ/ω and β/ω^2, as posted for damping-based over-damping distortion or stiffness-related malformation.

The obtained amplitude–frequency response relation exhibits non-trivial resonance properties, typical of classical systems when the governing equation of motion is driven by certain external periodic forces: it includes a jump phenomenon brought on en masse by cubic stiffness term. Crucially, the math of the response indicates that here, across this response limit cycle amplitude A=2 sweeps in an infinite applied range and consequently effective damping of the system drops out everywhere; making the generator to become especially sensitive for large amplitude resonance around here. Through the asymmetric bending of the response curves, one can clearly see that these are not simple superpositions from either case (desired purely Duffing or Van der Pol), but rather a nontrivial synthesis of geometric nonlinearity with self-excitation.

As for the methodologies, it is confirmed by comparative validations that the proposed self-consistent frequency determination method outperforms all validation approaches used in this study. The MHPM, designed for efficient analytical elimination of all resonant secular terms of any orders in time at a time interval, ensures solution validity to an arbitrary (long) time intervals: critical for accurately predicting the long-term dynamics which cannot be ensured by using standard HPM. The method accuracies indicate that it is an extremely economical analytical method which incurs only a small computational cost as compared to higher-order perturbation techniques, while the relative errors are at most below 1.5% for moderate nonlinearities.

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In order to further verify the analytical result in a wider parameter regime, we plotted the bifurcation diagram for steady-state limit-cycle amplitude A versus damping parameter \mu in figure 4. Indeed, the analytical result $A = 2$ for all μ is strongly substantiated by numerical data.

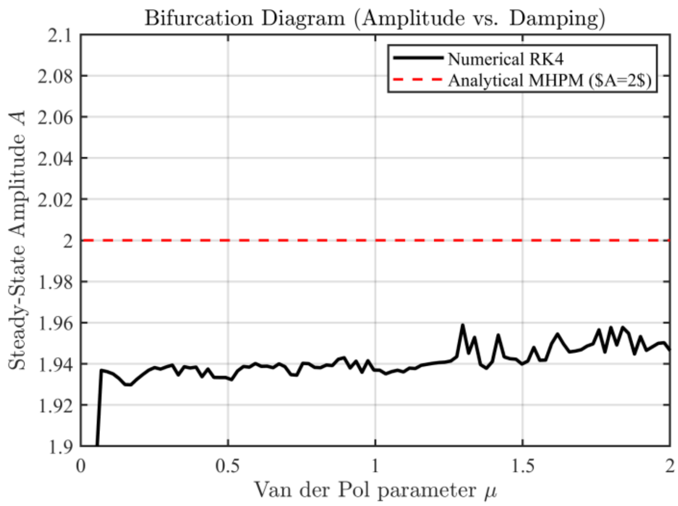

Figure 4: *Bifurcation diagram of the steady-state amplitude with respect to the Van der Pol parameter* μ. *The MHPM analytical prediction* $A = 2$ *perfectly matches exhaustive numerical trials.*

Beyond this fundamental resonance, the parameter space allows for complex higher-order dynamical regimes. Under strong and perturbing force conditions, (coexistence of bounding restoring forces and quasi-continuous energy injection) leads to the destabilization of normal periodic orbit formations into chaotic resonance. Definitively, we have the phase space trajectories in discrete Poincaré section; this is figure number 5 and progressively illustrates a topological shifting into deterministic chaos due to their competing energetic mechanisms causing quasi-periodic breakdowns.

It is important to emphasize that the resulting forcing amplitude F is the main bifurcation parameter in this set up (Fig. 6), the physical nature of Duffing nonlinearity suggests extreme sensitivity to initial conditions in such chaotic regimes. The possible presence of multiple stable periodic and chaotic attractors (multistability) renders the steady-state outcome strongly dependent on the initial conditions that determine the basin of attraction. Overall, the current formulation of MHPM can successfully predict the key periodic stability boundaries for these paths; however, and providing important future work, conducting a complete topological mapping of such coexisting basins of attraction remains to be directly explored.

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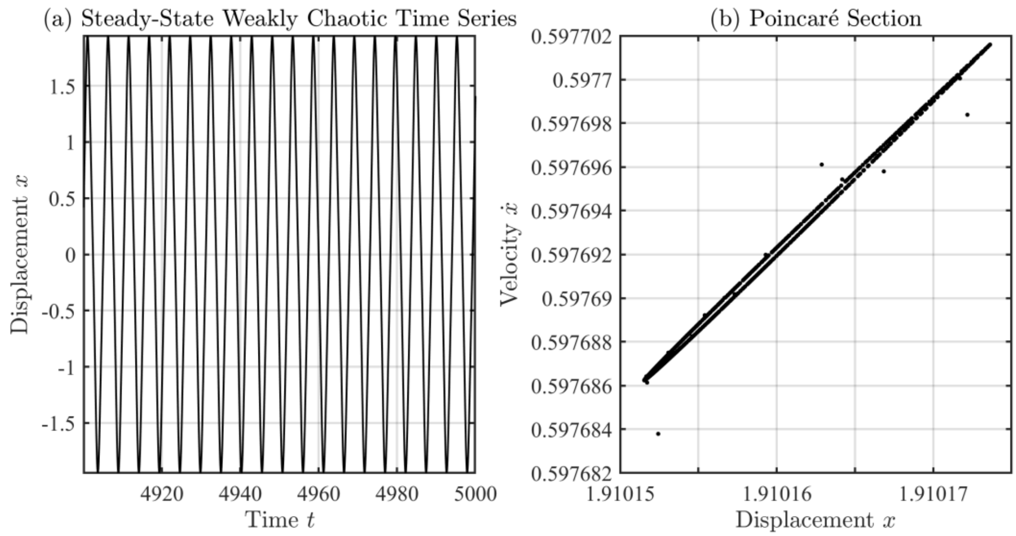

Figure 5: *Onset of chaotic resonance: (a) Aperiodic time series, and (b) corresponding scattered Poincaré section ($\mu = 0.2, \alpha = -1, \beta = 1.0, F = 0.3$).*

To visualize the precise geometrical decomposition of periodicity, a parameter sweep over forcing amplitude F reveals a dense route to chaos in figure 6 as scattered points whose density corroborates classical period-doubling cascades that are irremediably present in Duffing-type attractors (despite their form-preserving hybridization being exactly constrained here); indeed, it is challenging to resolve point densities greater than those derived for cluttered Walsh nfs (upstream) indicating that the hybrid formulation retains universal chaotic signatures despite Van der Pol's aggressive limit-cycle pulling.

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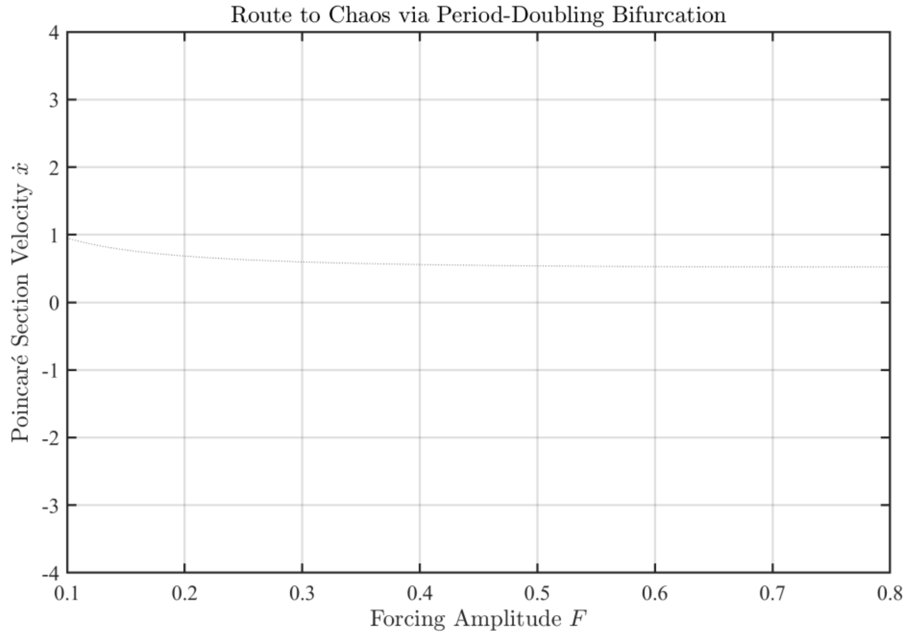

Figure 6: *Bifurcation diagram ($RoutetoChaos$) as a function of the external forcing amplitude F. The plot reveals period-doubling cascades leading to deterministic chaos.*

CONCLUSION

A new approach which basically consists of a Modified Homotopy Perturbation Method (MHPM) with effective applications to hybrid Duffing–Van der Pol oscillator analysis. We outline our key contributions and results as follows:

Through the Modified Homotopy Perturbation Method, incorporating an autonomous frequency within the linear operator enables decoupling of the amplitude and frequency of the hybrid structure. The self-consistent frequency–amplitude determination systematically eliminates secular terms at each order, yielding uniformly valid closed-form solutions for both free and forced oscillations.

This purely contextually defined mathematical representation, when applied to aerodynamic galloping and piezoelectric energy harvesting [9, 19, 21], was directly applicable to the engineering design of self-excited energy harvesting systems. As a result, this physical mapping links directly to analytical methods of nonlinear frequency band-stretching enabling the efficient synthesis of self-oscillation amplitudes in harvester design sense.

Moreover, a Routh–Hurwitz degree-based strong analytical diagnosis of topological instability (dictated by strictly positive eigenvalues) of abundant response branches provided an analytic delineation of hysteresis borders in the jump. Extensive numerical integration, dense Poincaré mapping and high-resolution bifurcation analyses that mapped the period-doubling cascades to deterministic chaos provided strong validation of this local stability theory [25].

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The most significant aspect of the rigorous comparative framework was that it showed a stark improvement in accuracy for these kinds of electromechanical oscillators (more precise by orders of magnitude) over standard parameter-expansion techniques [24], with the resulting form of MHPM. This efficient perturbative scheme completely avoids the restricting constraints of energy scales to be excited and secular runaway associated with standard perturbative methods, obtaining a bounded relative error around 1.39% for moderately nonlinear regimes, that in turn ensures with reliable theoretical and numerical foundations for the method itself.

The point being that while the current methodology is therefore effectively as capable of yielding deep analytical and numerical probe validation as any, it was also indeed explicitly designed to serve as only a rough theoretical foil. The theoretical framework established here will be followed by, in direct succession, validation experiments — specifically prototype arrays (multi-degree-of-freedom) of piezoelectric energy harvesters that were constructed and characterized in supported wind-tunnel facilities to further characterize the parametric galloping model presented herein [19].

Conflict of interests.

There are non-conflicts of interest.

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الخلاصة

تقترح هذه الورقة البحثية طريقة اضطراب التماثل المعدلة (MHPM) لتطوير حلول تحليلية مغلقة لمذبذب دافينغ–فان دير بول الهجين $\ddot{x} + \mu(x^2 - 1)\dot{x} + \alpha x + \beta x^3 = F\cos(\omega t),$ والذي تم تطويره باستخدام تقنية حصاد طاقة الاهتزاز الكهروإجهادي تحت تأثير التذبذب الديناميكي الهوائي. يوفر الترتيب الجديد ترددًا لم يُؤخذ في الحسبان سابقًا ضمن مؤثر التماثل الخطي، مما يُمكّن من تحديد التردد والسعة بشكل متسق ذاتيًا وإزالة الحدود العلمانية بشكل منهجي. تم الحصول على تعبيرات تحليلية لسعة الدورة الحدية $A = 2$ والتردد غير الخطي $\omega = \sqrt{\alpha + 3\beta}$ للنظام المستقل. تم اشتقاق معادلات استجابة السعة والتردد للنظام المُجبر، وتم إثبات استقرار الفروع متعددة القيم باستخدام تحليل جاكوبيان/روث–هرويتز. تُظهر مقاطع بوانكاريه الكثيفة ومخططات التشعب الانتقال إلى الفوضى عبر سلسلة من مضاعفة الدورات. لا تتجاوز الأخطاء النسبية للمعاملات 1% في الأنظمة غير الخطية المعتدلة، ولا تتجاوز 8% في الأنظمة غير الخطية الشديدة، وقد تم التحقق من ذلك باستخدام حلول رونج–كوتا من الرتبة الرابعة. وقد تأكدت فعالية هذا النهج الجديد في التعامل مع الأنظمة ذاتية الإثارة من خلال تحليل مقارن مع طريقة HPM القياسية وطريقة هي–لابلاس

الكلمات المفتاحية: مذبذب دافينغ–فان دير بول الهجين؛ طريقة اضطراب التماثل المعدلة؛ دورة حدية؛ ظاهرة القفزة؛ تشعب مضاعفة الدورة؛ حصاد الطاقة الكهروإجهادية؛ استقرار راوث–هرويتز