Transition Support Matrices and Local–Global Dynamics in Non-Abelian Group Cellular Automata

Sajjad Hayder Naji
College of Science, Hakim Sabzevari University, sajaadhaider1999@gmail.com, Iran, Sabzwar.

مصفوفات دعم الانتقال والديناميكيات المحلية–العالمية في الأوتوماتا الخلوية الزمرية على الزمر غير الإبدالية

سجاد حيدر ناجي
كلية العلوم، جامعة حكيم سبزواري، sajaadhaider1999@gmail.com , ايران، سبزوار.

Accepted:	25/5/2026		Published:	30/6/2026

ABSTRACT

In this paper, we study matrix-type structures associated with group cellular automata over finite non-abelian groups. We introduce a transition support matrix that encodes locally admissible transitions and provides a finite combinatorial description of local transition behavior. We show that the irreducibility of this matrix reflects local propagation properties and, under an additional realizability assumption, can imply topological transitivity. We also construct an explicit example over the symmetric group S3, showing that matrix irreducibility alone is not sufficient to guarantee transitivity in the non-abelian setting. These results illustrate the distinction between local transition structure and global dynamical behavior in non-abelian group cellular automata, and indicate some limitations of purely matrix-based approaches beyond the abelian case.

Keywords: Group cellular automata, Non-abelian groups, Transition support matrix, Topological transitivity, Orbit realizability

1. INTRODUCTION

Cellular automata (CA) provide a rich class of discrete dynamical systems that have been extensively studied from both algebraic and topological perspectives. They arise naturally as shift-commuting maps on symbolic spaces and have found applications in areas ranging from theoretical computer science to statistical physics. A classical approach to their study relies on symbolic dynamics, in which CA are interpreted as sliding-block codes acting on subshifts of finite type. In this framework, many fundamental dynamical properties, such as transitivity and mixing, can be characterized through combinatorial and algebraic structures. In particular, for shifts of finite type, irreducibility of the associated adjacency matrix is known to be equivalent to topological transitivity [1- 8]. This connection highlights the central role of matrix-based methods in understanding dynamical behavior in symbolic systems. A more algebraic viewpoint emerges in the study of additive and linear cellular automata. In these settings, the configuration space carries a natural vector space structure, and the global evolution can be described using matrices or linear operators. As a consequence, several dynamical properties, including injectivity and transitivity, can be analyzed and, in some cases, decided via algebraic and matrix-theoretic methods [9], [10]. These results demonstrate that, in abelian or linear contexts, matrix methods provide powerful algebraic tools for studying the dynamics of cellular automata. In contrast, cellular automata defined over finite non-abelian groups exhibit a markedly different behavior. Although significant progress has been made in understanding their structure and dynamics, including results on expansivity, transitivity, and decomposition into simpler components [11], [12]. The available techniques are predominantly group-theoretic rather than matrix-based. In particular, while algebraic decompositions and structural properties of the underlying groups play an important role, there is no direct analogue of the linear matrix framework that is available in the abelian setting. This disparity suggests a structural limitation: the lack of a compatible linear structure in non-abelian groups prevents the direct application of classical matrix techniques. Consequently, there is currently no unified matrix-based framework that captures key dynamical properties, such as transitivity, for group cellular automata over finite non-abelian groups. This paper aims to address this gap. We propose a matrix-type framework that is associated with a given group cellular automaton, a suitable algebraic structure capturing its transition behavior. Rather than relying on linearity, our approach extracts matrix-like information from the underlying group dynamics, allowing us to study transitivity through matrix-type structural conditions analogous to those appearing in the classical abelian setting. Our results provide a first step toward bridging matrix-based methods and the dynamics of non-abelian group cellular automata. They suggest that, despite the absence of a genuine linear structure, it is still possible to recover structurally meaningful matrix-type conditions that reflect the dynamical complexity of the system. The main contribution of this paper is not to provide a complete matrix characterization of transitivity, but rather to introduce a local matrix-type invariant together with an analysis of its limitations in the non-abelian setting. More precisely, we introduce the transition support matrix as a finite combinatorial object encoding locally admissible transitions. We show that its irreducibility reflects local propagation phenomena and can imply transitivity under additional realizability assumptions. On the other hand, we construct an explicit non-abelian example showing that irreducibility alone is insufficient to characterize global transitivity. This highlights the distinction between local transition connectivity and global dynamical realizability in non-abelian group cellular automata.

2 PRELIMINARIES

2.1 Discrete Dynamical Systems

Let X be a compact metric space and $f\colon X \to X$ a continuous map. The pair (X, f) is called a (discrete) dynamical system.

For $x \in X$, the *orbit* of x is defined by

$$O(x) = \{fn(x) : n \geq 0\}.$$

The system (X, f) is said to be *topologically transitive* if for any two nonempty open sets U, $V \subset X$, there exists $n \geq 0$ such that [1], [5]

$$fn(U) \cap V \neq \emptyset.$$

The map f is called *sensitive to initial conditions* if there exists $\delta > 0$ such that for every $x \in X$ and every neighborhood U of x, there exists $y \in U$ and $n \geq 0$ satisfying.

$$d\big(fn(x), fn(y)\big) > \delta.$$

2.2 Symbolic Dynamics and Shift Spaces

Let A be a finite alphabet, and consider the full shift space A^Z endowed with the product topology. The shift map $\sigma : A^Z \to A^Z$ is defined by

$$(\sigma(x))i = xi + 1, \forall i \in Z.$$

A *subshift* $X \subseteq A^Z$ is a closed, shift-invariant subset. A shift of finite type (SFT) is a subshift defined by finitely many forbidden patterns. Shifts of finite type can be represented by finite directed graphs or, equivalently, by adjacency matrices [1], [3], [6].

2.3 Cellular Automata

Let A be a finite alphabet. A map $F: A^Z \to A^Z$ is called a *cellular automaton* if it is continuous and commutes with the shift map σ, that is,

$$F \circ \sigma = \sigma \circ F.$$

By the Curtis–Hedlund–Lyndon theorem, every cellular automaton is defined by a local rule: there exists a finite radius $r \geq 0$ and a function $\varphi: A^{2r+1} \to A$ such that.

$$(F(x))i = \varphi(xi - r, \ldots, xi + r) for\ all\ i \in Z$$

[5], [10].

2.4 Group Cellular Automata

Let G be a finite group. A *group cellular automaton* (GCA) is a cellular automaton

$$F: G^Z \to G^Z$$

whose local rule is compatible with the group structure [5], [10]. In particular, in the linear or additive case (typically when G is abelian), the global map can often be described using algebraic or matrix representations. However, when G is non-abelian, such linear representations are no longer available. In general, the analysis relies primarily on group-theoretic methods.

2.5 Matrix Representations in Symbolic Systems

For shifts of finite type, a directed graph with adjacency matrix A provides a complete description of the allowed transitions. The shift is topologically transitive

if and only if the matrix A is irreducible [3].

This correspondence between matrices and dynamics plays a fundamental role in symbolic dynamics and motivates the search for similar representations in more general settings, including group cellular automata [1].

3 MATRIX CHARACTERIZATIONS IN ABELIAN CELLULAR AUTOMATA

In the study of cellular automata over abelian groups, a powerful algebraic structure emerges that allows the dynamics to be described using matrices and linear operators.

3.1 Linear and Additive Cellular Automata

Let G be a finite abelian group. A cellular automaton

$F\colon G^Z \to G^Z$ is said to be *additive* (or linear) if its local rule is compatible with the group operation, that is, $F(x \cdot y) = F(x) \cdot F(y)$ for all configurations $x, y \in G^Z$. In this setting, the configuration space inherits a module or vector space structure, and the global map F can often be represented by a matrix or a linear operator over a suitable algebraic structure. This representation enables the use of tools from linear algebra and matrix theory in the analysis of dynamical properties [7], [8].

3.2 Matrix-Based Dynamical Criteria

For additive cellular automata, several dynamical properties can be studied through algebraic and matrix-theoretic methods. In particular, properties such as injectivity and transitivity are closely related to algebraic invariants associated with the global rule, including matrix representations and characteristic polynomials. These approaches provide an effective algebraic framework for analyzing the dynamics of additive cellular automata [4], [7], [8].

3.3 Decidability and Computability

The algebraic structure of additive cellular automata allows several dynamical properties to be analyzed through finite algebraic data.

In particular, linear representations of the global rule provide effective methods for studying properties such as injectivity and transitivity [9], [10].

3.4 Limitations of the Abelian Framework

The matrix-based framework relies strongly on the presence of an underlying abelian or linear structure. In particular, the representation of the global rule as a linear operator is generally unavailable when the alphabet is a non-abelian group. Consequently, matrix methods that are effective in the abelian setting do not directly extend to non-abelian group cellular automata.

This motivates the search for matrix-type approaches capable of capturing aspects of the dynamics in the non-abelian setting [12].

4 GROUP CELLULAR AUTOMATA OVER NON-ABELIAN GROUPS

In this section, we consider cellular automata defined over finite non-abelian groups and review some of the known structural and dynamical results in this setting.

4.1 Definition and Basic Structure

Let G be a finite group, not necessarily abelian. A *group cellular automaton* (GCA) is a map

$$F\colon G^Z \to G^Z$$

that is continuous and commutes with the shift map.

As in the classical case, F is defined by a local rule, but in general, this rule is no longer linear, and the configuration space does not carry a natural vector space structure.

As a result, many of the algebraic tools available in the abelian case do not directly apply [13].

4.2 Algebraic Decomposition

One of the key approaches to the study of GCAs over finite groups relies on the internal structure of the group.

In particular, when G admits a decomposition into a direct product of simpler groups,

$$G = G_1 \times G_2 \times \cdots \times G_k,$$

The cellular automaton F can often be analyzed through its induced components

$$F_1, F_2, \ldots, F_k$$

acting on each factor.

Such decompositions allow one to reduce certain problems to simpler or more structured cases, especially when the factors G_i are simple or minimal groups [11].

4.3 Expansivity and Transitivity

Recent work has established strong connections between expansivity and transitivity in group cellular automata.

In particular, in certain classes of group cellular automata, expansivity appears to be strongly linked to topological transitivity, and, in some cases, the two notions coincide [11].

Moreover, expansivity can be interpreted as the propagation of local differences: small perturbations in the initial configuration spread under iteration of the system, leading to global divergence of trajectories.

This propagation-based viewpoint provides a geometric interpretation of dynamical complexity in GCAs.

4.4 Decidability and Structural Results

Another important feature of GCAs over finite groups is the possibility of deciding certain dynamical properties.

In particular, it has been shown that expansivity is a decidable property in this setting [11].

These results highlight that, despite the lack of linear structure, group-theoretic methods still allow for a systematic analysis of dynamical behavior.

4.5 Limitations of Existing Approaches

Although the above results provide significant insight into the dynamics of GCAs over non-abelian groups, they are predominantly based on group-theoretic and combinatorial techniques. In contrast to the abelian case, there is no general representation of the global map in terms of matrices or linear operators, and consequently, no direct matrix-based criteria for dynamical properties such as transitivity. This contrast between the abelian and non-abelian settings motivates the search for alternative frameworks capable of capturing matrix-type information in the absence of linearity.

5 THE MISSING MATRIX FRAMEWORK

The previous sections highlight the contrast between the abelian and non-abelian settings in the study of cellular automata.

In the abelian case, the presence of a linear structure allows the global dynamics to be described through matrices or linear operators. This leads to algebraic approaches for studying properties such as transitivity and injectivity [8].

In contrast, cellular automata over finite non-abelian groups are mainly studied using group-theoretic and combinatorial methods. Although several structural and dynamical properties have been established [11], there is no direct analogue of the matrix-based framework available in the abelian setting.

More precisely, there is currently no general matrix-type representation whose structural properties reflect the global dynamics of non-abelian group cellular automata.

This motivates the search for alternative frameworks capable of capturing aspects of the dynamics without relying on linearity. The central question addressed in this paper is therefore the following:

Can one construct a matrix-type framework that captures aspects of the dynamics of group cellular automata over finite non-abelian groups and provides structural conditions related to transitivity?

Rather than seeking a direct linear representation, we aim to extract matrix-type information from the underlying group dynamics. This includes finite combinatorial or algebraic structures encoding local transition behavior in a manner analogous to adjacency matrices in symbolic dynamics.

The goal is not to replicate the linear theory, but to develop a framework that retains some of its structural advantages while remaining applicable in the non-abelian setting.

In the next section, we introduce such a matrix-type construction and investigate its relation to transitivity in non-abelian group cellular automata.

6 A MATRIX-TYPE FRAMEWORK FOR NON-ABELIAN GROUP CELLULAR AUTOMATA

In this section, we introduce a matrix-type construction associated with a group cellular utomaton over a finite (not necessarily abelian) group. The goal is to extract finite algebraic data that reflects aspects of the transition structure of the system.

6.1 Local Interaction Structure

Let G be a finite group, and let

$$F\colon\ G^{Z} \to G^{Z}$$

be a cellular automaton with a local rule

$$\varphi\colon\ G^{2r+1} \to G.$$

For each position $i \in Z$, the value $(F(x))_i$ depends only on

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

$$(xi - r, \ldots, xi + r).$$

This locality allows us to interpret the evolution of the system in terms of interactions between neighboring states.

This property serves as a local compatibility condition and motivates the stronger realizability assumptions introduced later.

6.2 Transition Support Matrix

We associate with F a finite matrix that encodes the possible transitions induced by the local rule.

Let $S \subseteq G$ be the set of states. We define the *transition support matrix.*

$M = (m_{ab})_{a,b \in G}$

by

$$m_{ab} = \begin{cases} 1 & \text{ifthereexistsalocalconfiguration}(x_{-r}, \ldots, x_r) \in G^{2r+1} \\ & \text{suchthat} x_0 = a \text{and} \varphi(x_{-r}, \ldots, x_r) = b, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $m_{ab} = 1$ indicates that the state a can evolve into b under some local configuration.

6.3 Interpretation

The matrix M can be viewed as the adjacency matrix of a directed graph on G, where an edge $a \longrightarrow b$ exists if $m_{ab} = 1$.

This graph encodes the local transition structure of the cellular automaton and provides a finite combinatorial description of locally admissible transitions [1].

6.4 Irreducibility and Connectivity

We say that the matrix M is *irreducible* if for any $a, b \in G$, there exists $n \geq 1$ such that.

$$(M^n)_{ab} > 0.$$

Equivalently, the associated directed graph is strongly connected. This property reflects the existence of chains of locally admissible transitions connecting arbitrary states [1].

6.5 Local Independence Property

The transition support matrix records locally admissible single-site transitions. To relate this information to global dynamics, an additional compatibility condition is required.

Definition 6.1 (Local independence property)**.** Let F: $G^{Z} \longrightarrow G^{Z}$ be a cellular automaton with radius r. We say that F satisfies the *local independence property* if the following holds:

For every finite family of local neighborhoods

$$I_1, \ldots, I_k \subset Z,$$

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

whose pairwise distances are sufficiently large, and for every choice of locally admissible patterns on these neighborhoods, there exists a configuration

$x \in G^Z$

realizing all these patterns simultaneously.

Equivalently, sufficiently separated local patterns can be combined without creating global compatibility obstructions.

6.6 Classes where local independence holds

We now identify a class of cellular automata for which the local independence property holds under finite-radius separation.

Proposition 6.2 (Local independence under extension property)**.** Let G be a finite group and let

$$F: G^{\mathbb{Z}} \to G^{\mathbb{Z}}$$

be a cellular automaton with radius r.

Assume that F satisfies the following extension property: every finite pattern locally consistent with the defining rule of F extends to a configuration in $G^{\mathbb{Z}}$.

Let

$$I_1, \dots, I_k \subset \mathbb{Z}$$

be finite subsets satisfying

$$dist(I_i, I_j) > 2r, \qquad i \neq j.$$

Then any collection of locally admissible patterns on these sets can be realized simultaneously by a configuration

$$x \in G^{\mathbb{Z}}.$$

Proof. Since F has radius r, the value at each site depends only on coordinates within distance r.

For each I_i, define the dependency neighborhood

$$N(I_i) := \{j \in \mathbb{Z}: dist(j, I_i) \leq r\}.$$

The condition

$$dist(I_i, I_j) > 2r$$

implies that the sets $N(I_i)$ are pairwise disjoint.

Let p_i be a locally admissible pattern on I_i. Extend each p_i to a locally consistent pattern on $N(I_i)$. Since the neighborhoods $N(I_i)$ are disjoint, these patterns are mutually compatible and determine a finite pattern on

$\bigcup_i N(I_i).$

By the extension property, this finite pattern extends to a configuration

$x \in G^{\mathbb{Z}}.$

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

By construction, x realizes each pattern p_i on I_i.

Remark 6.3. The extension property is closely related to classical gluing conditions in symbolic dynamics, such as the specification property for shifts of finite type [1]. In systems satisfying such conditions, sufficiently separated patterns can often be combined without creating global compatibility obstructions.

Remark 6.4. The above proposition does not hold in general for cellular automata over non-abelian groups.

Even when dependency regions are disjoint, global algebraic constraints may prevent the simultaneous realization of locally admissible patterns. This indicates that certain global features of the dynamics are not detected by purely local transition data, as illustrated later in Example 8.6.

6.7 Example: additive cellular automata

The local independence property arises naturally for additive cellular automata over finite abelian groups. In this setting, the underlying linear structure allows sufficiently separated locally admissible patterns to be combined independently [8].

Remark 6.5 This property is not automatic in the non-abelian setting. Its role is to distinguish between:

1. the existence of allowed local transitions, encoded by the matrix M*;*
2. the simultaneous realization of such transitions within a single global configuration.

Thus, the transition support matrix describes local reachability, while local independence controls whether this reachability can be realized globally.

6.8 Heuristic Link to Transitivity

The irreducibility of M suggests the existence of chains of locally admissible transitions connecting arbitrary states. This is analogous to irreducibility conditions in symbolic dynamics, where irreducibility of the adjacency matrix is equivalent to topological transitivity for shifts of finite type.

In the non-abelian setting, however, the matrix M records only local transition data. Relating this local reachability to global dynamics requires an additional mechanism ensuring that sufficiently separated local transitions can be realized within a single configuration. This motivates the search for additional realizability conditions relating local transition chains to global orbit behavior [1].

6.9 Scope and Limitations

It is important to note that the matrix M records only local transition information and does not determine the global dynamics of F.

In particular, the non-abelian structure of G may introduce additional constraints that are not detected at the level of M.

Nevertheless, M provides a matrix-type description of local transition behavior and serves as a basis for studying dynamical properties in the absence of linearity [9].

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

7 MAIN RESULTS

In this section, we establish the first connections between the transition support matrix introduced above and the local dynamics of the cellular automaton.

7.1 Local-to-Global Propagation

We begin by relating powers of the transition support matrix to chains of locally admissible transitions.

Lemma 7.1 (Matrix paths correspond to chains of local transitions)**.** Let G be a finite group and let

$F: G^Z \to G^Z$

be a cellular automaton with radius r.

Let M be the transition support matrix defined by

$M_{ab} = 1$ if and only if there exists a locally admissible pattern

whose central state is a and whose image is b.

such that
for each $k = 0, \ldots, n - 1$,
there exists a locally admissible pattern realizing the transition
$a_k \to a_{k+1}$.

Proof. The condition $(M^n)_{ab} > 0$ means that there exists a sequence
$a = a_0, a_1, \ldots, a_n = b$
such that
$M_{a_k a_{k+1}} = 1$
for all $k=0, \ldots, n-1$
By definition of M, each condition $M_{a_k\, a_{k+1}} = 1$ guarantees the existence of a locally admissible pattern whose central value is a_k and whose image under the local rule is a_{k+1}. Thus, each step $a_k \to a_{k+1}$ is locally realizable, and the sequence defines a chain of locally admissible transitions connecting a to b.

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

Remark 7.2. *This lemma provides the basic link between the algebraic structure of the matrix M* and the local dynamics of the cellular automaton. In particular, it shows that powers of M encode concatenations of locally realizable transitions, although such chains do

not necessarily correspond to a single global orbit.

The local independence property provides a useful compatibility mechanism for combining separated local patterns. However, to obtain global transitivity, a stronger orbit realizability condition is required.

7.2 Irreducibility as a Propagation Condition

The previous lemma motivates the interpretation of irreducibility as a form of local transition connectivity.

7.3 Transitivity under Additional Conditions

We now show that irreducibility, together with additional realizability assumptions, leads to topological transitivity.

Definition 7.3 *(Pattern-compatible orbit realizability). Let* $F: G^Z \to G^Z$

be a group cellular automaton. We say that F satisfies the pattern-compatible orbit realizability property if the following holds. Whenever

$a_0 \to a_1 \to \dots \to a_n$

is a finite chain of locally admissible transitions, and $U, V \subset G^Z$*, are nonempty cylinder sets whose defining patterns contain* a_0 *and* a_n*, respectively, there exists* $x \in U$ *and* $i \in Z$

such that

$(F^k(x))_i = a_k$ *for all* $k=0,\dots,n,$ *and additionally*

$F^n(x) \in V.$

Theorem 7.4 (Transitivity under realizability assumptions). Let G be a finite group and let

$F: G^Z \to G^Z$ be a group cellular automaton with transition support matrix M.

Assume that:

1. M is irreducible;
2. F satisfies the pattern-compatible orbit realizability property.

Then F is topologically transitive.

Proof. Let $U, V \subset G^Z$ be nonempty cylinder sets.

Choose symbols

$a, b \in G$

appearing in the defining patterns of U and V, respectively.

info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com ISSN: 2312-8135 | Print ISSN: 1992-0652

Since M is irreducible, there exists $n \geq 1$

such that

$(M^n)_{ab} > 0$

By Lemma 7.1, there exists a chain of locally admissible transitions

$a = a_0 \rightarrow a_1 \rightarrow \ldots \rightarrow a_n = b.$

By the pattern-compatible orbit realizability property, there exist

$x \in U$ and $i \in Z$

such that

$(F^k(x))_i = a_k$ for all $k=0, \ldots, n$ and $F^n(x) \in V$.

Therefore,

$F^n(U) \cap V \neq \emptyset.$

Hence, F is topologically transitive.

Remark 7.5. The pattern-compatible orbit realizability property should be viewed as a strong realizability assumption relating local admissible transitions to global orbit structure.

7.4 Failure of the Criterion

The irreducibility of the transition support matrix is not sufficient in general to imply topological transitivity. This limitation is illustrated by the non-abelian example presented in Section 8.

7.5 Discussion

The results above show that the transition support matrix provides a meaningful description of local transition behavior. Irreducibility reflects local propagation properties, while additional assumptions are needed to obtain global dynamical conclusions.

This indicates that matrix-type constructions remain useful in the non-abelian setting, although their interpretation differs from the classical abelian case [8].

8 EXAMPLES AND SUPPORTING CASES

In this section, we present examples illustrating the role of matrix-type structures in cellular automata. The abelian examples highlight the classical matrix viewpoint, while the non-abelian examples illustrate the limitations of purely local transition data.

8.1 Example 1: An additive cellular automaton over Z/2Z**Z/2Z**

Consider the additive cellular automaton

$$F: (Z/2Z)^{Z} \rightarrow (Z/2Z)^{Z}$$

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

defined by the local rule

$$(F(x))_i = x_i + x_{i+1} (\text{mod } 2).$$

This is a standard example of an additive cellular automaton over a finite abelian group [8].

The set of states is

$$G = \{0, 1\}.$$

The associated transition support matrix is

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Hence, M is irreducible (indeed, primitive), since every state can transition locally to every other state in one step. This example illustrates how the transition support matrix reflects local transition behavior in the additive setting [7], [8].

8.2 Example 2: Shift spaces of finite type

Shift spaces of finite type. Consider the matrix.

Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

The associated edge shift X_A is the full two-symbol shift. Since A is irreducible, the corresponding shift is topologically transitive, and since A is primitive, it is topologically mixing.

This example illustrates the classical relation between adjacency matrices and dynamical properties in symbolic dynamics. The transition support matrix introduced in this paper is motivated by this viewpoint [1].

8.3 Example 3: Transitivity in symbolic dynamics

Transitivity plays an important role in symbolic dynamics and cellular automata. In particular, for one-dimensional cellular automata, transitivity on a non-trivial Bernoulli subshift of finite type implies the existence of dense periodic points and hence Devaney chaos in that setting.

This example illustrates the dynamical relevance of transitivity in symbolic dynamics and cellular automata [2], [3].

8.4 Example 4: Why the non-abelian case is subtler

The non-abelian setting is considerably more delicate. Recent work shows that for group cellular automata over finite groups, expansivity is decidable, and that expansive GCAs form a proper subclass of topologically transitive injective GCAs [11]. These results indicate that non-abelian GCAs can exhibit rich global dynamics, while different dynamical properties

مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية مجلة جامعة بابل للعلوم الصرفة والتطبيقية

ISSN: 2312-8135 | Print ISSN: 1992-0652

info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

remain structurally distinct. Accordingly, matrix-type constructions in the non-abelian setting should be interpreted as structural tools rather than complete descriptions of the dynamics [12], [14].

8.5 A cautionary perspective

Remark 8.1 (Why irreducibility alone is insufficient). The irreducibility of the transition support matrix reflects local transition connectivity, but in the non-abelian setting, this local information does not by itself determine the global dynamics.

Indeed, the matrix M records only locally admissible single-site transitions, while global configurations may still satisfy additional compatibility constraints that are invisible at the matrix level.

Consequently, the matrix M may be irreducible even when the global dynamics is confined to proper invariant subsets of G^{Z}.

This observation motivates the stronger realizability assumptions introduced in Section 7. The following example illustrates such a global obstruction in the non-abelian setting.

8.6 A genuinely non-abelian counterexample

Let $G = S_3$

and define a cellular automaton

$$F\colon\ G^{Z} \to G^{Z}$$

by the local rule

$(F(x))\, i = x_i x_{i+1} x_i^{-1},\ i \in Z.$

Proposition 8.2. *The transition support matrix M is irreducible.*

Proof. Fix $a, b \in G$. We solve

$$a\, x a^{-1} = b.$$

This gives

$$x = a^{-1} b a,$$

which always exists in G.

Hence, $M_{ab} = 1$ for all $a, b \in G$, and therefore M is irreducible.

Proposition 8.3. *The system F is not topologically transitive.*

Proof. Let $C \subset G$ be a conjugacy class in S_3, for example, the set of transpositions, and define $Y = \{x \in G^{Z} : x_i \in C \text{ for all } i \in Z\}$.

We show that Y is forward invariant under F.

Let $x \in Y$. Since $x_{i+1} \in C$ and conjugation preserves conjugacy classes, we have

$$(F(x))_i = x_i\, x_{i+1}\, x^{-1} \in C$$

, for every $i \in Z$.

Hence,

$F(Y) \subseteq Y$.

The set Y is nonempty, closed, and proper in G^Z. Now define

$$U = \{x \in G^Z : x_0 \in C\}, \qquad V = \{x \in G^Z : x_0 \notin C\}.$$

Then U and V are nonempty open sets, with $U \subseteq Y$ and $V \cap U = \emptyset$. Therefore

$F^n(U) \subset Y$ for all $n \geq 0$,

we obtain

$F^n(U) \cap V = \emptyset$. for all $n \geq 0$

Therefore, F is not topologically transitive.

Remark 8.4. In this example, the transition support matrix is irreducible, although the global dynamics is confined to a proper invariant subset of G^Z. The obstruction arises from conjugacy invariance, which is not detected by the local transition matrix. Therefore, irreducibility of the transition support matrix alone does not characterize topological transitivity in the non-abelian setting.

9 DISCUSSION

The results of this paper highlight the distinction between local transition structure and global dynamical behavior in group cellular automata over finite non-abelian groups. The transition support matrix provides a finite combinatorial description of locally admissible transitions and offers a matrix-type viewpoint on the local dynamics of the system. Our analysis shows that the irreducibility of this matrix reflects local transition connectivity, but does not by itself imply global dynamical properties such as topological transitivity. The non-abelian counterexample presented in Section 8 demonstrates that additional global constraints may remain invisible at the matrix level. Additional realizability assumptions were introduced to relate local transition chains to global dynamical behavior. This highlights the role of structural compatibility conditions in connecting local transition data with global orbit structure. These results suggest that matrix-type constructions remain useful in the non-abelian setting, although their interpretation differs from the classical abelian framework.

10 Conclusion

In this paper, we introduced the transition support matrix associated with group cellular automata over finite non-abelian groups. This construction provides a finite combinatorial description of locally admissible transitions and relates powers of the matrix to chains of local transitions. Our analysis shows that irreducibility of the transition support matrix

ISSN: 2312-8135 | Print ISSN: 1992-0652

info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com

reflects local transition connectivity, but does not by itself characterize global dynamical behavior. In particular, the non-abelian counterexample presented in Section 8 demonstrates that irreducibility alone does not imply topological transitivity. Additional realizability assumptions were introduced to connect local transition connectivity with global orbit structure. These results suggest that matrix-type constructions remain useful in the study of non-abelian group cellular automata, although additional global structure must be taken into account.

Future Directions

Several questions remain open in the non-abelian setting. Possible directions for future work include the study of stronger realizability and compatibility conditions, the construction of additional non-abelian examples, and the development of refined matrix-type structures incorporating global dynamical constraints. It would also be interesting to investigate whether related constructions can be extended to higher-dimensional cellular automata or to broader algebraic classes of group cellular automata.

Conflict of interests.

There are non-conflicts of interest.

References

[1] D. Lind, B. Marcus, An Introduction to Symbolic Dynamics and Coding, Cambridge University Press, 1995.

[2] F. Chen, G. Chen, W. Jin, “Transitivity and Chaoticity in 1-D Cellular Automata,” Applied Mathematics, Vol. 4, No. 2, 2013.

[3] R. Devaney, An Introduction to Chaotic Dynamical Systems, Westview Press, 2003.

[4] J. Kari, “Reversibility and Surjectivity Problems of Cellular Automata,” Journal of Computer and System Sciences, 48(1), 1994.

[5] G. Hedlund, “Endomorphisms and Automorphisms of Shift Dynamical Systems,” Mathematical Systems Theory, 1969.

[6] P. Kůrka, Topological and Symbolic Dynamics, Cours Spécialisés, Vol. 11, Société Mathématique de France, 2003.

[7] G. Manzini, L. Margara, “Invertible Linear Cellular Automata,” Theoretical Computer Science, Vol. 325, No. 1, pp. 183–204, 2004.

[8] A. Dennunzio, E. Formenti, D. Grinberg, L. Margara, “An Efficiently Computable Characterization of Stability and Instability for Linear Cellular Automata,” Information Sciences, Vol. 581, pp. 618–636, 2021.

[9] A. Dennunzio, E. Formenti, D. Grinberg, L. Margara, “From Linear to Additive Cellular Automata,” in 47th International Colloquium on Automata, Languages, and Programming (ICALP 2020), Leibniz International Proceedings in Informatics (LIPIcs), 2020.

[10] N. Castronuovo, A. Dennunzio, L. Margara, “Decidability and Characterization of Expansivity

for Group Cellular Automata," arXiv preprint arXiv:2510.14568, 2025

[11] T. Ceccherini-Silberstein, M. Coornaert, Cellular Automata and Groups, Springer Monographs in Mathematics, Springer, 2010.

[12] N. Castronuovo, A. Dennunzio, L. Margara, "Topological Transitivity of Group Cellular Automata Is Decidable," arXiv preprint arXiv:2601.16243, 2026.

[13] A. Dennunzio, E. Formenti, D. Grinberg, L. Margara, "Decidable Characterizations of Dynamical Properties for Additive Cellular Automata over a Finite Abelian Group with Applications to Data Encryption," Information Sciences, Vol. 563, pp. 183–195, 2021.

[14] L. Bartholdi, "Cellular Automata, Duality and Sofic Groups," New York Journal of Mathematics, Vol. 23, pp. 1417–1425,

ISSN: 2312-8135 | Print ISSN: 1992-0652
info@journalofbabylon.com | jub@itnet.uobabylon.edu.iq | www.journalofbabylon.com