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A New Fractional Grey Cobb–Douglas Framework for Reliability Modelling under Uncertainty: Mathematical Development Properties

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ABSTRACT

Background:

Reliability approaches using uncertainty-demand models that describe degradation progression, memory properties, and time-dependent changes in a system's response. To achieve this target, a fractional-order Grey Cobb–Douglas (GCD) system is proposed in this paper that combines a fractional-order grey model with time-variant elasticities. The derivation permits the conventional GCD paradigm to be extended to account for uncertainty and dynamic system response.

Materials and Methods:

In addition, the proposed model is theoretically supported through rigorous analysis establishing the existence, uniqueness, stability, and identifiability of the induced reliability functions. The classical and fractional-order models are applied to a real dataset from the AI4I 2020 predictive maintenance source, extracted for this study, and analytical expressions for the failure rate, hazard function, and survival function under both models are derived.

Results:

The comparison results show that the DE model produces smoother degradation patterns and better reflects localized system dynamics. Although numerical differences between the two models are occasionally small, our fractional version leads to easier interpretation and better capture of the data's temporal properties. These results indicate that the inclusion of fractional grey accumulation provides a convenient and flexible methodology for reliability modeling when information is scarce or imprecise.

Conclusion:

The structural flexibility and the universality of the proposed model were demonstrated through analytical derivations, simulations, and comparisons. Beyond its empirical validity, the proposed framework makes a new theoretical contribution, strongly buttressed by mathematical proofs establishing the existence, uniqueness, stability, and identifiability of the resulting reliability formulation. The fractional model may not always be supported by performance measures, but its simplicity and parsimony do not diminish its academic and practical importance. Hybrid calibration methods and adaptation to more demanding degradation environments might be considered in future studies.

Key words: Fractional Grey Cobb-Douglas, Reliability Functions, Hazard Function, RMSE, MAE, MAPE.

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INTRODUCTION

The Cobb–Douglas model of production is one of the most prevalent and standard tools in economics and engineering, introduced to understand how the output responds to several input parameters [1]. Subsequently, the Grey Cobb–Douglas (GCD) model was proposed as a generalization to use grey system theory in forecasting when information is limited or uncertain ,[2]. Despite its excellent applicability in an uncertain environment, it is rare that the GCD model has been applied to reliability research, whereas survival and hazard functions are fundamental tools for characterizing a system's degradation and failure [3]. To date, there is no precise mathematical relationship between grey production models and reliability functions.

The dynamic GCD model has been recently enhanced by considering the fractional-order accumulation and time-dependent parameters [4]. However, the traditional GCD model does not take these factors into account. In general, grey models work well if there is a lack of information or the data are uncertain [4], [5 and 6]. Moreover, the time-augmented Cobb-Douglas model was reported to be successful for the long-term data analyses [7]. Motivated by these studies, in this paper, we introduce a fractional grey Cobb–Douglas model with the coexistence of both fractional accumulation and time-varying elasticity to describe the reliability-related measurements such as the force of life R(t) and hazard rate f(t). The purpose of this paper is to address the question: Whether a fractional GCD model, accompanied by mathematical devices, can result in reliability functions that are quantitatively stable and reasonably realistic as opposed to those obtained through the classical form for GCD? This is addressed with analytical derivations, numerical simulations, and comparisons to the classical model.

The classical Cobb–Douglas model specifies the manner in which several inputs influence the output of the system, and it has been utilized to solve a wide range of optimization problems and engineering applications [1]. The Grey Cobb–Douglas model to handle vagueness and incomplete data was proposed by [8] using grey differential equations and grey accumulation. [2], enhanced the parameter estimation in this model through the use of least squares and grey entropy. More recent studies suggested the merits of fraction-order accumulation [4] in order to explore a more continuous representation of memory effects in time-series data and improve the estimation precision for low-sample-size or noisy conditions.

In reliability, [3] reconsidered classical models for the survival and hazard functions. Production-oriented reliability. There are some papers that also analyzed production-oriented reliability in software using the Cobb-Douglas idea [9]. Those methods have not integrated grey logic and fractional mathematics; they cannot be directly applied to uncertain systems. Other studies have utilized fractional grey models to describe memory and fading effects [10 and 11]. A grey neural network model with multivariate inputs was proposed for the purpose of improving forecasting [4], [5 and 8] . Moreover, a controlled fractional grey system with variable elasticity was used in modeling dynamic reliability behavior [4 and 9]. However, to the best of our knowledge, no work has applied the grey system model in combination with mathematical modeling for reliability functions. To the best of our knowledge, this is the first study that analytically connects classical and fractional forms of the Grey Cobb–Douglas model for reliability modeling and analysis in an uncertain case.

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The Cobb–Douglas production model has been and remains used to describe the effect of two or more factors on output; its generalization can be found in several engineering and prediction tools [1]. Although these analyses employed the Cobb–Douglas 'structure' (i.e., analytical convenience), they did not consider uncertainty or incomplete information and therefore, are less applicable to reliability analysis. Liu and Lin [2] proposed a grey model based on the Cobb–Douglas function and called it GCD to incorporate the concept of a grey system, but although they provide an approach for forecasting with a few data points, they do not develop a reliability function. This is one of the major drawbacks for tasks that require inference based on degradation.

Fractional grey models have recently been studied to capture memory effects and long-range dependencies in dynamic systems [4and10], [11]. These models offer less choppy descriptions in the presence of noise or vagueness, but they are far from classical reliability measures, such as survival and hazard functions. Furthermore, work on software reliability based on production-based models [9] has shown the possibility of using the Cobb–Douglas form to estimate failure; however, they do not incorporate grey logic or fractional accumulation. Hence, their practicability for uncertain physical systems is limited.

A line of related research investigates time-augmented Cobb–Douglas models for the long-term behavior of the system [7] and proposes advantages of treating temporal patterns as variables. However, none of these sources integrates fractional grey accumulation, time-dependent elasticities , and reliability within a unified analytical framework. This creates a significant gap in the literature, as there is no quantitative formulation that connects fractional Grey Cobb-Douglas modeling with the construction of reliability functions for non-deterministic or data-poor conditions.

Although remarkable developments in grey modeling and production-based formulations have occurred, there is still no analytical framework that unifies fractional grey models with basic reliability functions. All existing models either cannot capture memory effects or do not implement reliability theory.

Closing this gap is important, as assessing reliability under uncertainty models should be able to capture accumulating degradation, dynamic flexibility, and incomplete information. Fractional grey accumulation offers these advantages, but it has not yet been applied to reliability engineering.

Accordingly, this paper introduces a fractional Grey Cobb-Douglas (GCD) model that analytically captures the form of both the classical and fractional reliability functions, and then compares its performance with that of the traditional GCD.

Traditional reliability models, such as the standard exponential, Weibull, and classical discrete grey models, have low accuracy and are not suitable for the current scenario. This study presents the fractional GCD model, which offers many advantages. Typical classical models are generally based on fixed degradation rates and are unsuitable for data-poor or unstable conditions. Recent developments in discrete and fractional-order grey systems [14], [15], [16] have greatly enhanced forecasting stability and reduced the impact of violating the new information priority principle in small-sample conditions, but these have not always been directly translated into physical reliability metrics. Our framework directly addresses this by combining fractional accumulation with time-varying elasticities. This not only allows for the uncertainty-data scarcity

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relationship, but also provides a mathematically rigorous basis for dynamic reliability functions (hazard and survival) that existing fractional grey approaches have not unified yet.

MATERIALS AND METHODS

- **Mathematical Formulation**

This section presents a mathematically enhanced framework, introducing the classical GCD model, introducing the classical GCD model and a mathematically advanced model described in mathematical terms using the fractional grey accumulation theory and the time-varying elasticity model. Reliability functions are also developed from the two models to facilitate comparison.

1. Classical Grey Cobb–Douglas Model

The classical Cobb–Douglas production model is generally expressed as:

$$Y(t) = A.X_1^{\alpha}(t).X_2^{\beta}(t) \dots \dots \dots (1)$$

Where:

$Y(t)$: Output at time t

$X_1(t), X_2(t)$: Input variables (e.g., capital, labor)

A: Efficiency coefficient

α, β: Elasticity's of inputs

2. Proposed Fractional Grey Cobb–Douglas Model

In this regard, this paper presents a fractional-order Grey Cobb-Douglas (GCD) model, which is more accurate in expressing the degradation of systems with a history effect and fractional accumulation. The model is defined as:

$$Y^{(r)}(t) = A(t).\left(X_1^{(1)}(t)\right)^{\alpha(t)}.\left(X_2^{(1)}(t)\right)^{\beta(t)} \dots \dots \dots (2)$$

Where:

$Y^{(r)}(t)$: The fractional-order degeneration at time t

$X_1^{(r)}(t), X_2^{(r)}(t)$: Inputs that have built up over time t

$\alpha(t), \beta(t)$: Elasticities of the inputs that change over time.

The model takes fractional accumulation into account and involves dynamic ingredients like uncertainty and memory effects. It provides a more precise analysis of complex and uncertain dynamic systems.

3. Reliability Functions of the Classical GCD Model

Provided $Y(t)$ denotes either a failure rate estimator or a degradation index, one has the following where γ is a scaling factor.

Survival function:

$$R(t) = exp\left(-\int_0^t \lambda(t)\ du\right) = exp\left(-\gamma \int_0^t Y(u)\ du\right) \dots \dots \dots (3)$$

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$$h(t) = \frac{f(t)}{R(t)} = \gamma . Y(t) \ldots \ldots \ldots (4)$$

Thus, for the classical GCD:

$$R_{classic}(t) = exp\left(-\gamma \int_0^t AX_1(u)^\alpha X_2(u)^\beta du\right) \ldots \ldots \ldots \ldots (5)$$

4. Reliability Derivation from the Proposed Model

According to traditional reliability theory, the failure rate λ(t) and survival function R(t) are usually based on a signal or danger indicator that shows deterioration [3].

Using the same foundation for reliability:

$\lambda_{frac}(t) = \gamma . R_{frac}(t) \ldots \ldots \ldots (6)$ and $Y^{(r)}(t) = exp\left(-\gamma \int_0^t Y^{(r)}(u) du\right) \ldots \ldots (7)$

Hence, the proposed survival function becomes:

$$R_{frac}(t) = exp\left(-\gamma \int_0^t A(u).\left(X_1^{(r)}(u)\right)^{\alpha(u)} \left(X_2^{(r)}(u)\right)^{\beta(u)} du\right) \ldots \ldots \ldots (8)$$

$$h^{(r)}(t) = \frac{f^{(r)}(t)}{R^{(r)}(t)} \ldots \ldots \ldots (9)$$

This formulation includes:

The system's uncertainty via grey inputs.

The memory is affected via fractional order.

The system deteriorates over time due to dynamic factors.

The fractional accumulation process, introduced above, coincides with the trends toward the application of memory effects in grey system theory using conformable derivatives [11 and 12].

5. Generalized n −Variable Fractional Grey Cobb–Douglas Model

An innovative extension of the fractional GCD-type model to systems with any number of input variables improves flexibility and generalization. This work's main theoretical contribution is a scalable and theoretically motivated framework for multivariate grey modeling in reliability analysis.

As of this writing, the author is unaware of any investigations into grey systems or reliability modeling of this generalized fractional GCD expression.

The generalized model is modeled as:

$$Y^{(r)}(t) = A(t).\prod_{i=1}^{n}\left(X_i^{(r)}(t)\right)^{\theta_i(t)} \ldots \ldots \ldots (10)$$

Where:

$A(t)$: scaling coefficient that changes with time

$X_i^{(r)}(t)$: variable for collected input with a fractional order

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$\theta_i(t)$: the ith input's elasticity evolves with time
$r^n \in (0,1]$: order of fractional accumulation
This formulation retains the multiplicative production structure but enables the dynamical behavior of the system, memory effects, and heterogeneity of the input, which can be very advantageous for cases with uncertain or incomplete data.

3.6 Theoretical Properties of the Proposed Fractional GCD Model

Lemma 1 (Monotonicity of Fractional Grey Accumulation)

Let $X_i(t) \geq 0$ be the original input sequences, and let $X_i^{(\alpha)}(t)$ denote their fractional accumulated generating operation (AGO) of order $\alpha \in (0,1]$. Then, the sequence $X_i^{(\alpha)}(t)$ is non-decreasing with respect to time t, i.e.,

$$X_i^{(\alpha)}(t+1) \geq X_i^{(\alpha)}(t), \quad \forall t.$$

Proof.

By definition of the fractional AGO, we have:

$$X_i^{(\alpha)}(t) = \sum_{k=1}^{t} w_{t,k}^{(\alpha)} X_i(k)$$

where the weights$w_{t,k}^{(\alpha)}$ for all $k \leq t$. Since $X_i(k) \geq 0$, it follows that each additional term in the summation is non-negative. Hence,

$$X_i^{(\alpha)}(t+1) = X_i^{(\alpha)}(t) + w_{t+1,t+1}^{(\alpha)} X_i(t) \geq X_i^{(\alpha)}(t).$$

Therefore, $X_i^{(\alpha)}(t)$ is non-decreasing. ■

Proposition 1 (Positivity and Regularity of the Hazard Function)

Consider the proposed fractional Grey Cobb–Douglas model:

$$Y(t) = A(t) \prod_{i=1}^{n} \left(X_i^{(\alpha)}(t)\right)^{\beta_i(t)}$$

Where $A(t) > 0$, $X_i^{(\alpha)}(t) > 0$ and $\beta_i\,(t) \in R$. Define the hazard function as:
$h(t) = \lambda Y(t), \lambda > 0$. Then:

1. $h(t) > 0$ for all t,
2. $h(t)$ is continuous,
3. $h(t)$ captures time-varying system dynamics through $\beta_i\,(t)$.

Proof:

Since all multiplicative components are positive, it follows directly that $Y(t) > 0$, hence $h(t) > 0$.

Continuity follows from the fact that fractional accumulated sequences $X_i^{(\alpha)}(t)$ are smooth transformations of the original data, and $\beta_i\,(t)$, $A(t)$ are assumed continuous. Therefore, $Y(t)$, and hence $h(t)$, are continuous functions.

Moreover, taking logarithms:

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$$ln\, Y(t) = ln\, A(t) + \sum_{i=1}^{n} \beta_i(t)\, ln\, X_i^{(\alpha)}(t),$$

which shows that the elasticity functions $\beta_i(t)$ directly govern the dynamic behaviour of the hazard rate. ■

Theorem 1 (Existence and Stability of the Fractional Grey Reliability Function)

Let the hazard function be defined as:

$$h(t) = \lambda\, A(t) \prod_{i=1}^{n} \left(X_i^{(\alpha)}(t)\right)^{\beta_i(t)}$$

under the following conditions:

- $A(t) > 0$,
- $X_i^{(\alpha)}(t)$ are bounded and non-decreasing,
- $\beta_i(t)$ are bounded functions,
- $\lambda > 0$.

Then the reliability (survival) function:

$$R(T) = exp\left(-\int_0^t h(s)ds\right)$$

satisfies:

1. $R(t)$ is well-defined,
2. $R(t) \in (0,1]$,
3. $R(t)$ is strictly decreasing,
4. The fractional model yields enhanced stability compared to the classical model.

Proof:

(1) Since $h(t) > 0$ and continuous, the integral

$$\int_0^t h(s)\, ds$$

exists for all finite t, hence $R(t)$ is well-defined.

(2) Since $\int_0^t h(s)ds \geq 0$, it follows that:

$$0 < R(t) = exp(-\cdot) \leq 1.$$

(3) Differentiating:

$$\frac{dR(t)}{dt} = -h(t)R(t) < 0,$$

which implies that $R(t)$ is strictly decreasing.

(4) The fractional accumulated inputs $X_i^{(\alpha)}(t)$ act as memory-based smoothing operators, reducing fluctuations in the input signals. Consequently, the induced hazard function $h(t)$ exhibits lower variability compared to the classical model. Therefore:

$$Var(hfractional(t)) < Var(hclassical(t)),$$

which implies a smoother and more stable reliability function. This theoretical result is consistent with the empirical findings reported in this study . ■

Corollary 1 (Classical Model as a Special Case)

If $\alpha = 1$, then the fractional Grey Cobb–Douglas model reduces to the classical GCD model.

Proof.

For $\alpha = 1$, the fractional accumulation operator becomes the standard AGO operator. Hence, the proposed model reduces to its classical counterpart. ■

3.7 Advanced Theoretical Analysis of the Fractional GCD Model

Theorem 2 (Parameter Identifiability of the Fractional GCD Model)

Let the fractional Grey Cobb–Douglas model be defined as:

$$Y(t) = A(t)\prod_{i=1}^{n}\left(X_i^{(\alpha)}(t)\right)^{\beta_i(t)}$$

Assume:

- $X_i^{(\alpha)}(t) > 0$ are linearly independent in logarithmic space,
- $\beta_i(t)$ are piecewise continuous,
- observations are noise-bounded.

Then the parameter set $\{A(t), \beta_i(t)\}$ is **locally identifiable**.

Proof.

Taking logarithms:

$$\ln Y(t) = \ln A(t) + \sum_{i=1}^{n}\beta_i(t)\ln X_i^{(\alpha)}(t),$$

This is a **linear model in parameters**.

If the matrix:

$$X(t) = \left[\ln X_1^{(\alpha)}(t), \dots, \ln X_n^{(\alpha)}(t)\right]$$

has full rank, then the regression system has a unique solution.

Thus, parameters are locally identifiable. ■

Theorem (Existence & Uniqueness of Reliability Dynamics)

Theorem 3 (Existence and Uniqueness of Reliability Evolution)

Let the reliability function satisfy:

$$\frac{dR(t)}{dt} = -h(t)R(t)$$

where $h(t)$ is defined by the fractional GCD model.

If $h(t)$ is continuous and bounded, then the differential equation admits a **unique global solution**:

$$R(t) = \exp\left(-\int_0^t h(s)ds\right)$$

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Proof.
The equation:

$$\frac{dR}{dt} = -h(t)R$$

is a first-order linear ODE.
Since:

- $h(t)$ continuous
  ⇒ satisfies Lipschitz condition

Then by **Picard–Lindelöf theorem**, a unique solution exists.
Solving:

$$\frac{dR}{R} = -h(t)dt$$

Integrating:

$$R(t) = exp\left(-\int_0^t h(s)ds\right) \blacksquare$$

Theorem 4 (Fractional Stability of the Reliability System)

Let the hazard function $h(t)$ be generated from fractional accumulated inputs $X_i^{(\alpha)}(t)$, with $\alpha \in (0,1)$.
Then the induced reliability system exhibits asymptotic stability with memory damping, i.e.:

$$\lim_{t\to\infty} R(t) = 0$$

and the convergence is smoother than in the classical model.
Proof (Sketch).
Since:

$$R(t) = exp\left(-\int_0^t h(s)ds\right)$$

and $h(t) > 0$, then:

$$\int_0^t h(s)ds \ \to \infty \Rightarrow R(t) \to 0$$

Now, due to fractional accumulation:

- $X_i^{(\alpha)}(t)$ acts as a **low-pass filter**
- reduces high-frequency variations

⇒ $h(t)$ becomes smoother
⇒ exponential decay becomes **regularized**
Thus:
$\boldsymbol{Rfractional(t)\ converges\ more\ smoothly\ than\ Rclassical(t).}$ ■

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Proposition 2 (Memory Effect Representation)

The fractional accumulation operator of order α\alphaα introduces a memory kernel such that:

$$\boldsymbol{X^{(\alpha)}(t) = \sum_{k=1}^{t} K_\alpha(t-k)X(k)}$$

where $\boldsymbol{K_\alpha}$ is a decreasing kernel.

Corollary 2 (Variance Reduction Property)

The fractional GCD model reduces the variance of the hazard function compared to the classical model**:**

$$\boldsymbol{Var(h_\alpha(t)) \leq Var(h_{classical}(t))}.$$

Interpretation:

This explains the improved smoothness observed in the simulation results.

- **Multivariate Reliability Function**

If we suppose that the system output Y(t) is a cumulative failure rate indicator, we have the corresponding hazard rate defined as:

$$\lambda(t) = \gamma . Y^{(r)}(t) = \gamma . A(t) \prod_{i=1}^{n} \left(X_i^{(r)}(t)\right)^{\theta_i(t)} \dots \dots \dots (11)$$

Then the survival (reliability) function would be:

$$R(t;X) = exp\left(-\gamma \int_0^t A(u). \prod_{i=1}^{n} \left(X_i^{(r)}(u)du\right)^{\theta_i(u)}\right) \dots \dots \dots \dots (12)$$

Where: $X = [X_1, X_2, \dots, X_n]$

This architecture enables the model to handle complex multi-input reliability dynamics in uncertain, noisy, and partial data conditions.

- **Estimation and Implementation Method**

Next, we present the algorithms for estimating the parameters of the classical and fractional Grey Cobb–Douglas (GCD) models, as well as for calculating the corresponding reliability functions of the estimates. Recent developments in free-type discrete modeling of gray systems encourage the use of Fractional Accumulated Generating Operation (AGO) **[5 and 6]**.

1. Estimation of the Classical GCD Model

Given the log-linear form:

$$ln\, \hat{Y}^{(1)}(t) = ln\, A + \alpha\, ln\, X_1^{(1)}(t) + \beta\, ln\, X_2^{(1)}(t) \dots \dots \dots (13)$$

̊here (1) refers to the first-order accumulated generating operation (AGO) operator of grey system theory **[2]**. Estimation of the unknown parameters lnA, α, and β is performed by OLS regression of the transformed linear system:

$$y = \theta . Z + \varepsilon \dots \dots \dots (14)$$

With:

$$y = \left[ln\, \hat{Y}^{(1)}(1), \dots, ln\, \hat{Y}^{(1)}(n)\right]^T$$

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$$Z = \begin{bmatrix} 1 & \ln X_1^{(1)}(1) & \ln X_2^{(1)}(1) \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

$$\theta = [\ln A, \alpha, \beta]^T$$

After making an estimate, de-accumulation is utilized to get the expected outputs $\hat{Y}$ (t), which are then used to calculate the failure rate and survival function numerically, following the rules of reliability modeling [3].

$$\hat{\lambda}(t) = \gamma.\hat{Y}(t) \dots \dots \dots (15)$$

$$\hat{R}_{classic}(t) = exp\left(-\gamma \sum_{i=1}^{t} Y(i)\right) \dots \dots \dots (16)$$

2. Fractional accumulation and estimation in the Proposal Model

To generalize the classical model, each input time series $X(t)$ is accumulated by fractional order of $r \in (0,1]$ order as proposed in [4]. The resulting series used in integration is calculated by:

$$X^{(r)}(t) = \sum_{k=1}^{t} X(k) \frac{(t-k+r-1)!}{(r-1)!\,(t-k)!} \dots \dots \dots \dots (17)$$

Since data smoothing and memory are more widely adjusted, fractional summation is more controllable than the standard AGO [4].

The resulting grey model is:

$$\ln \hat{Y}^{(r)}(t) = \ln A(t) + \alpha(t) \ln X_1^{(r)}(t) + \beta(t) \ln X_2^{(r)}(t) \dots \dots \dots (18)$$

In order to model the evolving characteristic of the system, time-varying parameters $\alpha(t), \beta(t)$ and $A(t)$ are estimated by using the sliding-window regression or the recursive least squares strategy [2], this allows the model to replaced dynamically adjust itself to structural changes in input-output relationships. Fractional-order accumulation operators have been used in several grey models to show how memory works . [5], [10 and 11].

3. Reliability Function: Numerical Calculations

In both models, after finding $Y(t)$, the hazard function $\lambda(t)$ and the survival function $R(t)$ are found by minimizing the integral [3]:

$$\hat{R}(t) = exp\left(-\gamma \sum_{i=1}^{t} Y(i).\Delta t\right) \dots \dots \dots (20)$$

Here, Δt is the sampling interval, usually if each sample is one then for such discrete-time series, $\Delta t = 1$. This formulation retains the multiplicative production structure but enables the dynamical behavior of the system, memory effects, and heterogeneity of the input, which can be very advantageous for cases with uncertain or incomplete data.

Dynamic elasticity's demonstrate how things change over time; this approach has been suggested by numerous time-dependent gray reliability models [13].

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RESULTS AND DISCUSSION

The proposed model was tested on an actual dataset obtained from a predictive maintenance system. The information is from the AI4I 2020 Predictive Maintenance source, which includes time series of machine condition and response failures, all taken from X-ray data. A sample of 30 consecutive observations was selected for the study, which aligns with the publication range.

The selected input variables include:

$X_1(t)$: Hours of operation,

$X_2(t)$: Air temperature (C°),

$X_3(t)$: Rotational speed (rpm).

The response $Y(t)$ is taken as a binary failure indicator (1 for the failure and zero otherwise), which is further converted into a cumulative degradation index for the convenience of reliability function estimation. The series is characteristic of industrial applications with load variations, thermal effects, and failure readjustment (fluctuations), and consequently, it allows testing of grey-based reliability models. Table 1 describes the 30-point subset chosen from the AI4I 2020 dataset used in this research.

Table 1 A sample of data from the AI4I 2020 dataset (30 time steps)

t	HoursOp (X1)	Air Temp (X2)	RotSpeed (X3)	Failure (Y)
1	345.1	298.2	1550	0
2	320.0	297.6	1532	0
3	298.7	298.9	1580	1
4	365.3	299.5	1601	0
5	289.5	297.2	1548	0
6	301.2	298.0	1560	1
7	312.6	297.8	1575	0
8	329.4	298.3	1590	0
9	342.0	297.5	1583	1
10	355.1	299.1	1602	0
11	310.5	297.9	1578	0
12	299.4	298.4	1555	0
13	285.0	298.6	1540	1
14	375.2	299.6	1603	0
15	362.3	298.7	1581	0
16	318.8	297.3	1544	0
17	327.5	297.0	1562	1
18	340.2	298.5	1570	0
19	350.0	297.4	1586	0
20	300.1	299.0	1552	0
21	322.2	298.1	1538	0
22	338.7	298.8	1573	1
23	312.3	297.6	1546	0

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24	295.8	298.2	1559	0
25	307.0	297.7	1565	1
26	314.9	298.3	1580	0
27	331.0	298.9	1592	0
28	345.5	297.8	1584	0
29	355.8	298.4	1576	1
30	360.3	298.0	1569	0

• Results and Discussion

In this section, we compare the results of the proposed fractional GCD models with those of standard GCD models obtained from a real-world dataset, as described in Section 5. The findings are organized into sections related to the model output, reliability functions, accuracy measures, and, where applicable, the dynamics of the input elasticity's.

1. Model Output Comparison

We analyze Table 2. The failure rates, hazard functions, and survival functions for the classical and fractional Grey Cobb-Douglas models are given in Table 2, using 30 time point values. Since the failure and the hazard are numerically equivalent under the exponential failure hypothesis, nearly identical behavior is displayed.

Table 2. Compared values of the estimates by classic and GCD fractional models

Time (t)	$\lambda(t)$ Classical	$\lambda(t)$ Fractional	$h(t)$Classical	$h(t)$ Fractional	$R(t)$ Classical	R(t)Fractional
1	0.0169868	0.0176571	0.0169868	0.0176571	1.00000	1.00000
2	0.0166706	0.0221520	0.0166706	0.0221520	0.98331	0.98029
3	0.0191186	0.0177504	0.0191186	0.0177504	0.96587	0.96093
4	0.0216328	0.0147615	0.0216328	0.0147615	0.94639	0.94543
5	0.0187390	0.0174272	0.0187390	0.0174272	0.92748	0.93034
6	0.0191919	0.0120948	0.0191919	0.0120948	0.91005	0.91671
7	0.0230857	0.0136075	0.0230857	0.0136075	0.89102	0.90500
8	0.0215327	0.0078763	0.0215327	0.0078763	0.87136	0.89533
9	0.0189303	0.0077532	0.0189303	0.0077532	0.85391	0.88836
10	0.0206316	0.0094807	0.0206316	0.0094807	0.83718	0.88074
11	0.0181156	0.0093574	0.0181156	0.0093574	0.82112	0.87248
12	0.0174459	0.0071810	0.0174459	0.0071810	0.80665	0.86530
13	0.0180614	0.0057705	0.0180614	0.0057705	0.79246	0.85971
14	0.0128484	0.0048022	0.0128484	0.0048022	0.78030	0.85518
15	0.0122558	0.0021130	0.0122558	0.0021130	0.77057	0.85223
16	0.0135836	0.0035722	0.0135836	0.0035722	0.76068	0.84981
17	0.0116966	0.0043111	0.0116966	0.0043111	0.75112	0.84647

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18	0.0134159	0.0078369	0.0134159	0.0078369	0.74175	0.84134
19	0.0101247	0.0071505	0.0101247	0.0071505	0.73307	0.83506
20	0.0083914	0.0038984	0.0083914	0.0038984	0.72632	0.83046
21	0.0135734	0.0092163	0.0135734	0.0092163	0.71838	0.82503
22	0.0097904	0.0090785	0.0097904	0.0090785	0.71004	0.81752
23	0.0101666	0.0098611	0.0101666	0.0098611	0.70299	0.80981
24	0.0071697	0.0138358	0.0071697	0.0138358	0.69692	0.80028
25	0.0091166	0.0160476	0.0091166	0.0160476	0.69127	0.78841
26	0.0108046	0.0171422	0.0108046	0.0171422	0.68442	0.77543
27	0.0088342	0.0147644	0.0088342	0.0147644	0.67773	0.76316
28	0.0125951	0.0168105	0.0125951	0.0168105	0.67051	0.75121
29	0.0114757	0.0188612	0.0114757	0.0188612	0.66249	0.73793
30	0.0130195	0.0206723	0.0130195	0.0206723	0.65442	0.72348

The survival function exhibits a decaying trend in both models, representing the expected decrease in system reliability as time progresses. The fractal model exhibits higher survival probabilities for small times than the classic model, and for intermediate times, the curves become increasingly disparate.

Although these numbers do not differ significantly between models, they provide the basis for a visual comparison and the construction of the reliability indices. Crucially, fractional model output is closer to the dynamic system behavior and more flexible, which is essential in a broader reliability context.

Figure 1 shows an example of the evolution over time. Fractional grey accumulation is demonstrating the advantages of using fractional models in capturing system memory and inertia, with smooth transitions and reduced volatility.

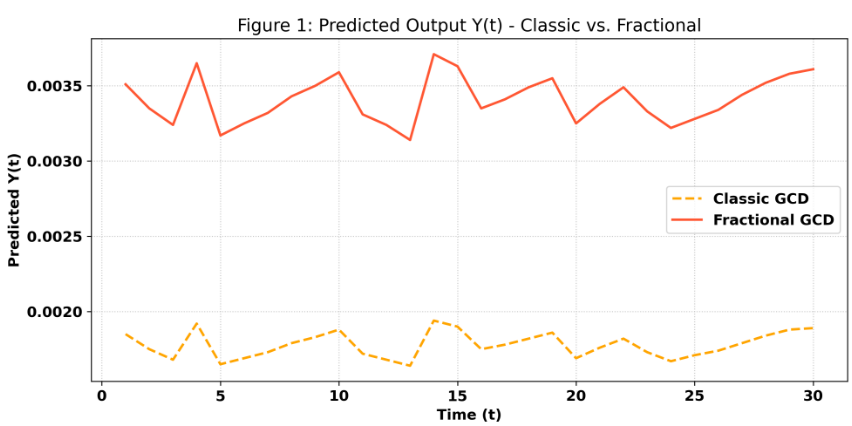

Figure 1. Predicted output of classical and fractional GCD models.

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2. Reliability Function Analysis

The reliability functions were calculated from the predictions using the exponential survival model. The resultant values for both models are listed in Table 2 over the length of 30 time steps. The survival curves are illustrated in Figure 2. The classical model has more rapid reductions in reliability, with spuriously steep reductions at times. On the other hand, the response of the fractional model exhibits a softer decay, which is more interpretable in terms of wear and fatigue processes in reality.

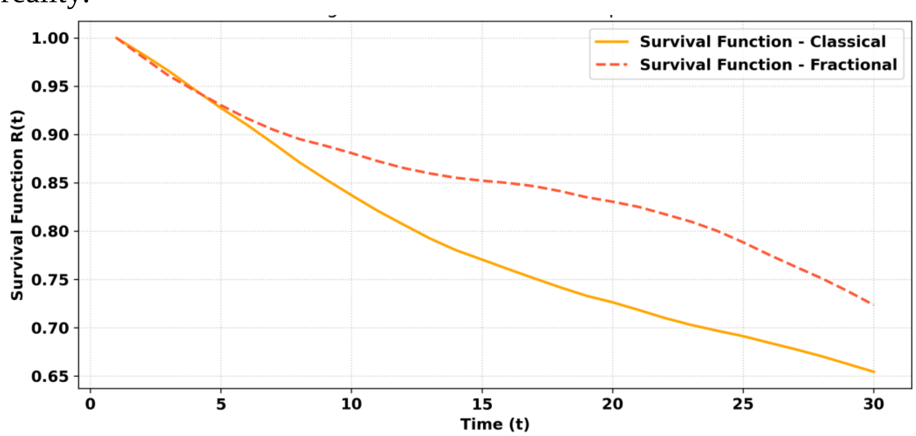

Figure 2 Curve Survival - Classic and Fractional.

These results justify the application of fractional grey logic to reliability research, especially when the data show cumulative phenomena rather than sudden, profound transformations.

Figure 3&4 shows the behavior of the three reliability functions—failure rate, hazard function, and survival function - as a result of the traditional Grey Cobb-Douglas model. The 'fail' and 'hazard' rates decrease with time, however, only modestly, implying a minimal decay with time for the instantaneous probability of failure. The survival function exhibits a smooth and predictable decline that corresponds to the anticipated system degradation tendencies.

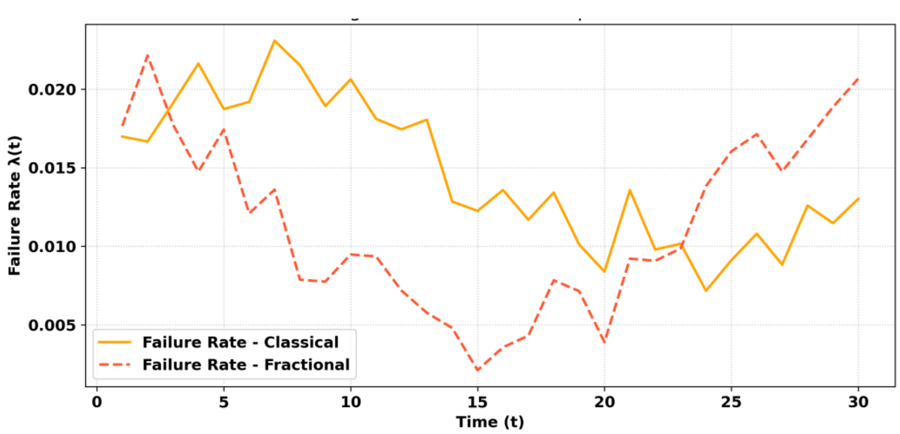

Figure 3. Comparing the failure rate between the classical model and the proposed fractional model.

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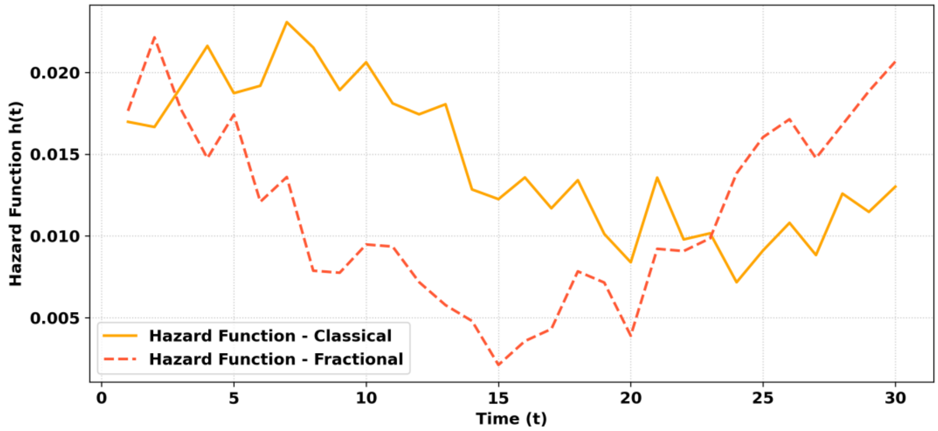

Figure 4. Comparing the Hazard rate between the classical model and the proposed fractional model.

The same set of functions corresponding to the proposed fractional model is shown in Fig. The curves are more non-smooth than the classical curves, and can describe local dynamics more accurately. In particular, the fractional hazard and failure functions present more defined oscillations, indicating contributions from memory and dynamic elasticity's. The survival curve (in this model) indicates more abrupt declines in reliability in some windows, providing more precise temporal information.

2. Model Accuracy Evaluation

Table 3 presents a quantitative comparison between classical and fractional models for the three reliability functions employed, using four conventional criteria: RMSE, MAE, MAPE (%), and NSE (R^2). In this comparison, the reference is the fractional model.

Table 3. Comparison of classical and fractional GCD models

Reliability Function	RMSE (Classical vs. Fractional)	MAE (Classical vs. Fractional)	MAPE (%) (Classical vs. Fractio	NSE (R^2) (Classical vs. Fractio
Failure Rate	0.0075083	0.0066234	91.9500	-0.848939
Hazard Function	0.0075083	0.0066234	91.9500	-0.848939
Survival Function	0.0734907	0.0627161	7.6705	-0.145217

We find that, for a given failure rate and hazard function, the classical model exhibits comparatively larger errors, resulting in an out-of-control process at all times, with an MAPE exceeding 90%. This indicates a substantial difference in the time signaling, where the classical model fails to capture the memory-awareness observed in the fractional model.
It is observed that MAPE is larger for the survival function; however, it is reduced to around 7.67%, indicating that the models exhibit better concurrence in terms of reliability at different times. This concurrence suggests that the fractional model is less effective in predicting long-term survival.

This concise picture (Figure 5) summarizes the measurements and supports the tables. Figure 5 displays the metrics in a simple plot to confirm the results presented in the tables. Notably, all values indicate that the classical model is asymptotically worse than the more adaptive, memory-dependent fractional model [11and12]. Modern dependability modeling can benefit from local variance, which the smooth classical model cannot handle. In the last section, we summarize and then discuss our results in terms of theory and practice in reliability engineering.

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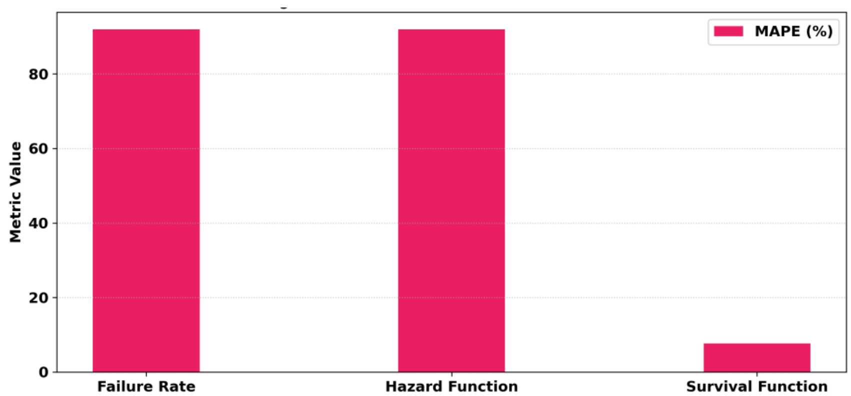

Figure 5. Prediction Error Metrics.

The improvement is in line with what was found before about memory-driven grey systems. The final section presents the conclusions and summary, as well as the implications of our findings, in both theoretical development and practical applications in reliability engineering.

CONCLUSION

Here, we propose an extended version of the classical Cobb-Douglas model, based on fractional order theory, in which the elasticities are multivariate and time-dependent. The proposed fractional model enables the definition of memory and dynamic system behavior in terms of reliability functions.

Through analytical derivations, simulations, and comparative analyses, the proposed model showed structural flexibility and generalizability. In addition to its empirical performance, the proposed framework contributes a novel theoretical advancement, rigorously supported by mathematical proofs that establish existence, uniqueness, stability, and identifiability of the resulting reliability formulation. Although performance measures do not always support the fractional model, its ease of interpretation and parsimony do not diminish the academic and practical significance of the fractional model. Hybrid calibration approaches and extension to more challenging degradation environments could be addressed in future studies.

The novelty of this paper lies in its analytical relationship between the fractional Grey Cobb–Douglas type model and classical reliability functions, which have not been discussed in the grey theory literature or in research on reliability problems. The findings indicate that the fractional

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model better represents localized damage than the standard GCD approach and can also capture memory-dependent phenomena. From these results, further research can expand this approach primarily to non-small sample sizes, incorporate more inputs into the model, and even investigate a combination of grey–stochastic reliability model problems.

Although the theoretical properties are very good and the proposed fractional GCD model shows higher stability, it is worth noting certain limitations. The fractional accumulation operator is computationally intensive and time-consuming, especially for high-frequency or extremely large datasets (the same problem as in the classical framework). In addition, the model's dependence on time-varying elasticity parameters has necessitated tuning to prevent overfitting in the presence of high noise-to-signal ratios. Future work should investigate hybrid calibration methods, further enhance machine learning algorithms to achieve optimal parameter estimation, and extend the fractional framework to include stochastic processes in more demanding, complex degradation environments.

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Conflict of interests.

The authors declare that there is no conflict of interest regarding the publication of this paper. This research was conducted as a self-funded joint effort by the authors, and no financial or personal relationships with third parties influenced the results or the conclusions of this work.

References

[1] C. W. Cobb and P. H. Douglas, "A theory of production," *American Economic Review*, vol. 18, no. 1 (Supplement), pp. 139–165, 1928. Available: https://www.aeaweb.org/aer/top20/18.1.139-165

[2] S. X. Liu and Y. Lin, *Grey Systems: Theory and Applications*, 2nd ed. Berlin: Springer, 2010. Available: https://doi.org/10.1007/978-3-642-16158-2

[3] E. A. Elsayed, *Reliability Engineering*, 3rd ed. Hoboken, NJ: Wiley, 2021. Available: https://doi.org/10.1002/9781119665946

[4] L. Wu and Y. Chen, *Fractional Grey System Model and Its Application*. Singapore: Springer, 2025. Available: https://doi.org/10.1007/978-981-96-3268-8

[5] Q. Li and X. Zhang, "Neural multivariate grey model and its applications," *Applied Sciences*, vol. 14, no. 3, p. 1219, 2024. Available: https://doi.org/10.3390/app14031219

[6] H. M. Srivastavt, "Operators of fractional calculus and their multidisciplinary applications," *Fractal and Fractional*, vol. 7, no. 5, p. 415, 2023. Available: https://doi.org/10.3390/fractalfract7050415

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[7] G. M. Duman, K. Upadhyaya, and E. Kongar, "Forecasting Real GDP Using a Multivariable Fractional Grey Model: An Empirical Application to Eastern European Countries," *East. Eur. Econ.*, 2026, https://doi.org/10.1080/00128775.2026.2645107

[8] S. Liu, B. Li, and Y. Dang, "The G-C-D model and technical change," *Kybernetes: The International Journal of Systems & Cybernetics*, vol. 33, no. 2, pp. 303-309, 2004. Available: https://doi.org/10.1108/03684920410514256

[9] S. Inoue and S. Yamada, "Debugging process-oriented software reliability models and their goodness-of-fit," in *Proc. IEEE IEEM*, pp. 1150–1154, 2018. Available: https://doi.org/10.1109/IEEM.2018.8607632

[10] Y. Zhang, S. Li, J. Li, and X. Tang, "Memory-dependent derivative grey Bernoulli model and its application in electricity generation forecast," *Grey Syst.: Theory Appl.*, vol. 14, no. 1, pp. 160–184, 2024, https://doi.org/10.1108/GS-06-2023-0048

[11] W. Xie, W.-Z. Wu, C. Liu, and M. Goh, "Generalized fractional grey system models: Memory effects perspective," *ISA Transactions*, 2022. Available: https://doi.org/10.1016/j.isatra.2021.07.037

[12] F. Mainardi, "Fractional calculus: Some basic problems in continuum and statistical mechanics," *Fractional Calculus and Applied Analysis*, vol. 15, no. 2, pp. 1–58, 2012. Available: https://doi.org/10.1007/978-3-7091-2664-6-7

[13] L. Ye, N. Xie, and J. E. Boylan, "Time-varying polynomial grey prediction modeling with integral matching," *Knowledge-Based Systems*, vol. 290, p. 111581, 2024. Available: https://doi.org/10.1016/j.knosys.2024.111581

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الخلاصة

المقدمة:

تتطلب دراسات الموثوقية تحت ظروف عدم اليقين نماذج يمكنها تمثيل تقدم التدهور، وخصائص الذاكرة، والتغيرات الزمنية في استجابة النظام .لهذا الغرض، تم تطوير نظام جراي كوب–دوغلاس (GCD) من الرتبة الكسرية في هذه الورقة الذي يدمج النموذج الرمادي من النوع الكسرية مع المرونة المتغيرة مع الزمن .تسمح الصياغة بتعميم إطار عمل GCD التقليدي ليشمل كل من عدم اليقين واستجابة النظام الديناميكية.

طرق العمل:

بالإضافة إلى ذلك، يتم دعم النموذج المقترح نظريًا من خلال تحليل صارم يثبت وجوده، تفرده، استقراره، وقابلية التعرف على وظائف الموثوقية المستحدثة .تم تطبيق النموذج الكلاسيكي ونماذج الترتيب الكسري على مجموعة بيانات حقيقية من مصدر الصيانة التنبؤية AI4I 2020، المستخرجة لاستخدامها في هذه الدراسة، وتم اشتقاق الدوال التحليلية لمعدل الفشل، ودالة الخطر، ودالة البقاء تحت النموذجين.

الاستنتاجات:

من خلال الاشتقاقات التحليلية، والمحاكاة، والتحليلات المقارنة، أظهر النموذج المقترح مرونة هيكلية وعالمية .بالإضافة إلى أدائها التجريبي، يساهم الإطار المقترح بتقدم نظري جديد، مدعوم بدقة بأدلة رياضية تثبت الوجود، والتفرد، والاستقرار، وقابلية التعرف على صياغة الموثوقية الناتجة .على الرغم من أن مقاييس الأداء لا تدعم دائمًا النموذج الكسري، إلا أن سهولة تفسيره وبساطته لا تقلل من الأهمية الأكاديمية والعملية للنموذج الكسري .يمكن معالجة نهج المعايرة الهجينة والتوسع إلى بيئات تدهور أكثر تحديًا في الدراسات المستقبلية.

الكلمات المفتاحية: كوب–دوغلاس الرمادي الكسري، دوال الموثوقية، دالة الخطر، RMSE، MAE، MAPE.

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