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Approximation Degree Analysis for a Special Functions Class

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تحليل درجة التقريب لفئة الدوال الخاصة

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ABSTRACT
This paper investigates the degree of approximation analysis of a special class of functions characterized by certain continuity conditions. The approximation of these functions is investigated using appropriate mathematical techniques under certain constraints. By exploring the behavior of the function approximation under different parameters, the study provides insights into the accuracy and limitations of existing approximation methods. The results contribute to a deeper understanding of how such functions can be approximated in preparation for their application in influential fields.

Keywords: Approximation analysis, special class of functions, Lip function, Fourier series, conjugate function.

INTRODUCTION

Alexits presented his study on the degree of approximation of Lip functions by the Cesaro mean of their Fourier series in the higher norm [1] . In a related vein, Das and Mohapatra extended Alexis's results by considering the Lip function's Fourier series transform (Z, a, 0) [2]-[5] . In this paper, we seek to prove the degree of approximation when the generating function satisfies a certain integration condition in preparation for obtaining distinctive results.

Let the periodic function with period 4n be denoted by h and be integrable in the Lebesque sense over [-π, π].

Also, let the Fourier series associated with h at y be

$$\frac{1}{4}a_0 + \sum_{k=1}^{\infty}(a_k \cos ky + b_k \sin ky)$$

And let the conjugate series be

$$\sum_{k=1}^{\infty}(b_k \cos ky - a_k \sin y)$$

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We know that

$$\emptyset_y(t) = \frac{1}{4}[h(y+t) + h(y-t) - 4h(y)]$$

Then we can write

$$s_k(y) - h(y) = \frac{1}{\pi}\int_0^{\pi} \frac{\emptyset_x(t)}{\sin\frac{1}{4}t} \sin\left(k + \frac{1}{2}\right) tdt$$

where $s_k(y)$ is the $k-th$ partial Fourier series sum. The conjugate function h is given by:

$$h(y) = \frac{1}{4\pi}\int_0^{\pi} \{h(y+t) - h(y-t)\}\cot\frac{t}{4}dt$$

which is interpreted as a Cauchy integral.

Let $A_n^{\alpha,\beta}$ be defined by the power series expansion ;

$$H_{\alpha,\beta}(z) = (1-z)^{-\alpha-1}\log\left(\frac{a}{1-z}\right)^{\beta} = \sum_n A_n^{\alpha,\beta}(z)$$

where (a) is a fixed constant > 2 and α, β are real numbers . We note the following results:

$$n_n^{\alpha-1,\beta} = \alpha A_{n-1}^{\alpha,\beta} + \beta A_{n-1}^{\alpha,\beta-1}$$
$$A_n^{\alpha,\beta} \sim \frac{n_n^{\alpha}}{r(\alpha+1)}(\log n)^{\beta}, (\alpha \neq -1, -2, \dots)$$
$$A_n^{\alpha,\beta} \sim \beta(-1)^{\alpha-1}(\alpha-1)!\, n^{\alpha}(\log n)^{\beta-1}, (\alpha = -1, -2, \dots,$$
$$A_n^{\alpha,\beta} = \sum_{k=0}^{n} A_k^{\alpha-1,\beta} = \sum_{k=0}^{n} A_{n-k}^{\alpha-1,\beta}$$

We write $A_n^{\alpha} = A_n^{\alpha,0}$. It can be seen that

$$A_n^{-1,1} = \begin{cases} \log a & (n = 0) \\ 1/n & (n \geq 1) \end{cases}$$

We refer to $\sigma_n^{\alpha,\beta}(h; y)$ the (Z, α, β)-transform of the Fourier series of f which is given by:

$$\sigma_n^{\alpha,\beta}(h; y) = \frac{1}{A_n^{\alpha,\beta}}\sum_{k=0}^{n} A_{n-k}^{\alpha-1,\beta} s_k(y)$$

The (Z, α, β) method is obtained called a generalized Cesàro-Harmonic mean. It may be noted that in the case β =0, the (Z, α, β)- transform reduces to the familiar Cesaro (C, α) transform and in case a = 0, it reduces to a special Norlund transform (N_β). The case is the familiar Harmonic mean. In this regard, we review the evidence and proofs that have been obtained:

Theorem 1:

Let

$$\tilde{h}'(y+t)+\tilde{h}'(y-t)=0(\wedge(t))$$

such that

$$\int_0^\pi \wedge(t)\frac{\log\frac{C}{t}}{t}dt<\infty$$

Then

$$\|h-\sigma_n^{0,1}(x)\|=0\left(\frac{1}{n(\log n)}\right)$$

Let $L_p[0,2\pi], p\geq 1$ be the class of all ph power integrable functions with its usual norm.

Let

$$K_p=\{h\in L_p:\tilde{h}\in Lip(1,p),1\leq p<\infty\}$$
$$K_\infty=\{h\in c_{2\pi}:\tilde{h}\in Lip\,1\}$$

where $C_{2\pi}$ is the set of all periodic continuous functions.

If $\tilde{f}\in Lip(1,p),1<p\leq\infty$, then it is known that $\tilde{f}$ is equivalent to the indefinite integral of a function belonging to L_p, [6], [7]. Thus if $f\in K_p$, then $\tilde{h}'\in L_p\ (1<p\leq\infty)$

Now we consider a subclass K_p for which

$$\|\tilde{h}'(y+t)+\tilde{h}'(y-t)\ \|_p=0(\wedge(t))\|$$

Auxiliary Results

Based on the following lemma we will to prove the next Theorems

We first write

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$$p_n = A_n^{\alpha-1,\beta}, P_n = A_n^{\alpha,\beta}$$

$$M_n(t) = \frac{1}{P_n \sin\frac{1}{4}t}\sum_{k=0}^{n} p_{n-k}\cos\left(k+\frac{1}{2}\right)t$$

$$Q_n(t) = \int_t^{\pi} t^t M_n(u)du$$

Lemma 2: Uniformly in $0 < t < \pi$

$$Q_n(t) = \begin{cases} o\left(\left(\frac{\log\left(\frac{c}{t}\right)}{\log n}\right)^{\beta}\left(\frac{1}{nt}\right)\right), (\alpha = 0, \beta > 0) \\ o\left(\left(\frac{\log\left(\frac{c}{t}\right)}{\log n}\right)^{\beta}\frac{1}{(nt)^{\alpha+1}}\right) \\ +0(1)\sum_{\theta=0}^{\alpha}\frac{1}{(nt)^{\theta}}\ (\alpha = 0, \beta \in R) \end{cases}$$

Proof: We know

$$p(z) = \sum_{n=0}^{\infty} P_n 2^n$$

which clearly converges for $|z| < 1$.

Now,

$$\sum_{k=0}^{n} P_k \cos\left(n-k+\frac{1}{2}\right)u = R_{ea1}\left\{e^{\left(n+\frac{1}{2}\right)iu}\sum_{k=0}^{n} P_k e^{-1ku}\right\}$$

$$= \quad \text{Real}\left[e^{\left(n+\frac{1}{2}\right)iu}\left\{\sum_{k=0}^{\infty} P_k e^{-1ku} - \sum_{k=n+1}^{\infty} P_k e^{-1ku}\right\}\right]$$

Thus,

$$Q_n(t) = \int_t^{\pi} M_n(u)du$$

$$= Q_{n,1}(t) = Q_{n,2}(t)$$

Where

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$$Q_{n,1}(t) = \int_t^{\pi} \left(P_n \sin\frac{1}{2}u\right)^{-1} \cdot \text{Real}\left\{e^{i\left(n+\frac{1}{2}\right)u} p\left(e^{-iu}\right)\right\} du,$$

$$Q_{n,2}(t) = \int_t^{\pi} \left(P_n \sin\frac{1}{2}u\right)^{-1} \cdot \text{Real}\left\{e^{i\left(n+\frac{1}{2}\right)u} \sum_{k=1}^{\infty} P_k\, e^{-iku}\right\} du,$$

we will compute the order estimation for $Q_{n,1}(t)$ based on , (see [8])

$$\text{Real}\ \{e^{\left(n+\frac{1}{2}\right)iu}\ P(e^{-iu})\}$$

$$= \frac{\{\log^2 \frac{a}{2\sin {}^{u}/_{2}} + \frac{1}{4}(\pi-u)^2\}^{\beta/_2}}{(2\sin\frac{1}{2}u)^{\alpha}} \cos\{\left(n+\frac{1}{2}\right)u+\theta\}$$

So by use of second mean value theorem twice

$$= \frac{\{\log^2 \frac{a}{2\sin {}^{u}/_{2}} + \frac{1}{4}(\pi-u)^2\}^{\beta/_2}}{P_n \sin\frac{t}{2}(2\sin\frac{1}{2}u)^{\alpha}} \cos\{\left(n+\frac{1}{2}\right)u+\theta\}$$

The application of the second mean-value theorem is valid as

$$\{\log^2 \frac{a}{2\sin {}^{u}/_{2}} + \frac{1}{4}(\pi-u)^2\}^{\beta/_2}$$

is slowly varying and $\left(\frac{1}{(\sin\frac{t}{2})^{\alpha+1}}\right)$ is decreasing as t increases.

Thus, for some constant $c > n$, we have

$$Q_{n,1}(t) = o\left(\frac{\log^2\left(\frac{c}{t}\right) + \frac{1}{4}(\pi-t)^2}{P_n t^{\alpha+1} n}\right)^{\frac{\beta}{2}}$$

$$= \left(\frac{\log({}^{c}/_{t})^{\beta}}{\log n}\ \frac{1}{nt^{\alpha+1}}\right)$$

As for compute the order estimation for $Q_{n,2}(t)$, there are two cases , where the case in which the values of $\alpha = 0, \beta > 0, G_{n,2}(t)$ will be of the following form:

$$Q_{n,2}(t) = o\left(\frac{(\log n)^{\beta-1}}{n^2(\log n)^{\beta} t}\right) = 0\left(\frac{1}{n^2(\log n)\, t}\right)$$

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But in the case where $\alpha = 0, \beta \in R$, the value of $Q_{n,2}(t)$, after making the required simplifications, is according to the following formula:

$$Q_{n,2}(t) = \sum_{\theta=0}^{r} 0\left(\frac{1}{(nt)^{\theta+1}}\right)$$

Hence,

$$Q_{n,2}(t) = \begin{cases} 0\left(\dfrac{1}{n^2(\log n)\, t}\right), \alpha = 0, \beta > 0 \\ \displaystyle\sum_{\theta=0}^{r} 0\left(\frac{1}{(nt)^{\theta+1}}\right), \qquad \alpha = 0, \beta \in R \end{cases}$$

Lemma 3: Let $P_n = A_n^{0,-\theta}, \theta > 0,$ then $Q_{n,2}(t) = o\left(\frac{(\log n)^{\theta}}{n^2 t^2}\right) + o\left(\frac{1}{nt}\right)$

Lemma 4: Let $P_n = \frac{1}{n \log n}$, then for some constant $c > \pi$,

$$Q_{n,2}(t) = o\left(\frac{\log\log\left({}^{c}/_{t}\right)}{nt(\log\log n)}\right) + o\left(\frac{1}{nt(\log\log n)}\right)$$

The Main Results

Theorem 5:

Let $h \in K_p$ such that

$$\int_0^{\pi} \frac{N(t)\left(\log\frac{k}{t}\right)^{\beta}}{t^{\alpha+1}} dt < \infty, \ B \geq 0$$

and

$$\int_0^{\pi} \frac{\Lambda(t)}{t^{\alpha+1}} dt < \infty, \ \beta < 0$$

Then

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$$\left\| h(y) - \sigma_n^{\alpha,\beta}(y) \right\|_p = \begin{cases} o\left(\frac{1}{n(\log n)^\beta}\right), & \alpha,\beta > o \\ 0\left(\frac{1}{n}\right), & \alpha,\beta \in R \end{cases}$$

Proof:

By lemma 2

Let $h \in k_p$, by M.Riesz theorem [9-12]

$h \in k_p(1 < p < \infty) \rightarrow \tilde{h} \in L_P \rightarrow \tilde{\tilde{h}} \in L_P$ and $\tilde{s}(\tilde{h}) = \tilde{s}\left(\tilde{\tilde{h}}\right)$ if $P = \infty$, then $-f + \frac{1}{2}a_o$ is equivalent to $\tilde{\tilde{h}}$.

Since $\sigma_n^{\alpha,\beta}\left(\tilde{s_n}(\tilde{h}, x) - \tilde{h}\right)$

$$= \frac{1}{\pi}\int_0^\pi \{\tilde{h}(y+t) - \tilde{h}(y+t)\} \sum_{k=0}^{n} \frac{A_{n-k}^{\alpha,\beta} \cos\left(k + \frac{1}{2}\right)t}{2\sin\frac{t}{2}}$$

$$h(y) - \sigma_n^{\alpha,\beta}(y) = \frac{1}{2\pi}\int_0^\pi \{\tilde{h}'(y+t) + \tilde{h}'(y-t)\} Q_n(t) dt$$

Hence, by generalized Minknowki's Inequality

$$\left\| h - \sigma_n^{\alpha,\beta\|\|} \leq \frac{1}{2\pi}\int_0^\pi \right\| \tilde{h}'(y+t) + \tilde{h}'(y-t)\ _p |Q_n(t)| dt$$

Case One: Let$\alpha = 0, \beta > 0$. Then

$$\begin{aligned} \| h(y) - \sigma_n^{\alpha,\beta}(y) \|_p &= 0\left(\int_0^\pi \frac{\Lambda(t)\left(\log\frac{c}{t}\right)^\beta}{n(\log n)^\beta t} dt\right) \\ &= \frac{1}{n(\log n)^\beta} 0\left(\int_0^\pi \frac{\Lambda(t)\log^\beta(c/t)}{t}\right) dt \\ &= 0\left(\frac{1}{n(\log n)^\beta}\right) \end{aligned}$$

Case Two: Let$\alpha = 0, \beta \in R$. Then

$$\left\| h(y) - \delta_n^{\alpha,\beta}(y) \right\|_p$$

$$=0\left(\frac{1}{n}\int_0^\pi \frac{\partial(t)}{t} dt\right) + 0\left(\frac{1}{n^2}\int_0^\pi \frac{\partial(t)}{t^2} dt\right) + \cdots + 0\left(\frac{1}{n^{r+1}}\int_0^\pi \frac{\partial(t)}{t^{r+1}} dt\right) + 0\left(\frac{1}{n^{\alpha+1}(\log n)^\beta}\int_0^\pi \frac{\partial(t)\log^\beta({}^c/_t)}{t^{\alpha+1}} dt\right)$$

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For $\beta \geq 0$, $\int_0^{\pi} \frac{\partial(t)\log^{\beta}(c/t)}{t^{\alpha+1}} dt < \infty$ that implies $\int_0^{\pi} \frac{\partial(t)}{t^{\alpha}} dt < \infty$ for $1 \leq \alpha \leq \alpha + 1$

Hence , for the above result we obtain

$$\left\| h(y) - \sigma_n^{\alpha,\beta}(y) \right\|_p = \begin{cases} o\left(\frac{1}{n(\log n)^{\beta}}\right), & \alpha, \beta > o \\ 0\left(\frac{1}{n}\right), & \alpha, \beta \in R \end{cases}$$

This completes the proof of Theorem 3.1. ■

Note: for the case $\alpha = 0, \beta = -\theta$ for $\theta > 0$, it will be covered in the following theorem:

Theorem 6:

Let $h \in K_p$ such that $\int_0^{\pi} \frac{\partial(t)}{t^2} dt < \infty$, then $\left\| h(y) - \sigma_n^{\beta}(y) \right\|_P = o(\frac{1}{n})$

Proof:

In the same manner as in proving theorem 5

$$\left\| h(y) - \sigma_n^{\alpha,\beta}(y) \right\|_P \leq \frac{1}{2\pi} \left\| \tilde{\tilde{h}}(y+t) - \tilde{\tilde{h}}(y-t) \right\|_P Q_n(t) dt$$

And since $h \in K_p$, we have by lemma 3

$$\left\| h - \sigma_n^{\alpha,\beta} \right\| = o\left(\int_0^{\pi} \frac{\partial(t)}{t^2} dt \right) + o\left(\frac{1}{n}\right) \int_0^{\pi} \frac{\partial(t)}{t} dt$$

$= o\left(\frac{1}{n}\right)$ ■

Theorem 7:

Let $h \in K_p$ such that $\int_0^{\pi} \frac{\partial(t)\log\log(c/t)}{t} dt < \infty$, then

$$\| h(y) - \sigma_n(y) \|_P = o\left(\frac{1}{n(\log n)(\log\log n)}\right)$$

Proof:

By lemma 4

$$\| h(y) - \sigma_n(h,y) \|_P \leq \frac{1}{2\pi} \left\| \tilde{\tilde{h}}(y+t) - \tilde{\tilde{h}}(y-t) \right\|_P Q_n(t) dt$$

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$$o\left(\int_0^{\pi} \partial(t)|Q_n(t)|dt\right)$$

$$= \frac{o(1)}{n(\log\log n)} \int_0^{\pi} \frac{\partial(t) \log\log\left({}^{c}/_{t}\right)}{t} dt + \frac{o(1)}{n(\log\log n)(\log n)} \int_0^{\pi} \frac{\partial(t)}{t}$$

$$= 0\left(\frac{1}{n \log\log n}\right) \qquad \blacksquare$$

Conflict of interests.

There are non-conflicts of interest.

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الخلاصة

تبحث هذه الورقة البحثية في درجة تحليل التقريب لفئة خاصة من الدوال تتميز بشروط استمرارية معينة. ويتم دراسة تقريب هذه الدوال باستخدام تقنيات رياضية مناسبة في ظل قيود محددة. ومن خلال استكشاف سلوك تقريب الدالة في ظل معايير مختلفة، تقدم الدراسة رؤى ثاقبة حول دقة وحدود طرق التقريب الحالية. وتساهم النتائج في فهم أعمق لكيفية تقريب هذه الدوال تمهيدًا لتطبيقها في مجالات مؤثرة.

الكلمات المفتاحية: تحليل التقريب، فئة خاصة من الدوال، دالة ليب، متسلسلة فورييه، الدالة المرافقة