Proposed Generalized Formula for Transmuted Distribution

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Abstract

In this paper proposed generalization formula for transmuted distribution is derived by the researcher , that can use to extend ,generalize, distribution ,and it is to be useful in wider applications in reliability, engineering and in other areas of research.

Keywords: Cumulative Distribution Function, Probability Density Function, Rank Transmuted Distribution, Transmuted Distribution, Distribution, Weibull Distribution.

الخلاصة

في هذا البحث اشتقت صيغه التعميم مقترحه للتوزيع المحول, من قبل الباحث لتكون مفيده وواسعه التطبيقات في مجال المعوليه, الهندسه, وفي مجالات البحث(والحياة) الآخرى.

الكلمات المفتاحية: دالة التوزيع التجميعيه, دالة الكثافه الأحتماليه, توزيع المحول برتبه, التوزيع المحول, توزيع ويبل.

1.1.Introduction

Transmuted distributions are extended models. Shaw & Buckley (2009) , used the rank transmutation map RTM , a tool for the construction of new families of non-Gaussian distributions. They used it to modulate a given base distribution for the purposes of modifying the moments, in particular the skew and kurtosis . They introduced the quadratic rank transmutation map (QRTM) that has been used by many authors to introduce different new important distributions, Weibull distribution Aryal & Tsokos (2011). generalization of the generalized inverse Weibull distribution by Merovci *et al.*, (2013),generalize the Rayleigh by Merovci(2013), the transmuted Lindley distribution by Merovci (2013), original articles on the transmuted Fréchet Distribution Mahmoud and Mandouh(2013), generalization of the exponentiated Lomax distribution by Ashour and Eltehiwy (2013), generalized the three parameter modified Weibull distribution by Khan and King(2013), linear exponential distribution of four-parameter generalized version of the transmuted generalized linear exponential distribution by Elbatal *et al.*, (2013), trasmuted additive Weibull distribution by I.Elbatal and Aryal (2013), beta transmuted Weibull distribution by Pal and Tiensuwan (2014), characterise the transmuted inverse Weibull distribution by Khan, *et al.*, (2014), and a new generalized version, transmuted exponentiated gamma by Hussian (2014). A transmuted exponentiated Weibull geometric distribution is studied by Saboor *et al.*, (2016). While in this paper we propose generalized formula for transmuted distribution, that can use to extend ,generalize, distribution .

2. Main Results

Definition 2.1.

(Proposed General Formula for Transmuted Distribution)

A random variable X is said to have transmuted distribution when its ditrbution function ,$G_2(x)$ is defined as follows:

$$G_2(x) = \begin{cases} (1+\lambda)G_1(x) - 2\lambda G_1^2(x) + \lambda \sum_{i=3}^{n} (-1)^{i+1} G_1^i(x) & n \text{ is odd} \\ G_1(x) + \lambda \sum_{i=1}^{n} (-1)^{i+1} G_1^i(x) & n \text{ is even} \end{cases} \quad (1)$$

where $|\lambda| \leq 1$

and $G_{R12}(u) = G_2\left(G_1^{-1}(u)\right), G_{R21}(u) = G_1\left(G_2^{-1}(u)\right)$ (2)

Formula(1), can be proved using the , Principle of Mathematical Induction D. B. Surowski(2011),, which is stated as

Let N be a set of positive integers, and assume that for each let $n \in N$ we have a property $P(n)$. Assume

1) $P(a)$ is true for some $a \in N,$
2) Whenever $P(m)$ is true for al $a \leq m < n$, then $P(n)$ is also true, and then $P(n)$ is also true $a \leq n$

That is

1) Let $a = 2$, then $G_2(x) = G_1(x) + \lambda \sum_{i=1}^{2} (-1)^{i+1} G_1^i(x) G_1(x)$
$G_2(x) = (1+\lambda)G_1(x) - G_1^2(x)$

which is called quadratic rank transmutation, then P_2 is true,

2) For any integer $m \geq 2$, if P_k is true, either k is even, that is
$G_2(x) = G_1(x) + \lambda \sum_{i=1}^{k} (-1)^{i+1} G_1^i(x)$ is true, then $k+1$ is odd
$G_2(x) = (1+\lambda)G_1(x) - 2\lambda G_1^2(x) + \lambda \sum_{i=3}^{k+1} (-1)^{i+1} G_1^i(x)$ is true
or k is odd, then
$G_2(x) = (1+\lambda)G_1(x) - 2\lambda G_1^2(x) + \lambda \sum_{i=3}^{k} (-1)^{i+1} G_1^i(x)$ is true,
and
$G_2(x) = G_1(x) + \lambda \sum_{i=1}^{k+1} (-1)^{i+1} G_1^i(x)$ is true.

Thus $P(m)$ is true, either m is even or odd , then $P(m+1)$ is true.

Then the statement $P(n)$ is true for all integers $n \geq a = 2$.

2.1.1.Properties of Proposed General Formula for Transmuted Distribution

From this formula,(1), we note the following:

1) If $\lambda = 0$, then we have the distribution of the base random variable,
2) If $u = G_1(x)$, then from(1), $G_2(x) = \begin{cases} 0, u = 0 \\ 1, u = 1 \end{cases}$,
3) $G_2(x) = \frac{1}{2}$ if $\lambda = 0,$
4) The even rank transmuted distribution is symmetric while the odd transmuted distribution is not, according to this formula

$G_{R12}(1-u) = 1 - G_{R12}(u).$ (3)

Definition2.1.1.1. (Fourth Rank Transmuted Distribution)

A random variable X is said to be fourth rank transmuted Weibull distribution random variable if it has the following cumulative distribution function ,cdf,

$G_2(x) = (1+\lambda)G_1(x) - \lambda G_1^2(x) + \lambda G_1^3(x) - \lambda G_1^4(x)$ (4)

and the pdf

$g_2(x) = g_1(x)[1+\lambda - 2\lambda G_1(x) + 3\lambda G_1^2(x) - 4\lambda G_1^3(x)]$ (5)

where $g_1(x), g_2(x)$ are the pdf's corresponding to the cdf's $G_1(x)$ and $G_2(x)$ respectively. We note

Firstly, when $\lambda = 0$, $G_2(x) = G_1(x)$

Secondly, (5) is a pdf where

$$g_2(x) = g_1(x)[1 - \lambda\{2G_1(x) - 3G_1^2(x) + 4G_1^3(x) - 1\}]$$

since

1) $g_2(x) > 0$ when

$g_1(x) > 0$, a) if $\lambda < 0$ and $[1 - \lambda\{2G_1(x) - 3G_1^2(x) + 4G_1^3(x)\}] < 0$

b) if $\lambda > 0$ and $1 > \lambda\{2G_1(x) - 3G_1^2(x) + 4G_1^3(x) - 1\}$

2) $\int_{-\infty}^{\infty} g_2(x)\,dx = 1, u = G_1(x), du = G_1(x)\, g_1(x)dx$

$\int_{-\infty}^{\infty} g_2(x)\,dx = [1 - \lambda\{u^2 - u^3 + u^4 - u \;\;_0^1\}] = 1$.

Which means $G_2(0) = 0, G_2(1) = 1$

Thirdly , since $G_{R12}(u) = G_2\,(G_1^{-1}(u))$, then

$G_{R12}\left(\frac{1}{2}\right) = \frac{(1+\lambda)}{2} - \frac{\lambda}{4} + \frac{\lambda G_1^3(x)}{8} - \frac{\lambda G_1^4(x)}{16}$

$G_{R12}\left(\frac{1}{2}\right) = \frac{1}{2}$,only when $\lambda = 0$, to say that rank transmutation is median-Preserving

Eample 2.1.1.2. (Fourth Rank Transmuted Weibull Distribution)

According to (1) fourth rank transmuted Weibull distribution,FRTWD, can be defined as

$F_{FRTWD}(x;\alpha,\beta,\lambda) =$

$= 1 - e^{-\alpha x^\beta} + 2\lambda e^{-\alpha x^\beta} - 4\lambda e^{-2\alpha x^\beta} + 3\lambda e^{-3\alpha x^\beta} - 4\lambda e^{-4\alpha x^\beta}$ (6)

Shape parameter $\beta > 0$, scale parameter $\alpha > 0$,

Now the pdf , reliability, hazard, cumulative hazard functions are respectively

$f_{FRTWD}(x;\alpha,\beta,\lambda) = \beta\alpha x^{\beta-1}e^{-\alpha x^\beta}[1 - 2\lambda + 8\lambda e^{-\alpha x^\beta} - 9\lambda e^{-2\alpha x^\beta} + 4\lambda e^{-3\alpha x^\beta}]$

(7)

$R_{FTWD}(x;\alpha,\beta,\lambda) = e^{-\alpha x^\beta}[1 - 2\lambda + 4\lambda e^{-\alpha x^\beta} - 3\lambda e^{-2\alpha x^\beta} + 4\lambda e^{-3\alpha x^\beta}]$

(8)

$h_{FTWD}(x;\alpha,\beta,\lambda) = \frac{\beta\alpha x^{\beta-1}[1-2\lambda+8\lambda e^{-\alpha x^\beta} - 9\lambda e^{-2\alpha x^\beta} + 4\lambda e^{-3\alpha x^\beta}]}{[1-2\lambda+4\lambda e^{-\alpha x^\beta} - 3\lambda e^{-2\alpha x^\beta} + 4\lambda e^{-3\alpha x^\beta}]}$ (9)

for $[1 - 2\lambda + 4\lambda e^{-\alpha x^\beta} - 3\lambda e^{-2\alpha x^\beta} + 4\lambda e^{-3\alpha x^\beta}] > 0$ if $\alpha, \beta > 0$

$H_{FRTW}(x;,\alpha,\beta,\lambda) = -\ln[\,R_{FTWD}(x;\alpha,\beta,\lambda)]$

It can be given as

$H_{FRTW}(x;,\alpha,\beta,\lambda) =$

$-\ln e^{-\alpha x^\beta}[1 - 2\lambda + 4\lambda e^{-\alpha x^\beta} - 3\lambda e^{-2\alpha x^\beta} + 4\lambda e^{-3\alpha x^\beta}]$ (10)

Therefore the rth moment about the origin is defined as

$$\mathrm{E}(X^r)=\frac{\Gamma\left(\frac{r}{\beta}+1\right)}{\alpha^{\frac{r}{\beta}}}\left[1-2\lambda+\frac{\lambda}{2^{\frac{r}{\beta}-2}}-\frac{\lambda}{3^{\frac{r}{\beta}-1}}+\frac{\lambda}{4^{\frac{r}{\beta}}}\right],r\in\mathbb{Z}^+,\alpha,\beta>0 \qquad (11)$$

And the rth moment about the mean is defined as

$$\mathrm{E}(X)=\mu=\mathrm{E}(X)=\frac{\Gamma(\frac{1}{\beta}+1)}{\alpha^{\frac{1}{\beta}}}[1-2\lambda+\frac{\lambda}{2^{\frac{1}{\beta}-2}}-\frac{\lambda}{3^{\frac{1}{\beta}-1}}+\frac{\lambda}{4^{\frac{1}{\beta}}}]$$

$$\mathrm{E}(X^2)=\mathrm{E}(X^2)=\frac{\Gamma(\frac{2}{\beta}+1)}{\alpha^{\frac{2}{\beta}}}[1-2\lambda+\frac{\lambda}{2^{\frac{2}{\beta}-2}}-\frac{\lambda}{3^{\frac{2}{\beta}-1}}+\frac{\lambda}{4^{\frac{2}{\beta}}}],\text{ when } r=2$$

Therefore from (12) we can find all central moments about the mean μ , the variance when $r=2$

$$\sum_{j=0}^{2}C_j^r\,(-\mu)^j\,\frac{\Gamma\left(1+\frac{r-j}{\beta}\right)}{\alpha^{\frac{r-j}{\beta}}}\left[1-2\lambda+\frac{\lambda}{2^{\frac{2-j}{\beta}-2}}-\frac{\lambda}{3^{\frac{2-j}{\beta}-1}}+\frac{\lambda}{4^{\frac{2-j}{\beta}}}\right]=$$

$$=\frac{\Gamma\left(1+\frac{2}{\beta}\right)}{\alpha^{\frac{2}{\beta}}}\left[1-2\lambda+\frac{\lambda}{2^{\frac{2}{\beta}-2}}-\frac{\lambda}{3^{\frac{2}{\beta}-1}}+\frac{\lambda}{4^{\frac{2}{\beta}}}\right]$$

$$-\frac{2\mu\Gamma(1+\frac{1}{\beta})}{\alpha^{\frac{1}{\beta}}}\left[1-2\lambda+\frac{\lambda}{2^{\frac{1}{\beta}-2}}-\frac{\lambda}{3^{\frac{1}{\beta}-1}}+\frac{\lambda}{4^{\frac{1}{\beta}}}\right]+\frac{\mu^2\Gamma(1)}{\alpha^0}[1+2\lambda-3\lambda+\lambda]$$

$$=\frac{\Gamma(1+\frac{2}{\beta})}{\alpha^{\frac{2}{\beta}}}\left[1-2\lambda+\frac{\lambda}{2^{\frac{2}{\beta}-2}}-\frac{\lambda}{3^{\frac{2}{\beta}-1}}+\frac{\lambda}{4^{\frac{2}{\beta}}}\right]-\mu^2 \qquad (13)$$

The moment generating function given by

$$M_x(t)=\int_0^\infty e^{tx}f_{FRTWD}(x;\alpha,\beta,\lambda)\,dx=\int_0^\infty\sum_{j=0}^{\infty}\frac{t^j}{j!}\beta\alpha x^{j+\beta-1}e^{-\alpha x^\beta}[1-2\lambda+8\lambda e^{-\alpha x^\beta}-9\lambda e^{-2\alpha x^\beta}+4\lambda e^{-3\alpha x^\beta}]$$

$$M_X(t)=\sum_{j=0}^{\infty}\frac{t^j}{j!}\frac{\Gamma\left(\frac{j}{\beta}+1\right)}{\alpha^{\frac{j}{\beta}}}[1-2\lambda+\frac{\lambda}{2^{\frac{j}{\beta}-2}}-\frac{\lambda}{3^{\frac{j}{\beta}-1}}+\frac{\lambda}{4^{\frac{j}{\beta}}}] \qquad (14)$$

We know that

$$\frac{d^{(r)}M_X(t)}{dt^r}\bigg\|_{t=0}=\mathrm{E}(X^r),\quad r\in\mathbb{Z}^+,\alpha,\beta>0 \qquad (15)$$

That is

$$\frac{d^{(1)}M_X(t)}{dt}\bigg\|_{t=0}=\frac{\Gamma(\frac{1}{\beta}+1)}{\alpha^{\frac{1}{\beta}}}\left[1-2\lambda+\frac{\lambda}{2^{\frac{1}{\beta}-2}}-\frac{\lambda}{3^{\frac{1}{\beta}-1}}+\frac{\lambda}{4^{\frac{1}{\beta}}}\right]+0+\cdots.+0=\mathrm{E}(X)$$

$$\frac{d^{(2)}M_X(t)}{dt^2}\bigg\|_{t=0}=0+\frac{\Gamma(\frac{2}{\beta}+1)}{\alpha^{\frac{1}{\beta}}}\left[1-2\lambda+\frac{\lambda}{2^{\frac{2}{\beta}-2}}-\frac{\lambda}{3^{\frac{2}{\beta}-1}}+\frac{\lambda}{4^{\frac{2}{\beta}}}\right]+0+\cdots.+0$$

$$=\mathrm{E}(X^2)$$

And so on. Now the following are the plots of the pdf,cdf, reliability, hazard, cumulative hazard functions are respectively

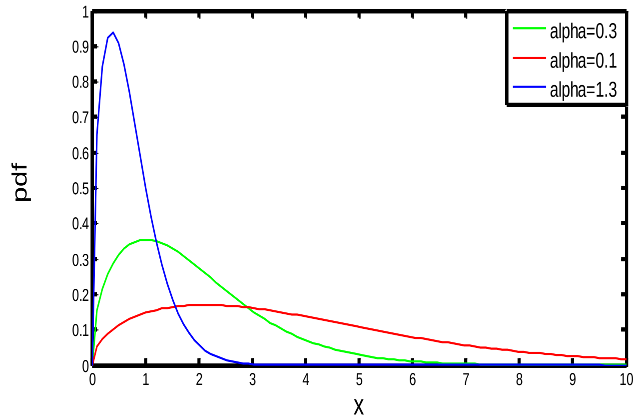

Figure (1) plot of the pdf of FRTWD at alpha $= \boldsymbol{\alpha} = \mathbf{0.3\ 0.1\ 1.3}, \boldsymbol{\beta} = \mathbf{1.5\ 1.3\ 2},\ \boldsymbol{\lambda} = \mathbf{0.1\ 0.4\ 0.2}$

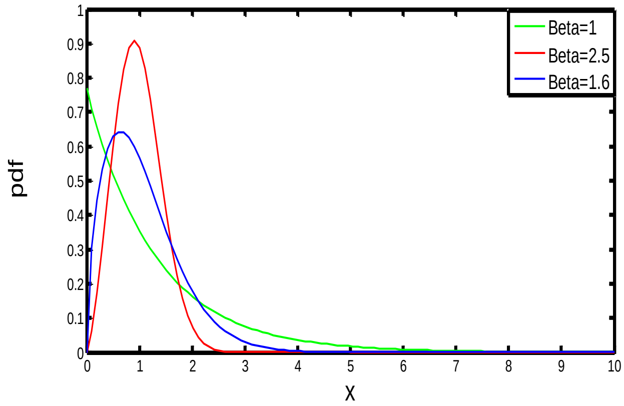

Figure (2) plot of the pdf of FRTWD at alpha $= \boldsymbol{\alpha} = \mathbf{0.7\ 0.2\ 0.5}, \boldsymbol{\beta} = \mathbf{1, 2.5, 1.6},\ \boldsymbol{\lambda} = \mathbf{0.1\ 0.4\ 0.2}$

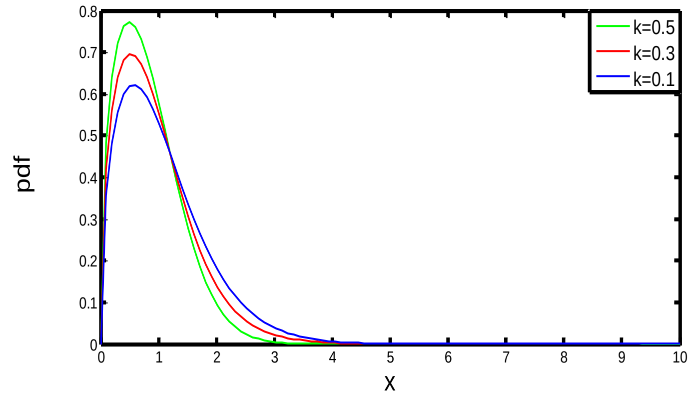

Figure (3) plot of the pdf of FRTWD at alpha $= \boldsymbol{\alpha} = \mathbf{0.7\ 0.2\ 0.5}, \boldsymbol{\beta} = \mathbf{1.5, 1.3, 2},\ \boldsymbol{\lambda} = \mathbf{0.5, 0.3, 0.1}$

From the figures 1,2,3, we note that shape of the pdf is affected by the change of the values of shape parameter $\beta > 0$ more than α, λ. The FRTWD is reduced to exponential distribution as in figure 2 at $\beta = 1$. Therefore the shape of the pdf is increased as $x \to 0$,and it is decreased as $x \to \infty$,with heavy right tail and has only mode at different values of the parameters

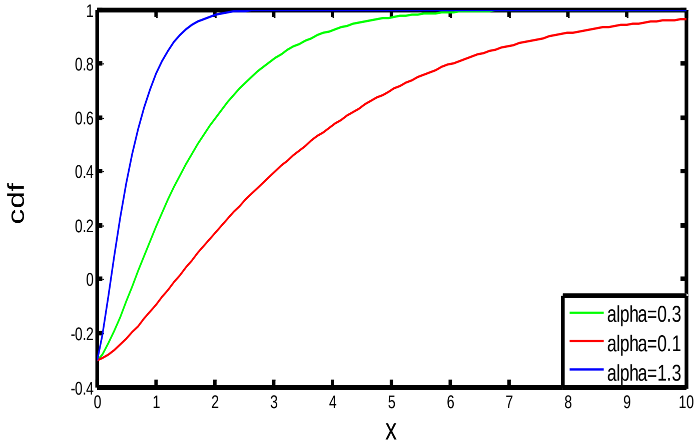

Figure (4) plot of the cdf of FRTWD at alpha $= \boldsymbol{\alpha} = \mathbf{0.3\ 0.1\ 1.3}, \boldsymbol{\beta} = \mathbf{1.5\ 1.3\ 2},\ \boldsymbol{\lambda} = \mathbf{0.1\ 0.4\ 0.2}$

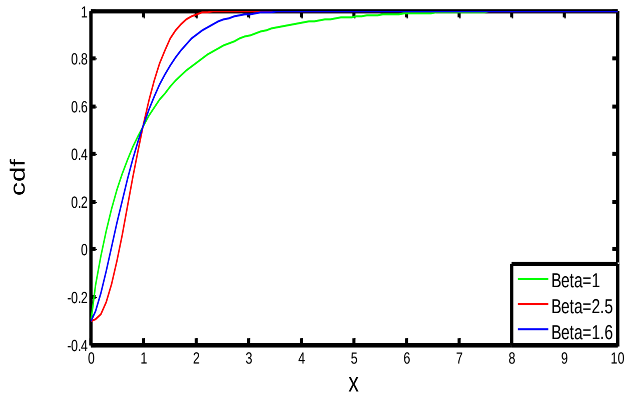

Figure (5) plot of the cdf of FRTWD at alpha $= \boldsymbol{\alpha} = \mathbf{0.7\ 0.2\ 0.5}, \boldsymbol{\beta} = \mathbf{1, 2.5, 1.6},\ \boldsymbol{\lambda} = \mathbf{0.1\ 0.4\ 0.2}$

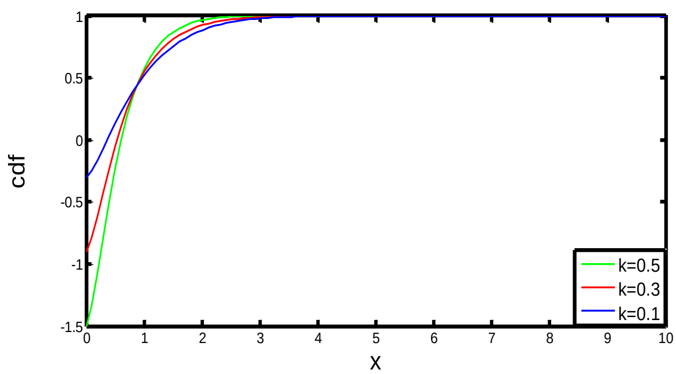

Figure (6) plot of the cdf of FRTWD at alpha $= \boldsymbol{\alpha} = \mathbf{0.7\ 0.2\ 0.5}, \boldsymbol{\beta} = \mathbf{1.5, 1.3, 2},\ \boldsymbol{\lambda} = \mathbf{0.5, 0.3, 0.1}$

From the figures 4,5,6, we note that shape of the cdf of the FRTWD is decreased as $x \to 0$,and it is increased as $x \to \infty$, at different values of the parameters

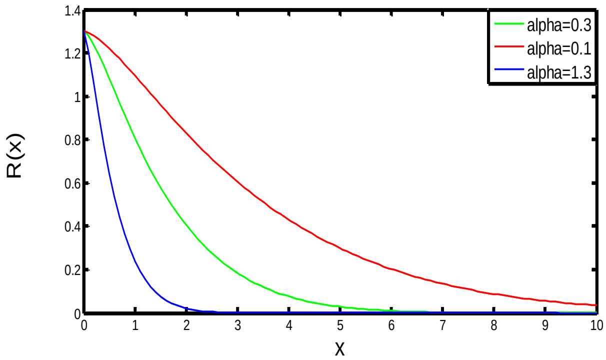

Figure (7) plot of the $R_{FTWD}(x;\alpha,\beta,\lambda)$ at alpha $= \alpha = 0.3\ 0.1\ 1.3, \beta = 1.5\ 1.3\ 2,\ \lambda = \ 0.1\ 0.4\ 0.2$

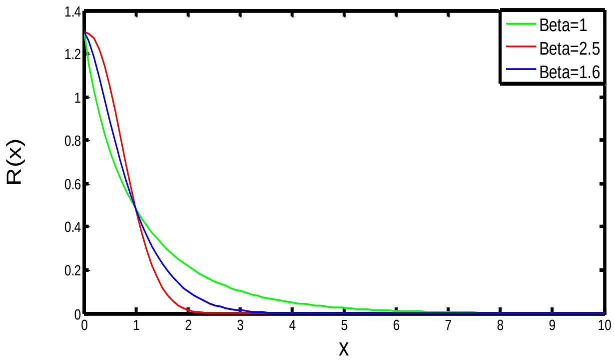

Figure (8) plot $R_{FTWD}(x;\alpha,\beta,\lambda)$at alpha $= \alpha = 0.7\ 0.2\ 0.5, \beta = 1, 2.5, 1.6,$ $\lambda = \ 0.1\ 0.4\ 0.2$

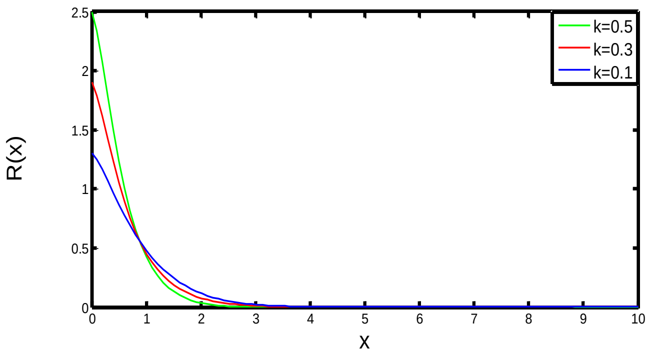

Figure (9) plot of the $R_{FTWD}(x;\alpha,\beta,\lambda)$at alpha $= \alpha = 0.7\ 0.2\ 0.5, \beta = 1.5, 1.3, 2,\ \lambda = \ 0.5, 0.3, 0.1$

From the figures 7,8,9, we note that shape of the $R_{FTWD}(.)$ of the FRTWD is increased as $x \to \infty$, and it is decreased as $x \to 0$, at different values of the parameters. Also there is the same note for the following figures 10,11,12 of $h_{FTWD}(.)$.

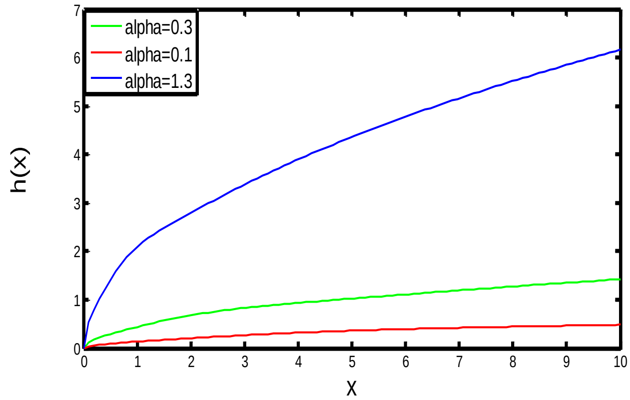

Figure (10) plot $h_{FTWD}(x;\alpha,\beta,\lambda)$ at alpha = $\alpha = 0.3\ 0.1\ 1.3, \beta = 1.5\ 1.3\ 2,$ $\lambda = \ 0.1\ 0.4\ 0.2$

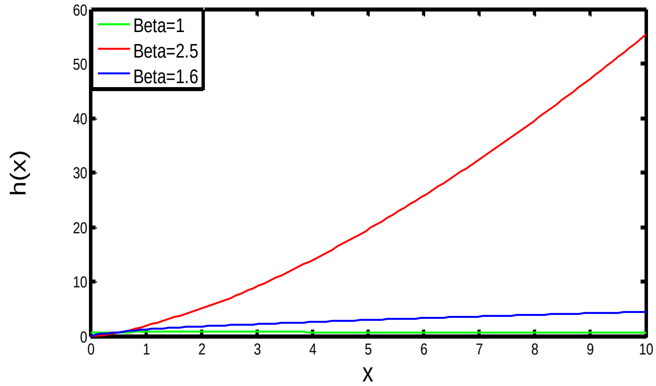

Figure (11) plot $h_{FTWD}(x;\alpha,\beta,\lambda)$at alpha = $\alpha = 0.7\ 0.2\ 0.5, \beta = 1, 2.5, 1.6,$ $\lambda = \ 0.1\ 0.4\ 0.2$

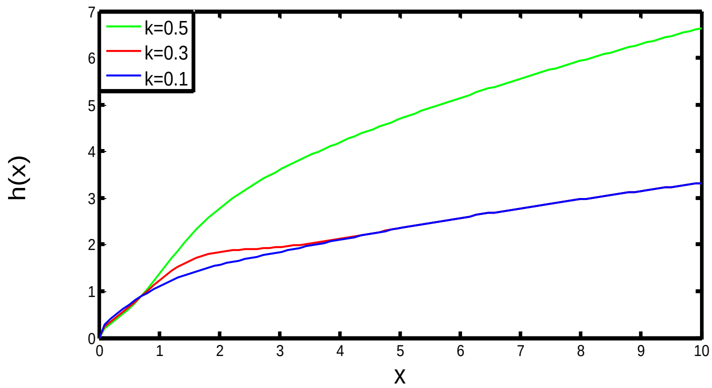

Figure (12) plot of the $h_{FTWD}(x;\alpha,\beta,\lambda)$at alpha = $\alpha = 0.7\ 0.2\ 0.5, \beta =$ $1.5, 1.3, 2, \ \lambda = \ 0.5, 0.3, 0.1$

From these figures we note that the shape of the pdf, cdf, reliability, hazard, cumulative hazard functions are closely similar for is of the Weibulll distribution.

Therefore we can find other statistical and mathematical properties of this distribution that may use to study some life data in many fields of life.

3.Conclusions

In this paper proposed general formula for transmuted distribution has interested properties, and it helps us in construction new distributions, models, as the fourth rank transmuted Weibull distribution, FRTWD

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