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The Distribution of a General Non-Centeral Quadratic From in Normal Variates

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Abstract

The distribution of the difference of two independent non-central chi-square variates along with the distribution of a general non-central quadratic from in normal variates are obtained. Robinson [2] obtained the distribution of a general quadratic from in normal variates.

Distribution of the Difference of two independent Non-Central chi-Square Variates

Denote by $\psi_m(t,\lambda) = (1-2it)^{-m/2} \exp\{\lambda/2((1-2it)^{-1}-1)\}$,

the charateristic function of a non-central chi-square variate with m degrees of freedom and non-centrality parameter λ. The chracterisitic function of $-X_n^2(\theta)$ will be:

$\psi_n(-t,\theta) = (1+2it)^{-n/2} \exp\{\theta/2((1+2it)^{-1}-1)\}$.

for a given indepenedent non-central chi-square variates, the probability density function of $Z_{mn}(\lambda,\theta) = X_m^2(\lambda) - X_n^2(\theta)$ is :

$$f_{mn}(z/\lambda,\theta) = 1/2\,(Z/2)^{1/2(m+n)-1}\exp\{-1/2(\lambda+\theta+Z)\}\sum_{u=0}^{\infty}\frac{(\lambda z/4)^u}{\Gamma(1/2m+u)u!}\sum_{s=0}^{\infty}\frac{(\theta z/4)^s}{s!}$$

$$F\,(n/2+s;1/2(m+n)+u+s;z)\,, \qquad z>0,$$

$$f_{mn}(z/\lambda,\theta) = 1/2(-Z/2)^{1/2(m+n)-1}\exp\{-1/2(\lambda+\theta-Z)\}\sum_{u=0}^{\infty}\frac{(-\lambda z/4)^u}{u!}\sum_{s=0}^{\infty}\frac{(-\theta z/4)^s}{\Gamma(n/2+s)s!}$$

$$F\,(m/2+u;1/2(m+n)+u+s;-z) \qquad z<0,$$

$$f_{mn}(z/\lambda,\theta) = 2^{-1/2(m+n)}\exp\{-1/2(\lambda+\theta)\}\sum_{u=0}^{\infty}\frac{(\lambda/4)^u}{\Gamma(m/2+u)u!}\sum_{s=0}^{\infty}\frac{(\theta/4)^s}{\Gamma(1/2n+s)s!}$$

$$\Gamma(1/2(m+n)+u+s-1) \qquad z=0,$$

where F (a,b;x) is the confluent hypergeometric function .

Proof

From the inversion theorem

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الحل المقترح لمشكلة تحليل المتوالية في فضاء منته في تطبيقات نظام تشفير المفتاح المعلن

ستار بدر سد خان   عبد الستار سلمان

الخلاصة

يطرح هذا البحث مقترحاً لطريقة تتجاوز مشكلة تحديد قيمة رمز ليجندر والتي تعتمد على مفهوم اختزالية متوالية الحدود ، وذلك بالاعتماد على استخدام القاسم المشترك الاعظم . وبذلك تجاوزت الطريقة المقترحة التحديدات التي تفرضها طريقة رابين . تم بناء برنامج على الحاسبة الالكترونية والتحقق من نجاح هذه الطريقة المقترحة على بعض الامثلة .

$$f_{mn}(z/\lambda,\theta) = 1/(2\pi) \int_{-\infty}^{\infty} \exp\{-itz\} \psi_{mn}(t,\theta,\lambda)dt, \qquad (2.1)$$

$$= (2\pi)^{-1} \exp\{(-1/2(\theta+\lambda)\} \sum_{u=0}^{\infty}(\lambda/2)^{u}/u! \sum_{s=0}^{\infty}(\theta/2)^{s}/s! \int_{-\infty}^{\infty} (1-2it)^{-(m/2+u)} (1+2it)^{-(n/2+s)} \exp\{-itz\}dt, \quad (2.2)$$

where $\psi_{mn}(t,\theta,\lambda)$ represents the characteristic function of $Z_{mn}(\lambda,\theta)$, and equal to $\psi_m(t,\lambda)\ \psi_n(-t,\theta)$. Now, for $Z_{mn}(\lambda,\theta) > 0$, the intergral of (2.2) will be :

$$\frac{\exp(-z/2)}{(2\pi i)2^{1/2(m+n)+u+s)}} \int_{1/2+i\infty}^{1/2-i\infty} w^{-(m/2+u)} (1-w)^{-(n/2+s)} \exp\{wz\}dw, \qquad (2.3)$$

where $w = 1/2(1-2it)$.

Consider the contour consisting of the line from $-iR+(1/2)$ to $-R+(1/2)$, the circular arc from $-iR+(1/2)$ to $-R+(1/2)$, a loop in the positive direction around the origin from $-R+(1/2)$ to $-R+(1/2)$ and the circular arc from $-R+(1/2)$ to $iR+(1/2)$. As R tends to ∞, the integrals around the circular arces tend to zero. Put $w=h\exp(-i\pi)$, then (2.3) will be:-

$$\frac{\exp(-z/2)}{2^{1/2(m+n)+u+s)}(2\pi i)} \int_{\infty}^{0+} h^{-(m/2+u)} (1+h)^{-(n/2+s)} \exp\{hz\} \exp\{-i\pi(1-(m/2)-u)\}dh. \qquad (2.4)$$

$$\frac{\exp(z/2)}{\Gamma(m/2+u)2^{1/2(m+n)+u+s}} F(1-m/2-u, 2-1/2(m+n)-u-s;z) \qquad (2.5)$$

$$= \frac{\exp(z/2)}{2\Gamma(m/2+u)} (z/2)^{1/2(m+n)+u+s-1} F(n/2+s, 1/2(m+n)+u+s;z), \qquad (2.6)$$

where

$$F(a,b;x) = (2\pi i)^{-1} \exp(-a\pi i)\ \Gamma(1-a) \int_{\infty e^{i\varnothing}}^{0+} \exp\{-xy\}y^{a-1}(1+y)^{b-a-1}dy, \quad (2.7)$$

$-1/2\ \pi < \arg x\ \varnothing < 1/2$, $\arg z = \varnothing$ at the begining of the loop and $F(a,b;x) = x^{1-b}\ F(a-b+1, 2-b;x)$, see Erdelyi [1].

Thererfore

$$f_{mn}(z/\lambda,\theta) = 1/2(z/2)^{1/2(m+n)-1} \exp\{-1/2(\lambda+\theta+z)\} \sum_{u=0}^{\infty} \frac{(-\lambda z/4)^{u}}{\Gamma(m/2+u)u!}$$

$$\sum_{s=0}^{\infty} \frac{(\theta z/4)^s}{s!} F(n/2+s, 1/2(m+n)+u+s; z) . \quad (2.9)$$

Similarly, for Zmn $(\lambda,\theta) <0$, if we put $w=1/2(1+2it)$, we obtain:-

$$f_{mn}(z/\lambda,\theta) = 1/2(-z/2)^{1/2(m+n)-1} \lambda \exp\{-1/2(\lambda+\theta-z)\} \sum_{u=0}^{\infty} \frac{(-\lambda z/4)^u}{u!}$$

$$\sum_{s=0}^{\infty} \frac{(-\theta z/4)^s}{(n/2+s)s!} F(m/2+u, 1/2\ (m+n) +u+s; -z) \quad (2.10)$$

for $z=0$, it can be easily seen that

$$f_{mn}(z/\lambda,\theta) = \exp\{-1/2(\lambda+\theta)\} \sum_{u=0}^{\infty} \frac{(\lambda/2)^u}{u!} \sum_{s=0}^{\infty} \frac{(\theta/2)^s}{s!} f_{(m+2u)(n+2s)}(z) \quad (2.11)$$

where

$f_{(m+2u)(n+2s)}(z)$: represents the distribution of the difference of two independent chi-square variates (see Robinson[2]), which is by continuing, for $z=0$, defdined to be :-

$$\frac{\Gamma(1/2(m+n)+u+s-1)}{(2)^{1/2(m+n)+u+s}\Gamma(m/2+u)\Gamma(n/2+s)}$$

and the theorem is proved.

Distribution of a Non-Central Quadratic Form

The characteristic function of the variate a $X_m^2(\lambda)$, for any constant $a>=1$, is:-

$$\psi_1(t)=(1-2iat)(-m/2 \exp\{\lambda/2((1-2iat)^{-1}-1)\} \quad (3.1)$$

$$=\exp\{-\lambda/2\} \sum_{r=0}^{\infty} \frac{(\lambda/2)^r}{r!}(1-2iat)^{-(m/2+r)} \quad (3.2)$$

Put $w = (1-2it)^{-1}$, then

$$(1-2iat)^{-(m/2+r)} = [a/w(1-w(1-1/a))]^{-(m/2+r)} \quad (3.3)$$

also

$$(1-w(1-1/a))^{-(m/2+r)} = \sum_{j=0}^{\infty} [(j)^{-(m/2-r)} (-w(1-1/a))$$

$$=\sum_{j=0}^{\infty}(-1)^j (j)^{-(m/2+r+j-1)}(-w(1-1/a))^j$$

$$=\sum_{j=0}^{\infty} \frac{(m/2+r+j)(w(1-1/a))^j}{\Gamma(m/2+r)j!} \qquad (3.4)$$

Hence, from (3.4) and (3.3), the characteristic function in (3.2) will be:-

$$\psi_1(t)=\sum_{r=0}^{\infty}\sum_{j=0}^{\infty} A_{rj}\ (1-2it)^{-(m/2+r+j)}$$

where

$$A_{rj} = a^{-(m/2)} \exp\{-\lambda/2\} \frac{\Gamma(m/2+r+j)(\lambda/2a)^r(1-1/a)^j}{\Gamma(m/2+r)r!j!}$$

Similarly, we can get that the characteristic function of $-bx^2_n(\theta)$, for any constant $b>=1$, to be:-

$$\psi_2(t)=\sum_{r=0}^{\infty}\sum_{j=0}^{\infty} B_{rj}\ (1+2it)^{-(n/2+r+j)} \qquad (3.6)$$

where

$$B_{rj} = b^{-(n/2)} = \exp\{-\theta/2\} \frac{\Gamma(n/2+r+j)(\theta/2b)^r(1-1/b)^j}{\Gamma(n/2+r)r!j!}$$

Now we may extend theorem 1 to a general quadratic form.

Theorem2

Denote $Z= a_1X^2_{m1}(\lambda_1) + \ldots + a_pX^2_{mp}(\lambda_p)-b1X^2n1(\theta_p)-\ldots-bqX^2nq(\theta_q)$ where the non-central chi-square variates are independent and $a1,\ldots,ap,b1,\ldots,bq$ are positive constants such that

$a\,l>=1(l=1,\ldots,p),\ bk>=1(k=1,\ldots,q)$

Define

$$A^1_{rj}(x)=a_1^{-(m1/2)}\exp\{-\lambda_1/2\} \frac{\Gamma(m1/2+r1+j1)\ (\lambda x/4a1)^{r1}(x/2(1-1/a1))^{j1}}{\Gamma(m_1/2 + r_1)r1!j1!}$$

$l=1,\ldots,p,$

$$B^k_{rj}(x)=b_k^{-(nk/2)}\exp\{-\ \theta_k/2\} \frac{\Gamma(n_k/2 +r_k+j_k)\theta_k x/4a_k)^{r1}[x/2(1-1/a_k)]^{jk}}{\Gamma(n_k/2+r_k)r_k!j_k!}$$

$k=1,\ldots,q.$

Let $N=n1+\ldots+nq$, $M=m1+\ldots+mp$,

$R = r1+\dots+rp$, $J = j1+\dots+jp$,

,$R^1 = r1+\dots+rq$, $J^1 = j1+\dots+jq$

$\Lambda = \lambda 1+\dots+\lambda p$, $\Theta = \theta 1+\dots+\theta q$

then

$fmn(x/\underline{\lambda},\underline{\theta}) = 1/2(x/2)^{1/2(M+N)-1}\exp\{-1/2(\Lambda+\Theta+x)\} \Sigma_{r1}\Sigma_{j1}A^1_{rj}(x)\dots \Sigma_{rp}\Sigma_{jp}A^p_{rj}(x)\Sigma_{r1}\Sigma_{j1} B^1rj(x)\dots\Sigma rq\Sigma jq B^q rj(x)F(N/2+R+J , (M+N)/2+R+J + R' +J',X)/\Gamma(M/2+R+J)$ $x>0$

$fmn(x,\lambda,\theta) = 1/2(-x/2)^{\{(M+N)/2\}-1} \exp\{-1/2(\Lambda+\Theta+x)\} \Sigma_{r1}\Sigma_{j1}A^1_{rj}(-x)\dots \Sigma_{rp}\Sigma_{jp}A^p_{rj}(-x)_{r1}\Sigma_{j1}B^1_{rj}(-x)\dots\Sigma_{rq}\Sigma_{jq} B^q_{rj}(-x)$
$F(M/2 +R + J , 1/2 (M + N) + R + J +R'+J',-x) / \Gamma(N/2+R+J)$ $x<0$

$fmn(x/\underline{\lambda},\underline{\theta}) = (2)^{1/2(M+N)-1}\exp\{-1/2(\Lambda+\Theta+x)\} \Sigma_{r1}\Sigma_{j1}A^1_{rj}(\quad)\dots$

$\Sigma_{rp}\Sigma_{jq}A^p_{rj}(\quad)$ $\Sigma_{r1}\Sigma_{j1}B^1_{rj}(\quad)$
$\Sigma_{rq}\Sigma_{jq}B^q_{rj}(\quad)$

$\Gamma(1/2(M+N)+R+J+R'+J'-1)/\Gamma(N/2+R+J)\ \Gamma(M/2+R+J)$

$X=0$

From (3.5) and (3.6), the characteristic function of x is :

$$\psi(t)=\Pi_{l=1}^{p}\left[\sum_{r_1=0}^{\infty}\sum_{j_1=0}^{\infty} A^1_{rj}\,(1-2it)^{-(m1/2+r1+j1)}\right]\ \Pi_{k=1}^{q}\left[\sum_{r_k=0}^{\infty}\sum_{j_k=0}^{\infty} B^k_{rj}\,(1-2it)^{-(nk/2+rk+jk)}\right] \quad (3.7)$$

$$=\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}A^1_{rj}\ \dots\ \sum_{rp=0}^{\infty}\sum_{jp=0}^{\infty}A^p_{rj}\ \sum_{r1=0}^{\infty}\sum_{j1=0}^{\infty}B^1_{rj}\dots\ \sum_{rq=0}^{\infty}\sum_{jq=0}^{\infty}B^q_{rj}\ (1-2it)^{-(M+2R+2J)/2}\,(1+2it)^{-(N+2R+2J)/2} \quad (3.8)$$

So, if we use the same methods of proof of theorem1. we get the result of the theorem.

References

1- Erdeyli, A., *et al* (1953). Higher Transcendental Functions. Bateman Manuscript Project, Califorina Institute of Technology.Mc. Graw-Hill, New York.

2- Robinson, J. (1965). "The distribution of ageneral quadratic form in normal variates". Austral. J. Statist., 7(3), 110-114.